JOSEP MARIA FONT

Generalized Matrices in Abstract Algebraic Logic

Abstract. The aim of this paper is to survey some work done recently or still in progress that applies *generalized matrices* (also called *abstract logics* by some) to the study of sentential logics. My main concern will be to emphasize the links between this line of research and other existing frameworks in Algebraic Logic, either well-established ones (such as the old *theory of logical matrices* and the younger theories of *protoalgebraic logics*, *algebraizable logics* and the associated hierarchy) or really new ones (such as the theory of *algebraizability of Gentzen systems* or the *model theory of equality-free logic*). I would like to convey the idea that the interaction between these neighbouring fields may be specially fruitful, as it seems to be one of the leading forces in the shaping of this emerging field called *Abstract Algebraic Logic*.

Abstract Algebraic Logic (AAL) may be described as today's view of Algebraic Logic. The term began to be used around the mid-nineties, promoted by WIM BLOK and DON PIGOZZI (the expression "universal algebraic logic" was also used for a short time). In 1997 a *Workshop on Abstract Algebraic Logic* [35] was organized as the closing event of a *Semester in Algebraic Logic and Model Theory* of the *Centre de Recerca Matemàtica* of the *Institut d'Estudis Catalans*, located in Bellaterra, near Barcelona. Selected from its more than thirty talks, twelve papers have been published in two special issues of *Studia Logica* (number 1 of volume 65, 2000, and numbers 1/2 of volume 74, 2003). One of them [38] is a long survey of the field. In the meantime, the encyclopedic monograph [22] by JANUSZ CZELAKOWSKI has appeared. Also the long paper [26] by CZELAKOWSKI and PIGOZZI contains a readable, substantial introduction to the field; finally, a forthcoming long paper surveying an important part of the subject is BLOK and PIGOZZI's [6]. With this material now available, I feel safe in limiting my aim to the topic indicated by the title and in not giving too many details; interested readers now have several sources where to find whatever they want to know about the undefined notions I will deal with and about some theorems I am going to mention.

Let me begin with some general reflections.

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1. On Abstract Algebraic Logic

In my view, most of the work done in this area can be organized around two main questions:

Q1. What do we mean when we say that a logic $\mathcal{L}$ can be *algebraized* and that the class K is *its algebraic counterpart*?

Q2. What can we learn about the logic $\mathcal{L}$ or about the class K when they are related as in question (Q1)?

Question (Q1) concerns *the process of algebraization* while question (Q2) concerns the *consequences* of this process. Works on question (Q1) are easily recognized, as they address the problem rather explicitly. Also, many works on question (Q2) have an explicit format; for instance, the Budapest group (ANDRÉKA, NÉMETI, SAIN, etc.) has coined the expression *bridge theorems* for those that take the typical form

$$\mathcal{L} \text{ has property } \boldsymbol{P} \iff \mathsf{K} \text{ has property } \boldsymbol{P'}, \tag{1}$$

where $\boldsymbol{P}$ is a typically logical property, what one usually calls a *metalogical property* (e.g., interpolation), and $\boldsymbol{P'}$ is a typically algebraic property (e.g., amalgamation). There are also papers whose contents is (almost) entirely algebraic, but which can be considered as dealing with question (Q2) if we put them in the proper context.

One of the most interesting facts one recognizes in the historical development of Algebraic Logic, specially in the last ten or twenty years, is that answers to these and related questions have been organized and *have to be organized* at several *levels*.

Through the works of BLOK, PIGOZZI, CZELAKOWSKI and HERRMANN a *hierarchy of logics*, also called *the Leibniz hierarchy*, has emerged. This hierarchy is organized by taking into account the strength of the relationship Logic$\leftrightarrow$Algebras, or more generally Logic$\leftrightarrow$Models, that is, by their degree of algebraizability; the term *algebraizable* in the technical sense is applied only to those more restricted classes of logics where these links are very strong, and still with several qualifications: *weak, finite, strong, regular.* The most common answer to question (Q1) is that each algebra of K has a distinguished element 1 such that the following completeness theorem holds.

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathsf{K}} \varphi \approx 1 \tag{2}$$

This is a particular case of the technical notion of the class K being an *algebraic semantics* for $\mathcal{L}$ [3], where a finite set of equations in one variable replaces the single equation $x \approx 1$ exemplified here. Having an algebraic semantics has been shown to be a rather weak kind of algebraizability, and actually not playing a particular rôle in the hierarchy unless it is complemented by stronger conditions; roughly speaking, representability of $\models_{\mathsf{K}}$ inside $\boldsymbol{\mathcal{L}}$ as a kind of converse of (2); the following is the most common particular case:

$$\{\delta_i \approx \varepsilon_i : i \in I\} \models_{\mathsf{K}} \delta \approx \varepsilon \iff \{\delta_i \leftrightarrow \varepsilon_i : i \in I\} \vdash_{\boldsymbol{\mathcal{L}}} \delta \leftrightarrow \varepsilon \tag{3}$$

(in general, a finite set of formulas in two variables replaces the formula $x \leftrightarrow y$ used here). Only if the two representations are inverse to one another in a certain precise, natural sense, we get *algebraizability*, a situation where interesting and strong bridge theorems can be obtained. All this is explained in [38].

Analysing the hierarchy one recognizes that the different levels of algebraizability are reflected in different *levels of complexity of the models*: In the best-behaved cases (the different modalities of algebraizability) we can work with just *algebras* $\boldsymbol{A}$, without further structure. Since algebras carry in themselves their equations, we can work with the equational consequence $\models_{\mathsf{K}}$ *relativized* to a class K of algebras; and it is this consequence that makes the connections and helps in finding the bridge theorems. But in wider classes of logics (equivalential, protoalgebraic) we have to work with *matrices*, i.e., structures $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$. Then the bridge theorems will concern classes of matrices rather than classes of algebras. There is now a reasonable evidence that the widest class of logics for which this scheme seems to work smoothly enough in order to obtain significant bridge theorems concerning matrices is the class of *protoalgebraic logics*. By way of illustration, in [4, 21] it is shown that for protoalgebraic logics the LDT (Local Deduction Theorem) is equivalent to the FEP (filter extension property) for the class of reduced matrix models of the logic; if the logic is finitely algebraizable then the FEP reduces to the more algebraic RCEP (relative congruence extension property) for the equivalent quasivariety K.

When trying to enlarge the domain of applicability of the methods of Abstract Algebraic Logic outside the class of protoalgebraic logics, one can put more structure in the model, and consider *generalized matrices*, that is, systems $\langle \boldsymbol{A}, \mathcal{C}\rangle$ where $\boldsymbol{A}$ is an algebra and $\mathcal{C} \subseteq \mathcal{P}(A)$ is a family of subsets of A. This will be the proper subject of the paper.

So far I have recalled two hierarchies: One among logics, and one among their models. I would like to consider another process of enlarging the scope of Algebraic Logic that has also created a kind of hierarchy: the *notion of sentential logic* itself can also be considered at several levels.

The first level, which I call the *old style*, is to define a logic as a set of *formulas* (satisfying some conditions, of course; for instance invariance under substitutions). At this level, one identifies a logic $\mathcal{L}$ with a set of *theorems*.

The second level, which I prefer to call the *Tarski style* (and resist to call "Polish style"), is to define a logic $\mathcal{L}$ as a *consequence relation* between sets of formulas and formulas, denoted by $\vdash_{\mathcal{L}}$, also satisfying some minimal requirements, for instance invariance under substitutions, or (optionally) finitarity. The consequences of the empty set of assumptions are called *theorems*, and constitute a "logic" in the old style. In this paper the word "logic" will refer to a logic in the second level sense, unless otherwise specified. Considering logics at this level has become absolutely necessary in some situations; for instance, if we want to study the algebraizability and several bridge properties of modal logics: It is well-known that in this domain there are different consequence relations with the same set of theorems, each showing a different behaviour both from the metalogical and from the algebraic points of view; this is for instance the case of the two consequences one can associate with each class of Kripke-style or relational frames, a "global" or strong consequence and a "local" or weak one.

In the finitary case a logic at this level is determined by a set of rules of the form $\varphi_0, \ldots, \varphi_{n-1} \vdash \varphi_n$ through the well-known syntactical notion of *proof*. In particular such a proof system generates all the theorems of the logic. Indeed, in the old style days this was the only purpose of a proof system, and the analysis of the rules that preserve the set of "theorems" led in a further step to the creation of the second level conception. Tarski in [47] mentions "the axiomatic method" along with "the matrix method" as the two forms of definition of a logic in his sense, and in the first case he assumes substitution is one of the rules of the proof system. In [57] his investigation takes the most general form and he does not assume substitution-invariance; this condition was explicitly incorporated by Łoś and Suszko [46] in 1958. Rasiowa does not include invariance under substitutions in her general definition of a logic in the second level sense, but she explicitly assumes such a logic as given by an axiomatic system where axioms and rules are closed under substitution instances; [50, § VIII.5].

There is a third level that has been a natural step when trying to answer question (Q1) for non-protoalgebraic logics. I will call it the *Gentzen style*. Its main object of study are the so-called *Gentzen style rules*, that is, rules of the form

$$\frac{\{\Gamma_i \vdash \varphi_i : i < n\}}{\Gamma_n \vdash \varphi_n} \tag{4}$$

where the Γ_i can be either finite sets or finite sequences of formulas. This style appeared by purely proof-theoretic motivations: the investigation of the "rules" that preserved the set of *sequents* of the logic in the second level sense, that is, the pairs $\langle \Gamma, \varphi \rangle$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$. By using these rules, the old notion of proof, now acting on sequents (commonly written as $\Gamma \vdash \varphi$) as basic linguistic objects, defines the so-called *Gentzen systems*, that is, consequence relations between sets of sequents and sequents, again satisfying some minimal requirements such as invariance under substitutions; additionally, they can satisfy some of the so-called structural rules (idempotency, exchange, weakening, cut, etc.). These are the "logics" in the third level sense.

The relations between the second and the third levels closely parallel those between the first and the second. Moreover, if all structural rules are (at least) admissible, then the "theorems" of a Gentzen system $\mathfrak{G}$, i.e., its *derivable sequents*, define a logic in the sense of the second level by means of

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{L}} \varphi_n \iff \begin{array}{l}\text{the sequent } \varphi_0, \ldots, \varphi_{n-1} \vdash \varphi_n \\ \text{is derivable in } \mathfrak{G}\end{array} \tag{5}$$

or a similar stipulation; and two different Gentzen systems can have the same derivable sequents but different metalogical properties (for instance, one can be cut-free while the other is not).

In recent years the increased interest on the so-called *substructural logics* has put Gentzen *calculi* in a more central position. Again, we can take a broader view and consider them as just one of the ways of presenting a Gentzen system, the abstract notion whose semantical and algebraic sides can also be studied. This trend began in [40]; among the most recent contributions let me mention [1, 41, 42]. Its study, very interesting by itself, has been also useful for the algebraic study of sentential logics (in the second level sense) that do not successfully fit into the hierarchy of protoalgebraic logics. Moreover, the abstract theory can be generalized and applied not only to sequents of the form $\Gamma \vdash \varphi$ but to arbitrary kinds of sequents $\Gamma \vdash \Delta$, and even to more complicated syntactical objects that have been recently considered, such as *m-sided sequents* and *hypersequents*. The study of the relationship between the abstract algebraic study of such systems and their proof-theoretic properties has just begun, as shows [41], where the connection between protoalgebraicity (applied to m-sided Gentzen systems) and the presence of the cut rule is studied.

Here I will consider only the work done with the restricted kind of sequents first considered, and assume all logics are finitary. In this case, generalized matrices provide one of the most natural notions of model of a logic in the third level sense, and at the same time they are also models of logics in the second level sense in an extremely natural way. Actually, I consider that one of the most interesting facets of the research with generalized matrices is the interaction between these two rôles they can play. I will try to show some results that emphasize this interaction.

2. Prehistory

To be precise, let $\boldsymbol{Fm}$ denote the formula algebra of a certain arbitrary, fixed, algebraic type. A ***sentential logic*** $\boldsymbol{\mathcal{L}}$ is given by a finitary and substitution-invariant consequence relation $\vdash_{\boldsymbol{\mathcal{L}}}$ between sets of formulas and formulas (i.e., $\vdash_{\boldsymbol{\mathcal{L}}} \subseteq P(Fm) \times Fm$) which will be written in infix notation as usual. The set of all ***theories*** of the logic is denoted by $\mathcal{T}h\boldsymbol{\mathcal{L}}$, and the associated closure operator by $\mathrm{Cn}_{\boldsymbol{\mathcal{L}}}$; that is, for each set Γ of formulas, $\mathrm{Cn}_{\boldsymbol{\mathcal{L}}}(\Gamma) = \{\varphi \in Fm : \Gamma \vdash_{\boldsymbol{\mathcal{L}}} \varphi\}$ is the theory generated by Γ.

The traditional algebraic model theory of sentential logics is matrix-based. A ***matrix*** is a pair $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra of the appropriate type, and $F \subseteq A$, and it is a ***model of*** $\boldsymbol{\mathcal{L}}$ when F is an $\boldsymbol{\mathcal{L}}$***-filter***, that is, when for any $\varphi_0, \ldots, \varphi_n \in Fm$, if $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{L}}} \varphi_n$ and $h(\varphi_i) \in F$ for all $i < n$ then $h(\varphi_n) \in F$, for any interpretation h in $\boldsymbol{A}$ $\big($i.e., for any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\big)$. The set of all $\boldsymbol{\mathcal{L}}$-filters on $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$.

The most general notion of ***generalized matrix*** is a pair $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ where $\boldsymbol{A}$ is an algebra (of the required type) and $\mathcal{C} \subseteq \mathcal{P}(A)$ is an arbitrary collection of subsets of A. In this sense, they were conceived around 1969 by Wójcicki (in [59, 60]; see also [61, § IV.4]) as generalizations of the idea of matrix, more precisely, as an alternative presentation of the closely related notion of a ***bundle of matrices***, which refers to the family $\big\{\langle \boldsymbol{A}, F\rangle : F \in \mathcal{C}\big\}$. Most model-theoretic or universal-algebraic ideas and constructions on matrices can be reproduced or adjusted for generalized matrices. Also, set-theoretically, as a set with a family of subsets, they can be considered under topological intuitions. An ordinary matrix can be identified with the generalized matrix of the form $\big\langle \boldsymbol{A}, \{F, A\}\big\rangle$; thus the new theory can contain the classical one to a certain extent. Generalized matrices have reappeared recently in [28] under the name of ***atlases***, where its use is extended to deal with "multiple-conclusion logics" or "Scott consequence relations".

Generalized matrices have a well-known *dual presentation* as pairs $\langle \boldsymbol{A}, \mathrm{C} \rangle$ where $\mathrm{C} : \mathcal{P}(A) \to \mathcal{P}(A)$ is a ***closure operator*** on the power set of A. This was SMILEY's presentation in [56] and MAGARI's in [48] (he used the term *calcoli generali*). The first in-depth study of generalized matrices under this presentation was BROWN's 1969 dissertation [13], where SUSZKO was the principal advisor, and then their joint papers with BLOOM [11, 10, 14]; the last two were published together in 1973 with an interesting preface by SUSZKO. In these they coined the term *abstract logics*. While [11, 14] present the general theory with short examples, [10] studies the abstract logics consisting of a Boolean algebra and the closure operator of filter generation and shows these are characterized, roughly speaking, by the same metalogical properties that determine classical logic, viz. finitarity, the Deduction Theorem[1], and having the classical tautologies as theorems (that these properties characterize classical sentential logic is due to TARSKI). Similar characterizations were obtained in [8, 9] for several natural fragments of intuitionistic logic containing conjunction. This approach was extensively followed by the Barcelona group, see [33] and the many references therein. However, except for [56], all these works prior to 1990 do not try to provide a precise explanation of the relationship between the abstract logics they were considering and the sentential logics they had in mind. They explored the *analogies* between a logic $\boldsymbol{\mathcal{L}}$ presented as the abstract logic $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\boldsymbol{\mathcal{L}}} \rangle$ and the abstract logics of the form $\langle \boldsymbol{A}, \mathrm{Fi}^{\boldsymbol{A}} \rangle$ where $\boldsymbol{A}$ is an algebra of the class naturally associated with the logic (for instance, Boolean algebras or Heyting algebras for classical or intuitionistic logic, respectively) and $\mathrm{Fi}^{\boldsymbol{A}}$ is the corresponding closure operator of filter generation; but again the analogy did not rest on a technical notion of an abstract logic being a model of a sentential logic such as that of WÓJCICKI's.

SMILEY's 1962 discussion [56, pp. 433–435] of the insufficiency of ordinary matrices to model some logics contains the first proposal to use algebras with a closure operator in order to *model the deducibility relation* rather than theoremhood. Unfortunately, SMILEY's proposal was only followed briefly by HARROP in [44, 45] but did not attract any attention from the algebraic logic community (even SMILEY's well-known monograph with SHOESMITH [55] uses only ordinary matrices) and WÓJCICKI did not further develop the theory of generalized matrices beyond his first completeness results in [59, 60]. The stream reappears only in 1991 in the paper [40] published in *Studia Logica*, in a *Special Issue on Algebraic Logic* edited by BLOK and PIGOZZI; indeed, the personal contact between these editors and the authors

[1]See the Appendix for formal statements of some properties.

of [40] was a key point in the change of perspective.

Note that in order to obtain a one-to-one correspondence between the two dual presentations one has to restrict attention to the generalized matrices where the family of subsets $\mathcal{C}$ is a ***closure system*** (it is closed under intersections of arbitrary subfamilies). Moreover, given the context of this discussion in the model theory of sentential logics as defined above, I will also assume that $\mathcal{C}$ is ***inductive*** (closed under unions of upwards-directed subfamilies); dually, this means that C is ***finitary***. These assumptions will hold throughout the paper.

The duality is effected by the following constructions: Given an inductive closure system $\mathcal{C}$ on a set A one obtains a finitary closure operator by defining, for each $X \subseteq A$, $\mathrm{C}(X) = \bigcap\{T \in \mathcal{C} : X \subseteq T\}$. Conversely, for each such C, the family $\mathcal{C} = \{T \subseteq A : \mathrm{C}(T) = T\}$ is an inductive closure system. Bull's 1976 paper [15] exploits this duality as the basis of an abstract analysis of classical logic from the perspective of algebra and topology.

This duality is very convenient. In the development of the general theory, each notion and each situation can be expressed or understood in the presentation that best suits it. We are going to see this in the next sections.

3. Generalized matrices as models of sentential logics

As an easy example of the duality of presentations of generalized matrices mentioned in the previous section, consider the notion of model of a logic: Wójcicki's original definition, which uses the closure system, is certainly compact:

Definition 1. *A generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a* ***model of*** *a sentential logic $\boldsymbol{\mathcal{L}}$ when $\mathcal{C} \subseteq \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$; that is, when the closure system is just a collection of $\boldsymbol{\mathcal{L}}$-filters in the traditional sense.*

However concise this is, an equivalent definition using the closure operator gives a different and more intuitive view[2]:

Proposition 2. *A generalized matrix $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of $\boldsymbol{\mathcal{L}}$ if and only if it is a* ***model of all Hilbert-style rules of*** $\boldsymbol{\mathcal{L}}$*, in the sense that*

$$\text{if} \quad \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{L}}} \varphi_n \quad \text{then} \quad h(\varphi_n) \in \mathrm{C}\big(h(\varphi_0), \dots, h(\varphi_{n-1})\big) \tag{6}$$

for any interpretation h in $\boldsymbol{A}$ and for any $\varphi_0, \dots, \varphi_n \in Fm$.

[2] I write $\mathrm{C}(a, \dots, z)$ for $\mathrm{C}\big(\{a, \dots, z\}\big)$ and $\mathrm{C}(X, a)$ for $\mathrm{C}\big(X \cup \{a\}\big)$ when $a, \dots, z \in A$ and $X \subseteq A$, and use other similar customary abbreviations.

Thus, behind this notion of model as a collection of $\boldsymbol{\mathcal{L}}$-filters, which might look as an innocent generalization, there is a very natural notion of an interpretation satisfying a sequent: We can say that h ***satisfies*** a sequent $\varphi_0, \dots, \varphi_{n-1} \vdash \varphi_n$ when the right-hand half of (6) holds. As the reader is already guessing, this will be used later on to define the notion of a model of a Gentzen-style rule or of a Gentzen system.

It may be convenient to consider another example. A mapping h is a ***strict homomorphism*** between two generalized matrices $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathcal{D} \rangle$ when it is an algebraic homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ such that $\mathcal{C} = \{h^{-1}[G] : G \in \mathcal{D}\}$. The dual expression, however, is logically speaking more significant: A homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ is strict between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ if and only if for every $X \cup \{a\} \subseteq A$, $a \in \mathrm{C}(X) \iff h(a) \in \mathrm{D}\big(h[X]\big)$. Of particular interest are the strict *surjective* homomorphisms, which are called ***bilogical morphisms*** in [10, 33]: among other properties, they make the closure systems $\mathcal{C}$ and $\mathcal{D}$ to be isomorphic as complete lattices.

A common way of describing Abstract Algebraic Logic [6, 38] is to say that this theory starts from an abstraction of "the traditional Lindenbaum-Tarski process". As is well-known, this process consists in factoring the formula algebra by congruences associated with theories of the logic, in order to generate a class of algebras that can deserve the title of *the algebraic counterpart of the logic*, i.e., the class of *Lindenbaum-Tarski algebras.* The abstraction can be extended to make it a particular case of a ***process of reduction*** that can be applied to an arbitrary generalized matrix. To implement it we first consider the following

DEFINITION 3. *The* ***Frege relation*** $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}})$ *of a generalized matrix* $\boldsymbol{\mathcal{A}}$ *is defined by:* $\langle a, b \rangle \in \boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}}) \iff \mathrm{C}(a) = \mathrm{C}(b)$, $\forall a, b \in A$. *The* ***Tarski congruence*** *of* $\boldsymbol{\mathcal{A}}$, $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})$, *is the largest congruence of* $\boldsymbol{A}$ *that is below* $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}})$. *A generalized matrix* $\boldsymbol{\mathcal{A}}$ *is* ***reduced*** *when* $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) = Id_A$. *The* ***reduction*** *of* $\boldsymbol{\mathcal{A}}$ *is* $\boldsymbol{\mathcal{A}}^* = \big\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}), \mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) \big\rangle$, *where* $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) = \{F/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) : F \in \mathcal{C}\}$.

The Frege relation is an abstract analogue of the interderivability relation $\dashv\vdash_{\boldsymbol{\mathcal{L}}}$ of sentential logics, and is always an equivalence relation, but may not be a congruence. This is why in general it is strictly necessary to introduce the Tarski congruence, which is the analogue for generalized matrices of the *Leibniz congruence* of an ordinary matrix (and reduces to it when the generalized matrix is an ordinary one). In some respects, generalized matrices and the Tarski operator $\widetilde{\boldsymbol{\Omega}}$ behave on models of arbitrary logics as ordinary matrices and the Leibniz operator $\boldsymbol{\Omega}$ behave on (matrix) models of protoalgebraic logics; for instance, the Tarski operator is always monotonic with respect to the natural ordering between generalized matrices in their closure operator form, while the Leibniz operator is monotonic only on the matrix models of a protoalgebraic logic.

In the literature one can find (at least) the following three general processes for associating a class of algebras with a sentential logic $\boldsymbol{\mathcal{L}}$ using this reduction process:

- If we identify $\boldsymbol{\mathcal{L}}$ with the generalized matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\boldsymbol{\mathcal{L}} \rangle$, where $\mathcal{T}h\boldsymbol{\mathcal{L}}$ is the family of its *theories*, we can consider its Tarski congruence $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{L}})$ and the so-called ***Lindenbaum-Tarski algebra of*** $\boldsymbol{\mathcal{L}}$, i.e., the quotient algebra $\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{L}})$. This algebra generates the variety $\mathsf{V}(\boldsymbol{\mathcal{L}}) = \mathbb{V}\big(\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{L}})\big)$, considered e.g., in [51, 52], where it is called *the variety associated with* $\boldsymbol{\mathcal{L}}$.
- In the classical theory of matrices we find the class $\mathsf{Alg}^*\boldsymbol{\mathcal{L}}$ of *algebraic reducts of reduced matrix models* for $\boldsymbol{\mathcal{L}}$.
- And finally there is the class $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ of *algebraic reducts of reduced (generalized matrices) models of* $\boldsymbol{\mathcal{L}}$. They are called ***$\boldsymbol{\mathcal{L}}$-algebras***.

The class $\mathsf{Alg}^*\boldsymbol{\mathcal{L}}$ has been traditionally regarded as the algebraic counterpart of $\boldsymbol{\mathcal{L}}$; however, as I will remark in a moment, there are reasons for considering that it is $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ that better does the job in general. But first let me use the reduction process again to define the notion of *full model*, a special kind of model that shows a remarkably nice behaviour.

Definition 4. *A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is a **full model of** $\boldsymbol{\mathcal{L}}$ when the closure system of its reduction consists of all the $\boldsymbol{\mathcal{L}}$-filters of the quotient algebra: $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) = \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})\big)$. The class of all full models of $\boldsymbol{\mathcal{L}}$ is denoted by $\mathsf{FGMod}\boldsymbol{\mathcal{L}}$. The set of all full models of $\boldsymbol{\mathcal{L}}$ over a fixed algebra $\boldsymbol{A}$, which is denoted by $\mathcal{FGM}od_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$, is ordered under the pointwise ordering of the corresponding closure operators (which is dual to the set-inclusion ordering of the corresponding closure systems).*

Among general properties of this notion, let me highlight:

1. Full models are *full* in the sense that $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full model of $\boldsymbol{\mathcal{L}}$ if and only if $\mathcal{C} = \{F \in \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) \text{ is compatible with } F\}$.

   Recall that a congruence θ is *compatible* with a subset F when $a\theta b$ implies $a \in F \iff b \in F$; or, equivalently, when $\theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$.
2. Full models are *full* in the sense that the class $\mathsf{FGMod}\boldsymbol{\mathcal{L}}$ is the smallest class of models of $\boldsymbol{\mathcal{L}}$ closed under the construction of images and inverse images by strict surjective homomorphisms that contains all models of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A} \rangle$ (the so-called *basic full models*).

3. The so-called **Isomorphism Theorem:**
   For any logic $\boldsymbol{\mathcal{L}}$ and any algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}$ establishes an order isomorphism between the complete lattices $\mathcal{FGM}od_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ and $\mathrm{Co}_{\mathsf{Alg}\boldsymbol{\mathcal{L}}}\boldsymbol{A} = \{\theta \in \mathrm{Co}\boldsymbol{A} : \boldsymbol{A}/\theta \in \mathsf{Alg}\boldsymbol{\mathcal{L}}\}$.

What is most remarkable in this theorem is its extreme generality: it holds for *any logic* and *any algebra* (of the relevant similarity type). It can be considered an extension of the isomorphism between filters and relative congruences for finitely algebraizable logics; this last isomorphism, established in 1989 by Blok and Pigozzi [3, Thm. 5.1], generalizes the well-known correspondence between filters and congruences in Boolean algebras or in Heyting algebras, and between open filters and congruences in topological or monadic Boolean algebras. Other particular instances of the isomorphism were known before long to algebraic logicians: in Rasiowa's [50] the isomorphism is not explicitly stated, but follows easily from the correspondences between filters and congruences there established, and in 1981 Czelakowski says, just before proving his Theorem II.2.10 of [17], that it "generalizes some observations made independently by several people". This isomorphism has been an inspiring source of ideas about the relationships between logical and algebraic properties, as well as a fundamental tool for showing non-algebraizability of certain logics.

4. The class $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ is also the class of algebraic reducts of the reduced full models of $\boldsymbol{\mathcal{L}}$.
5. The algebras in $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ can be obtained by factoring a formula algebra of suitable cardinality by the Tarski congruence of some full model; more precisely,

   $$\mathsf{Alg}\boldsymbol{\mathcal{L}} = \mathbb{I}\big\{\boldsymbol{Fm}_\kappa/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) : \boldsymbol{\mathcal{A}} \text{ full model of } \boldsymbol{\mathcal{L}} \text{ over } \boldsymbol{Fm}_\kappa\ ,\ \kappa \in \mathbb{C}\mathrm{ARD}\big\}$$

   where $\boldsymbol{Fm}_\kappa$ is the formula algebra over κ variables.

Concerning the relationships between the three classes of algebras considered so far there are several general facts and some particular ones:

6. In general $\mathsf{Alg}^*\boldsymbol{\mathcal{L}} \subseteq \mathsf{Alg}\boldsymbol{\mathcal{L}} \subseteq \mathsf{V}(\boldsymbol{\mathcal{L}})$.
7. $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ is the class of subdirect products of algebras in $\mathsf{Alg}^*\boldsymbol{\mathcal{L}}$. Hence the two classes generate the same quasivariety and the same variety.
8. The variety generated by $\mathsf{Alg}^*\boldsymbol{\mathcal{L}}$ and by $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ is $\mathsf{V}(\boldsymbol{\mathcal{L}})$.
9. If $\boldsymbol{\mathcal{L}}$ is protoalgebraic then $\mathsf{Alg}^*\boldsymbol{\mathcal{L}} = \mathsf{Alg}\boldsymbol{\mathcal{L}}$.

Therefore, $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ coincides with the class of "$\boldsymbol{\mathcal{L}}$-algebras" introduced by Rasiowa for the logics she studied in [50] (now commonly called *implicative* *logics*), and with the *equivalent quasivariety* of the finitely algebraizable logics introduced by Blok and Pigozzi in [3].

Two big classes of logics, independent of the just mentioned ones, for which the definition of $\boldsymbol{\mathcal{L}}$-algebras gives a sound result are the following.

10. Let $\boldsymbol{\mathcal{L}}$ be *selfextensional* (see Section 6.1) and have a Conjunction or satisfy the uniterm Deduction Theorem. Then $\mathsf{Alg}\boldsymbol{\mathcal{L}} = \mathsf{V}(\boldsymbol{\mathcal{L}})$.

For non-protoalgebraic logics it is $\mathsf{Alg}\boldsymbol{\mathcal{L}}$ that gives the "right" or expected result, while $\mathsf{Alg}^*\boldsymbol{\mathcal{L}}$ can be a strictly smaller class with no particular metalogical significance. This has been shown to be the case, for instance, in three of the most natural examples of non-protoalgebraic logics, represented in Table 1: The conjunction-disjunction fragment of classical logic $\mathcal{CPC}_{\wedge\vee}$ [40], Belnap's four valued logic (with truth and falsity constants) $\mathcal{B}$ [32],

$\boldsymbol{\mathcal{L}}$	$\mathsf{Alg}\boldsymbol{\mathcal{L}}$	$\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ for $\boldsymbol{A} \in \mathsf{Alg}\boldsymbol{\mathcal{L}}$
protoalgebraic		
$\mathcal{IPC}_{\rightarrow}$	Hilbert algebras	{implicative filters}
$\mathcal{CPC}$	Boolean algebras	{filters}
global $\mathcal{S}5$	monadic Boolean algebras	{open filters }
local $\mathcal{S}5$	monadic Boolean algebras	{filters}
$Ł_\infty$	MV-algebras	{implicative filters}
$\mathcal{R}$	R-algebras	$\{F : F$ is a lattice filter and $a \rightarrow a \in F \ \ \forall a\}$
$\mathcal{BCK}$	BCK-algebras	{implicative filters}
non-protoalgebraic		
$\mathcal{CPC}_{\wedge\vee}$	distributive lattices	$\{$lattice filters$\} \cup \{\emptyset\}$
$\mathcal{B}$	De Morgan algebras	{lattice filters}
$\mathcal{IPC}^*$	pseudo-complemented distributive lattices	{lattice filters}
$\mathcal{WR}$	R-algebras	$\{$lattice filters$\} \cup \{\emptyset\}$

Table 1. Some logics, their $\boldsymbol{\mathcal{L}}$-algebras and their reduced full models. See the text for notations and references.

or the implication-less fragment of intuitionistic logic $\mathcal{IPC}^*$ [3, 53]. However, property 9 is not a characterization of protoalgebraicity: there are non-protoalgebraic logics where the two classes coincide, like the inferential version $\mathcal{WR}$ of the well-known system $\mathcal{R}$ of relevance logic, suggested in [62, §2.10] and studied in [39]; this coincides with the semantic consequence one can associate with the ternary relational semantics. While in most of the non-protoalgebraic cases the class $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ has been found to be a variety and thus to coincide with $\mathbf{V}(\boldsymbol{\mathcal{L}})$, for some subintuitionistic logics studied in [12] the situation is quite different; more precisely, the three classes of algebras associated with the global consequence of the class of transitive "intuitionistic-like" Kripke models are all different (this solved one of the open problems in [33]) and both $\mathbf{Alg}^*\boldsymbol{\mathcal{L}}$ and $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ are not even quasivarieties.

It is easy to see that the ***reduced full models*** have a very precise form, namely they are exactly the generalized matrices $\langle \boldsymbol{A}, \mathcal{C}\rangle$ such that $\boldsymbol{A} \in \mathbf{Alg}\boldsymbol{\mathcal{L}}$ and $\mathcal{C} = \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$. They are also characterized as the *reduced basic full models*, i.e., the generalized matrices of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}\rangle$ that are reduced. Hence, $\boldsymbol{A} \in \mathbf{Alg}\boldsymbol{\mathcal{L}}$ iff $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}\rangle$ is reduced. In examples, one sees that these generalized matrices have both a logical and an algebraic natural interpretation; the reader will not find any surprise in Table 1[3]. The empirical evidence it offers together with some of the previous results, seem to be good enough reasons for considering that reduced full models and $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ give a satisfactory account of the algebraization of (almost ?) any logic, regardless of whether it is protoalgebraic or not, and for calling $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ ***the class of Lindenbaum-Tarski algebras of*** $\boldsymbol{\mathcal{L}}$.

Quite different and a more interesting problem is the characterization of *arbitrary full models*; they exist on any algebra, for instance $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}\rangle$ is always the finest one, but in general they are not so neatly characterized as the reduced ones. One way will be dealt with in the next section.

One kind of the intended characterizations is to describe the full models as certain families of $\boldsymbol{\mathcal{L}}$-filters of some particular form (in contrast to general models, that can be obtained from arbitrary families of $\boldsymbol{\mathcal{L}}$-filters closed under intersection). Experience shows that for most of the best-behaved logics full models are principal filters in the lattice of all $\boldsymbol{\mathcal{L}}$-filters on each algebra. In the next two results from [33] we see that this form of full models can even characterize two main classes in the hierarchy:

[3]The two modal logics are the global and the local consequences associated with the class of Kripke models for the logic, $\mathcal{S}5$ in this case; they are sometimes called the normal and the quasi-normal logics. A similar situation arises for every normal "system" (in the old style sense) of modal logic

THEOREM 5. *A logic $\boldsymbol{\mathcal{L}}$ is protoalgebraic if and only if every full model of $\boldsymbol{\mathcal{L}}$ has the form $\langle \boldsymbol{A}, (\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A})^F \rangle$ for some $F \in \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$, where $(\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A})^F = \{G \in \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A} : F \subseteq G\}$ is the principal filter of $\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ generated by F.*

However, not every $\boldsymbol{\mathcal{L}}$-filter generates a full model in the above way; those that do have been studied in [34], where they are called *Leibniz filters* because they can be characterized as the smallest ones among all those having the same Leibniz congruence.

THEOREM 6. *A logic $\boldsymbol{\mathcal{L}}$ is weakly algebraizable if and only if the full models of $\boldsymbol{\mathcal{L}}$ are exactly all models of the form mentioned in the previous theorem.*

The class of weakly algebraizable logics has been studied in depth in [23]; it includes all algebraizable logics, but also others which are not algebraizable, related to certain subtractive varieties. For these logics the construction of Theorem 5 yields a lattice isomorphism between $\mathcal{FG}\mathcal{M}od_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ and $\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Thus to some extent working with full models amounts to working with filters. Moreover, from this isomorphism and the Isomorphism Theorem of item 3 above an alternative proof of BLOK and PIGOZZI's isomorphism is obtained.

4. Generalized matrices as models of Gentzen systems

This is a specially fruitful line of research concerning the problem of characterization of full models. Notice that generalized matrices, in their disguise as closure operators, satisfy all the so-called structural rules; therefore we will restrict our attention to *Gentzen systems having all structural rules.*

DEFINITION 7. *A generalized matrix $\langle \boldsymbol{A}, \mathrm{C} \rangle$ is a* ***model of a Gentzen-style rule***

$$\frac{\{\Gamma_i \vdash \varphi_i : i < n\}}{\Gamma_n \vdash \varphi_n} \tag{7}$$

when every interpretation h in $\boldsymbol{A}$ that satisfies all sequents in the antecedent (in the sense given after Proposition 2), also satisfies the sequent in the consequent.

$\boldsymbol{\mathcal{A}}$ is a ***model of a Gentzen system*** *$\mathfrak{G}$ when it is a model of all its derivable rules. The class of all such models is denoted by* $\mathsf{Mod}\mathfrak{G}$.

A rule like (7) *is a* ***rule of $\boldsymbol{\mathcal{L}}$*** *when the generalized matrix $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\boldsymbol{\mathcal{L}}} \rangle$ is a model of* (7) *in the above sense; that is, when for every substitution σ such that $\sigma\Gamma_i \vdash_{\boldsymbol{\mathcal{L}}} \sigma\varphi_i$ for all $i < n$, also $\sigma\Gamma_n \vdash_{\boldsymbol{\mathcal{L}}} \sigma\varphi_n$.*

The previous experience of the Barcelona group with particular logics and particular classes of algebras has shown that some logics characterize their full models in terms of properties that can be expressed by Gentzen-style rules (which are, of course, rules of the logic). To mention just the easiest example: The full models of $\mathcal{IPC}_{\rightarrow}$ are the generalized matrices that satisfy the Deduction Theorem. This has led naturally to the question: *When is the class of full models of a logic completely described by some set of Gentzen-style rules?* The associated technical notion (called "strong adequacy" in [33]) is the following.

DEFINITION 8. *A Gentzen system* $\mathfrak{G}$ *is* ***fully adequate*** *for a sentential logic* $\boldsymbol{\mathcal{L}}$ *when* $\mathbf{FGMod}\boldsymbol{\mathcal{L}} = \mathbf{Mod}\mathfrak{G}$*; in case* $\boldsymbol{\mathcal{L}}$ *has no theorems, one has to assume that all generalized matrices considered here have no theorems, either.*

Among the consequences of $\mathfrak{G}$ being fully adequate for $\boldsymbol{\mathcal{L}}$ let me mention:

1. The Gentzen system $\mathfrak{G}$ is ***adequate*** for the sentential logic $\boldsymbol{\mathcal{L}}$, that is, the derivable sequents of $\mathfrak{G}$ define $\boldsymbol{\mathcal{L}}$ as in (5).
2. $\boldsymbol{\mathcal{L}}$ is the weakest logic satisfying the rules of $\mathfrak{G}$.
3. $\mathbf{Alg}\boldsymbol{\mathcal{L}} = \mathbf{Alg}\mathfrak{G}$, the class of algebraic reducts of reduced models of $\mathfrak{G}$.
4. $\boldsymbol{A} \in \mathbf{Alg}\boldsymbol{\mathcal{L}}$ if and only if $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A} \rangle$ is a reduced model of $\mathfrak{G}$ (and it is the only one on $\boldsymbol{A}$).

Therefore we have a description of $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ in terms of Gentzen-style rules: Algebras in $\mathbf{Alg}\boldsymbol{\mathcal{L}}$ are exactly those supporting a generalized matrix that is a model of the Gentzen system and is reduced. In case the Gentzen system has a nice presentation, these characterizations can be useful and significant. Examples can be found in Chapter 5 of [33].

Not every logic has a fully adequate Gentzen system, but if one exists then it is unique, so it seems to have some kind of distinctness among all those that are adequate for the logic; however, the deep significance of the notion of full adequacy has not been thoroughly investigated. The question of existence of a fully adequate Gentzen system for a sentential logic has generated other questions, and has opened several lines of research; in particular, it turns out that this problem is related to other, more classical problems of Abstract Algebraic Logic, as I am going to show.

One line of research is to study the full adequacy problem in relation with several *closure properties* of the class $\mathbf{FGMod}\boldsymbol{\mathcal{L}}$ of all full models of $\boldsymbol{\mathcal{L}}$ or of the lattice $\mathcal{FGM}od_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ of all full models of $\boldsymbol{\mathcal{L}}$ on a fixed, arbitrary algebra $\boldsymbol{A}$. Some relevant results obtained so far are:

THEOREM 9 ([36]). *For any logic* $\boldsymbol{\mathcal{L}}$ *the following properties are equivalent:*

(i) $\boldsymbol{\mathcal{L}}$ *has a fully adequate Gentzen system.*

(ii) $\mathbf{FGMod}\boldsymbol{\mathcal{L}}$ *is closed under sub-generalized matrices and under reduced products of generalized matrices.*

(iii) *For any algebra $\boldsymbol{A}$, the lattice $\mathcal{FGM}od_{\mathcal{L}}\boldsymbol{A}$ is closed under arbitrary intersections (hence it is a complete sublattice of the lattice of all generalized matrices over $\boldsymbol{A}$).*

This belongs to the kind of results called bridge theorems at the beginning, as the existence of a fully adequate Gentzen system can be regarded as a metalogical property of the logic while the other items are properties of models.

Another line of research is to relate full models with Deduction Theorems of several kinds.

THEOREM 10 ([36] and [37]). *Let $\mathcal{L}$ be a finitely algebraizable logic. Then:*

1. $\mathsf{FGMod}\mathcal{L}$ *is closed under sub-generalized matrices if and only if $\mathcal{L}$ has the Local Deduction Theorem (LDT).*
2. *$\mathcal{L}$ has a fully adequate Gentzen system if and only if $\mathcal{L}$ has the Deduction Theorem.*

Besides their interest for the full adequacy problem, these results add to the extensive literature on the several forms of the Deduction Theorem. One of the key tools in this analysis has been the study of Leibniz filters and their relation with full models as described in Theorems 5 and 6. The other key approach is to exploit the presentation of generalized matrices as first-order structures that are the models of a strict universal Horn theory without equality, further extending BLOOM's idea [7], already generalized by BLOK and PIGOZZI in [5]. Briefly: given a generalized matrix $\langle \boldsymbol{A}, \mathrm{C}\rangle$ one considers the structure $\mathfrak{A} = \big\langle \boldsymbol{A}, \{R_m^{\mathfrak{A}} : m \geq 1\}\big\rangle$ with algebraic part $\boldsymbol{A}$ and where R_m is an m-ary relation symbol interpreted as

$$\langle a_1, \ldots, a_m\rangle \in R_m^{\mathfrak{A}} \iff a_m \in \mathrm{C}\big(a_1, \ldots, a_{m-1}\big)$$

for $m > 1$, and $R_1^{\mathfrak{A}} = \mathrm{C}(\emptyset)$. This allows to take advantage of general results on the *model theory of equality-free logic*, recently developed mainly by ELGUETA, DELLUNDE, CZELAKOWSKI and JANSANA. The papers [16, 27, 29, 30] contain a sample of such results and witness the mutual influence between Algebraic Logic and Model Theory.

The preceding results belong to the research line devoted to the theoretical analysis of full adequacy. One can also investigate it for particular logics or groups or logics. One key problem here is, in case we know that certain logic does have a fully adequate Gentzen system, to find a *calculus* or presentation for it with Gentzen-style rules that have a relevant metalogical significance. In Subsection 6.3 I mention two big classes of logics for which the full adequacy problem has a positive solution.

5. Transfer properties

Let's return to the beginning of the paper. There I mention *bridge theorems* (1) in connection with question (Q2). Now I am interested in a more specific class of typical answers to this question, which have been called *transfer theorems.* They are similar in structure to bridge theorems, but in transfer theorems the property ***P'*** of (1) is the same as property $\boldsymbol{P}$, and the right-hand side of the equivalence has a local character; thus, their typical form is

$$\boldsymbol{\mathcal{L}} \text{ has property } \boldsymbol{P} \quad \Longleftrightarrow \quad \text{Every object in } \mathsf{K} \text{ has property } \boldsymbol{P}. \tag{8}$$

Obviously, this can only make sense for properties $\boldsymbol{P}$ that can be equally predicated of the logic and of its models; accordingly, one has to shape models in an appropriate way or consider one of their presentations or aspects that can be successfully unified with those of the logic itself.

I think it is no exaggeration to say that *one can view an important number of recent works in Abstract Algebraic Logic as transfer theorems*; to be more precise, as works concerning transfer of properties of the lattice $\langle \mathcal{T}h\boldsymbol{\mathcal{L}}, \cap, \vee \rangle$ of the theories of $\boldsymbol{\mathcal{L}}$ to the lattices $\langle \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}, \cap, \vee \rangle$ of the $\boldsymbol{\mathcal{L}}$-filters on arbitrary algebras; or, equivalently, to the lattices $\langle \mathcal{C}, \cap, \vee \rangle$ of the closure system of arbitrary full models. Some transfer properties hold in general, while some hold only for restricted classes of logics; in some cases equivalent conditions have been found for certain transfers to hold.

To mention only one example: That the lattice of theories is distributive does not imply in general that all filter lattices, or all full models, are distributive; however, this is true inside the class of protoalgebraic logics, as proved in 1984 by Czelakowski [18][4]. Other celebrated results by Czelakowski, Blok, Pigozzi, Herrmann and Jansana are actually transfer theorems concerning *properties of the Leibniz operator*, such as monotonicity, definability, continuity, injectivity, commutativity with inverse endomorphisms, etc. Some of these properties transfer from $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ on $\mathcal{T}h\boldsymbol{\mathcal{L}}$ to $\boldsymbol{\Omega}_{\boldsymbol{A}}$ on $\mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}$ in general, while some do only for protoalgebraic logics; moreover the logics that satisfy some of them constitute well-known classes in the hierarchy: protoalgebraic, equivalential, finitely equivalential, weakly algebraizable, algebraizable, etc. See [38] for more details and references.

Transfer theorems also arise in the context of generalized matrices: if one considers the logic itself as the generalized matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\boldsymbol{\mathcal{L}} \rangle$ then it

[4] He stated and proved it for equivalential logics, but his proof actually works for all protoalgebraic logics.

makes sense to ask whether properties of $\boldsymbol{\mathcal{L}}$ expressed by a Gentzen-style rule do transfer from $\langle \boldsymbol{Fm}, \mathcal{T}h\boldsymbol{\mathcal{L}}\rangle$ to models like $\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{L}}}\boldsymbol{A}\rangle$, or, equivalently, to full models $\langle \boldsymbol{A}, \mathcal{C}\rangle$. This has the supplementary interest that properties expressed by Gentzen-style rules usually have an intuitive and direct interpretation at a metalogical level. Well-known examples are having the Deduction Theorem, having a Disjunction, or the Introduction of Modality. Czelakowski explicitly proved in [19, 20] that Disjunction and the Deduction Theorem always transfer, that is, that a logic satisfies any of them *if and only if* all its full models satisfy it[5]. In the best-behaved cases (such as the finitely algebraizable logics), the whole full model structure is encoded in the algebraic structure, so that after being transferred to the models the property may become equivalent to a typically algebraic property (and then we have a typical bridge theorem). This is the case, for instance, of the Deduction Theorem, which transfers to the equational consequence relative to K, and there it becomes equivalent to the familiar EDPRC property, [6]. Some of the results establishing bridge theorems between several interpolation and amalgamation properties can be formulated as transfer theorems for carefully formulated forms of amalgamation for matrices; the general theory of these correspondences has been developed in the context of equivalential and algebraizable logics in [24].

I am not aware of much work on the transfer problem *in general*; let me mention a few results. Trivially, all properties expressible as a Hilbert-style rule transfer. Moreover, Czelakowski's proofs of the transference of the Deduction Theorem or Disjunction can be generalized to obtain the following.

Theorem 11. *Let $\gamma(x, \overrightarrow{y}) \in Fm$ be a formula, and let $\{x_n : n \in \omega\}$ be a denumerable set of variables disjoint from those in $\overrightarrow{y}$. Call $\boldsymbol{P}(\gamma)$ the property represented by the following set of Gentzen-style rules.*

$$\left\{ \frac{x_0, \ldots, x_{n-1} \vdash x_n}{\gamma(x_0, \overrightarrow{y}), \ldots, \gamma(x_{n-1}, \overrightarrow{y}) \vdash \gamma(x_n, \overrightarrow{y})} \quad : \quad n \in \omega \right\} \tag{9}$$

Then the property $\boldsymbol{P}(\gamma)$ transfers (from the logic to all full models).

The case $\gamma(x, y) = y \to x$ essentially accounts for the transfer of the Deduction Theorem, and the case $\gamma(x, y) = x \vee y$ covers the case of Disjunction.

Another very general, specially remarkable fact is the following result by Czelakowski and Pigozzi:

[5] He did not use the notion of full model, but his results amount to this one.

THEOREM 12 (see [22], § 1.7). *If $\boldsymbol{\mathcal{L}}$ is a protoalgebraic logic, then any property expressed by a universal sentence of the first-order language of lattices transfers (that is, it holds in the lattice of theories if and only if it holds in the lattice of the closure system of every full model).*

We thus see that the transfer of distributivity mentioned before is not an isolated phenomenon, but a particular instance of a more general fact.

A few more transfer results and some problems concerning the congruence property will be mentioned in the next section.

6. Some open problems

This is a relatively new research area, so it is surely full of open problems and ideas for further work. For instance, in Section 4 I have already said that the notion of full adequacy has not been deeply investigated. Here, rather than formulating very specific *problems*, I am going to comment on a number of *topics* where there are several questions still to be scrutinized and deserving, in my opinion, special attention. Some of them are intimately connected, at least judging from the already known facts and the methods of proof hitherto used; but perhaps new ideas will make them differentiate significatively.

6.1. The transfer of congruence properties

This is a particular case of the general problem of transfer of any metalogical property that can be formulated in terms of generalized matrices.

Extending the notation introduced in Theorem 5, for a generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ and an $F \in \mathcal{C}$ we put $\mathcal{C}^F = \{G \in \mathcal{C} : F \subseteq G\}$ and $\boldsymbol{\mathcal{A}}^F = \langle \boldsymbol{A}, \mathcal{C}^F \rangle$. If $\boldsymbol{\mathcal{L}}$ is a sentential logic and $T \in \mathcal{T}h\boldsymbol{\mathcal{L}}$ is a theory closed under substitutions then $\boldsymbol{\mathcal{L}}^T$ is what is commonly called an axiomatic extension of $\boldsymbol{\mathcal{L}}$; but here closure under substitutions is not required.

DEFINITION 13. *A generalized matrix $\boldsymbol{\mathcal{A}}$ has* ***the congruence property*** *when $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}}) \in \mathrm{Co}\boldsymbol{A}$, that is, when $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) = \boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}})$; equivalently, when $\boldsymbol{\mathcal{A}}$ is a model of the* ***congruence rules****:*

$$\frac{\{x_i \vdash y_i \ , \ y_i \vdash x_i : i < n\}}{\varpi x_0 \dots x_{n-1} \vdash \varpi y_0 \dots y_{n-1}} \tag{10}$$

for each basic operation symbol ϖ in the language (where n is its arity).

A sentential logic $\boldsymbol{\mathcal{L}}$ is ***selfextensional*** *when as a generalized matrix it has the congruence property in the just defined sense. That is, when the* *interderivability relation* $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{L}})$ *of the logic has the* ***replacement property****.*

A generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ ***has the Fregean property*** *when for each* $F \in \mathcal{C}$ *the generalized matrix* $\boldsymbol{\mathcal{A}}^F$ *has the congruence property.*

A logic is ***Fregean*** *when as a generalized matrix it has the Fregean property, that is, when the interderivability relation modulo any theory of the logic has the replacement property.*

Selfextensional logics were first considered by WÓJCICKI (see [62, §5.6.5] for references); he has shown that they correspond to the local consequence defined by any class of *referential matrices*, a special kind of generalized matrices that abstract from several kinds of relational semantics. Fregean logics were first considered, in the present sense, in [31]; protoalgebraic Fregean logics were being studied independently at the same time by CZELAKOWSKI and PIGOZZI but their results have been published only recently in [22, 26]. Every Fregean logic is selfextensional; among Fregean logics we find classical and intuitionistic logic, their implicative or equivalential fragments, their axiomatic extensions, and all two-valued logics. Among non-Fregean but selfextensional logics there are the local consequences of normal modal logics, Belnap's logic, and the inferential system $\mathcal{WR}$ of relevance logic already mentioned in Section 3.

One can consider the transfer problem of these two properties: being selfextensional and being Fregean.

The interest in the transfer of the congruence property is not merely abstract. Having it makes life much easier; a generalized matrix is reduced and has the congruence property if and only if $\mathrm{C}(a) = \mathrm{C}(b)$ implies $a = b$; topologically this means being T_0 ($a \neq b \implies \exists F \in \mathcal{C}$ such that $a \in F$ and $b \notin F$, or $a \notin F$ and $b \in F$). In generalized matrices with the congruence property the equations valid in its reduction are obtained by computing "modulo C" in their algebra reduct. If $\boldsymbol{\mathcal{L}}$ is a selfextensional logic then an equation $\varphi \approx \psi$ holds in $\mathbf{V}(\boldsymbol{\mathcal{L}})$ if and only if φ and ψ are interderivable modulo $\boldsymbol{\mathcal{L}}$. Fregean logics seem to have rather strong properties and exhibit a nice algebraic behaviour; see next subsection.

The general transfer problems of the congruence property and of the Fregean property, stated in [33], have been open for some time, but were recently solved in the negative independently by BABYONISHEV [2] and by BOU [12]. Since the counterexamples they found are non-protoalgebraic logics, it is still interesting to investigate these transfer problems restricted to certain classes of logics, either to identify classes of logics where the transfer problem has a positive solution, or to find more counterexamples inside these classes. Some results have already been obtained:

THEOREM 14 ([26]). *If $\mathcal{L}$ is a protoalgebraic logic then the Fregean property transfers to all its full models.*

This is actually a consequence of a more general result of the same paper, which roughly speaking says that for protoalgebraic logics any property that can be formulated in terms of Gentzen-style rules that admit an arbitrary number of "side formulas" in the antecedent does transfer from the logic to all its full models; being Fregean is a property of this kind.

THEOREM 15 ([33]). *Let $\mathcal{L}$ be a logic satisfying the uniterm Deduction Theorem, or having a Conjunction. Then the congruence property transfers to all its full models.*

Selfextensional logics are much weaker than Fregean ones, and perhaps adding protoalgebraicity is not enough for the transfer. Actually, every logic with the Deduction Theorem is protoalgebraic, but the proof of the above result works only if the Deduction Theorem holds for a single term connective and cannot be extended to arbitrary protoalgebraic logics. But at present no counterexample is known even inside the much restricted class of finitely algebraizable logics. Hence the transfer problem of the congruence property is still open for the classes in the Leibniz hierarchy.

6.2. The two hierarchies

The main hierarchy that organizes the universe of logics as studied by Abstract Algebraic Logic, the so-called *Leibniz hierarchy*, can be described under several points of view, all concerning notions of the classical theory of matrices: the lattice-theoretic behaviour of the Leibniz operator on $\mathcal{L}$-filters, the definability of the $\mathcal{L}$-filters of the reduced matrices, the closure properties of the class of all reduced matrices of the logic, etc. In view of some already obtained results it seems that another, less complicated hierarchy of logics deserves some attention. It is based on the congruence property. Besides the classes of *selfextensional* and of *Fregean* logics it comprises the following two:

DEFINITION 16. *A logic is **fully selfextensional** when all its full models have the congruence property, and is **fully Fregean** when all its full models have the Fregean property.*

Thus the transfer problems of the congruence property can be paraphrased as the questions of whether every selfextensional logic is fully selfextensional, and of whether every Fregean logic is fully Fregean.

Here the (partly) open problems concern the clarification of the structure of this hierarchy and of its relationships with the Leibniz hierarchy. Every fully Fregean logic is both Fregean and fully selfextensional, but it is not known whether it equals the intersection of these two classes. The class of selfextensional logics includes the other classes, and most probably it does not equal their union (i.e., most probably there are selfextensional logics which are neither Fregean nor fully selfextensional). From the negative solutions to the transfer problems mentioned in Subsection 6.1 it follows that the four classes are different.

In principle it seems that selfextensionality is a property independent of the Leibniz hierarchy: There are selfextensional and non-selfextensional examples in all its subclasses, except for a few cases. The situation is very different for Fregean logics: due to Theorem 14 inside protoalgebraic logics (that is, inside the whole Leibniz hierarchy) the classes of Fregean and of fully Fregean logics coincide. And a few more things are known:

THEOREM 17.

1. *Every protoalgebraic Fregean logic with theorems is regularly algebraizable, that is, it is algebraizable and the designated set of its reduced matrices consists of a single element* [26, 31].
2. *A weakly algebraizable logic is Fregean if and only if it is fully selfextensional* [33].
3. *A fully selfextensional logic is algebraizable if and only if it is weakly algebraizable* [33].

Thus inside weakly algebraizable logics (hence, inside algebraizable logics) the only interesting remaining question is whether every selfextensional such logic is Fregean. This problem was already formulated in [31], and as far as I know, no counterexamples have been presented. In contrast with the strength that being Fregean seems to acquire when this property is coupled with algebraizability, as is clarified in [26, 49], selfextensionality seems much weaker; this makes the absence of counterexamples particularly noteworthy.

A related remaining task, which is interesting by itself but which can also shed some light over the mentioned open problems, is the classification in the two hierarchies of as many logics as possible. This forcefully includes the determination of the full models of these logics. The fact that the majority of logics (partially) classified up to now are either selfextensional, or algebraizable, or both, may just indicate that these logics are among the best known (and easier to study?) ones. The validity and usefulness of this kind of general theories and general classification schemes depends on their ability to deal not only with the well-behaved, central examples, but with the more marginal or even pathological ones[6]. I am convinced that the ex-

[6]For instance, those that are neither protoalgebraic nor selfextensional.

amination of a large number of logics coming from very different contexts will strengthen our understanding of these hierarchies and of the real significance of their subclasses; and as a bonus some of the problems might be solved with natural rather than *ad hoc* counterexamples. Of course, it can also turn out that some solution is affirmative, which would require a general proof; but even in this case the analysis of examples and of why they have certain property or why they do not have it, might help to shape a better view of the situation.

6.3. The variety problem

This problem arises in the theory of algebraizable logics, but has ramifications concerning generalized matrices. If $\mathcal{L}$ is a finitary, finitely algebraizable logic, then the general theory guarantees that its algebraic counterpart $\mathsf{Alg}\mathcal{L}$ (its largest equivalent algebraic semantics) is a quasivariety. However, experience shows that for a very large number of such algebraizable logics this class is indeed a variety (when this happens the logics are called ***strongly algebraizable***). The interesting questions is: *why is it so?* What do these logics have in common? Is there a characterization of such logics? Can we find necessary or sufficient conditions that explain this behaviour? More generally, one can also wonder why is it that $\mathsf{Alg}\mathcal{L}$ is a variety for many other logics, not necessarily algebraizable, not even protoalgebraic.

Some sufficient conditions have already been found in connection with the proof of Theorem 15:

THEOREM 18 ([33]). *Let $\mathcal{L}$ be a selfextensional logic satisfying the uniterm Deduction Theorem, or having a Conjunction. Then $\mathsf{Alg}\mathcal{L}$ is a variety, that is, $\mathsf{Alg}\mathcal{L} = \mathsf{V}(\mathcal{L})$.*

COROLLARY 19. *If $\mathcal{L}$ is a selfextensional and algebraizable logic satisfying the uniterm Deduction Theorem, or having a Conjunction, then $\mathcal{L}$ is strongly algebraizable.*

We have already seen (Theorem 17.1) that if the logic is not merely selfextensional but protoalgebraic and Fregean then it is also regularly algebraizable. Some deep results concerning this case are contained in [26]; all the logics analysed in this paper are shown to be definitionally equivalent to certain axiomatic expansions of the $(\to)$-fragment and of the $(\to,\wedge)$-fragment, respectively, of the intuitionistic propositional logic; see [38, § 4.4] for an overview. In [25] it is shown that the condition that the Deduction Theorem is effected by a single term is essential in the preceding results, by constructing an *ad hoc* example of a Fregean and algebraizable logic with a two-term Deduction Theorem that is not strongly algebraizable.

These are partial results that apply only to selfextensional or Fregean logics, but the phenomenon has been equally observed in non-selfextensional logics. Some of these cases might also be covered by the theory of the *strong version* of a protoalgebraic logic started in [34]: There it is shown that certain protoalgebraic logics that are not algebraizable engender a "strong version" which is at least weakly algebraizable, and has the same class of associated algebras. If in addition the weaker logic falls under one of the previous results, then this class is a variety and the strong version happens to be strongly algebraizable, without falling itself under the previous results. But again this theory is of limited application (it applies only to protoalgebraic logics), and moreover it only gives a sufficient condition. Thus there is plenty of room here for investigation.

Another interesting research would be to obtain more direct proofs of the preceding results, particularly of Theorem 18. The proof in [33] is obtained in a broader context, where it is also shown that the full adequacy problem has a positive solution in these cases:

THEOREM 20. *Let $\mathcal{L}$ be a selfextensional logic satisfying the uniterm Deduction Theorem, or having a Conjunction. Then $\mathcal{L}$ has a fully adequate Gentzen system.*

Actually the finding of such fully adequate Gentzen system is the key point in the proof that the associated class of algebras (which is also associated to the Gentzen system) is a variety. Remark that the statements of Theorems 15 and 18 concern generalized matrices understood as models of sentential logics, but not necessarily as models of Gentzen systems. While this fact emphasizes the usefulness of this double rôle of generalized matrices, it also calls for new proofs that can be developed entirely within the "sentential" level of the theory.

6.4. The notion of G-algebraizability

The just mentioned proof in [33] of Theorem 18 through the existence of a fully adequate Gentzen systems goes farther away from the classical theory of matrices and of generalized matrices. Actually, what one does to infer Theorem 18 from Theorem 20 is to consider the fully adequate Gentzen system under the light of a "third level" version of the notion of algebraizability. Such extension allows one to deal with substructural Gentzen systems, and was introduced by GIL, REBAGLIATO, TORRENS and VERDÚ [42, 43, 53, 54, 58]; a summary appears in Section 4.2 of [38]. In [33] it is shown that the obtained Gentzen system is finitely algebraizable in the sense of REBAGLIATO and VERDÚ, and that its largest equivalent algebraic semantics is precisely the class $\mathbf{V}(\mathcal{L})$; but on the other hand the general theory says this largest equivalent algebraic semantics has to be $\mathsf{Alg}\mathfrak{G}$, which by full adequacy equals $\mathsf{Alg}\mathcal{L}$.

While this situation is a nice example of how to combine work at several levels of the hierarchy of notions of sentential logic described in Section 1, it also suggests the consideration of a new notion of algebraizability for sentential logics:

DEFINITION 21. *A sentential logic is* ***G-algebraizable*** *when it has a fully adequate Gentzen system that is algebraizable in the sense of* [53, 54].

Note that this new concept is not an extension of the notion of algebraizable logic; there are logics which are algebraizable in the usual sense but do not have a fully adequate Gentzen system (this can be shown by Theorem 10) and hence are not G-algebraizable; and conversely, some of the known G-algebraizable logics are not algebraizable in the usual sense (actually they are not even protoalgebraic). Examples of the first kind are $\mathcal{BCK}$, $\mathcal{R}$ or $Ł_\infty$, and examples of the second kind are $\mathcal{CPC}_{\wedge\vee}$ and $\mathcal{B}$; see Table 1 for notations and references.

In G-algebraizable logics the link with algebraic properties of their associated class of algebras is effected by properties expressible as Gentzen-style rules, while in classically algebraizable logics the link is effected by the equational consequence relative to the class of algebras and two substitution-invariant translations. The in-depth study of the new notion and its interaction with the two hierarchies of Subsection 6.2 will surely be a rich area of research.

6.5. Characterizing the class $\mathsf{Alg}\mathcal{L}$

The most common perspective in Algebraic Logic is to view it from the logical side; i.e., one has a logic and is interested in knowing more things about it through the study of the algebraic properties of a certain class of algebras that bears to the logic some specific relation. Up to now the closest kind of relation is that of being the equivalent algebraic semantics for the logic, in case this one is algebraizable.

But there is a dual perspective, which is to start from a class of algebras and to look for a certain logic that bears to it the said relationship; this could be called *the problem of the logification* of a class of algebras. The theory of algebraizability has found one solution to it: For instance, one can specify necessary and sufficient conditions for a quasivariety to be the equivalent algebraic semantics of a finitary, finitely algebraizable logic, see [22, § 4.6]. An even finer work can be done if the focus is restricted to protoalgebraic Fregean logics: the corresponding algebras form the so-called congruence-orderable relatively point-regular quasivarieties; see [38, § 4.4] for more details and references. Sometimes, also the isomorphism theorem for algebraizable logics can be used to show that a certain class of algebras can never be the equivalent algebraic semantics of any algebraizable logic, as is done in [32] for the variety of De Morgan algebras and in [40] for the variety of distributive lattices.

I have already said that the notion of G-algebraizability just discussed can account for the algebraic character of logics not falling under the algebraizable paradigm. Conversely, it can also give rise to a broader notion of "algebra of logic". What has not been investigated yet, is the dual perspective on this new concept: Is there a characterization of the classes of algebras that can be shown to be the algebraic counterpart of a G-algebraizable logic? My conjecture is that this should be relatively easy if one restricts the search to fully selfextensional logics.

Finally, one can also consider the very general problem: Is there a characterization of those classes of algebras that are of the form $\mathbf{Alg}\mathcal{L}$ for some sentential logic $\mathcal{L}$ of the most arbitrary kind? It is probably too difficult, given its extremely generality, so one can also restrict the search to classes of logics with a good behaviour in one sense or another (the results mentioned two paragraphs above correspond to the restrictions to algebraizable and to protoalgebraic and Fregean logics).

Appendix

Let $\mathcal{L}$ be a sentential logic. It satisfies the ***Deduction Theorem*** when there is a finite set $\Delta(p,q) \subseteq Fm$ such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$, $\Gamma, \varphi \vdash_{\mathcal{L}} \psi \iff \big(\Gamma \vdash_{\mathcal{L}} \delta(\varphi,\psi)$ for all $\delta(p,q) \in \Delta(p,q)\big)$. If the set Δ contains just one formula, we speak of a *uniterm Deduction Theorem*, in contrast to the *multiterm* case, where the set Δ may contain more than one formula. In the literature this property is also called the *Deduction-Detachment Theorem (DDT).*

$\mathcal{L}$ has a ***Conjunction*** when there is a binary term (i.e., a formula in two variables) $p \wedge q$ such that $\{\varphi,\ \psi\} \dashv\vdash_{\mathcal{L}} \varphi \wedge \psi$ for all $\phi, \psi \in Fm$.

$\mathcal{L}$ has a ***Disjunction*** when there is a binary term (i.e., a formula in two variables) $p \vee q$ such that $\Gamma, \varphi \vee \psi \vdash_{\mathcal{L}} \xi \iff \big(\Gamma, \varphi \vdash_{\mathcal{L}} \xi$ and $\Gamma, \psi \vdash_{\mathcal{L}} \xi\big)$ for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$.

$\mathcal{L}$ satisfies the ***Introduction of Modality*** for a unary connective or term $\Box p$ when $\Gamma \vdash_{\mathcal{L}} \varphi$ implies $\Box\Gamma \vdash_{\mathcal{L}} \Box\varphi$, for all $\Gamma \cup \{\varphi\} \subseteq Fm$.

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Josep Maria Font
Faculty of Mathematics, University of Barcelona
Gran Via 585
E-08007 Barcelona, Spain