Refutability and Post Completeness

TOMASZ SKURA

Abstract

The goal of this paper is to give a necessary and sufficient condition for a multiple-conclusion consequence relation to be Post complete by using proof/refutation systems.
Keywords. refutation systems, Post completeness, multiple-conclusion logic.

1 Introduction

When we deal with the set L of propositional formulas valid in some structures, we are usually interested in laws and inference rules generating valid formulas. However, we can also consider non-laws and refutation rules generating non-valid formulas. As a result, we have a pair $\mathbf{S} = (\mathcal{T}, \mathcal{F})$ of complementary inference systems ($\mathcal{T}$ for L and $\mathcal{F}$ for $-L$), which can be called a proof/refutation system. Such systems enable provability and refutability on the propositional level. In a manner of speaking, we have two engines rather than one.

The idea of syntactic refutability was introduced by Łukasiewicz [3] (but it was already known to Aristotle), and it is now used in computer science (see e.g. [5], [2], [1]). For an introduction to refutation systems see [11].

In this paper we show that this method provides a natural characterization of Post completeness.

Post completeness can be defined in a couple of ways. Here, in view of possible generalizations, it is convenient to define it as follows. A logic (or a logical theory[1]) L is Post complete iff L is consistent and L has no consistent proper extension. This definition was adopted by (among others) McKinsey [4] and Segerberg [8]. (Note that this definition is different from that in [9].)

[1]A logical theory is a set of formulas closed under a consequence relation.

The fact that Classical Logic (that is, the set CL of Boolean laws) is Post complete can be established by showing the following.

For any formula $A \notin CL$ there is a substitution s such that

$s(A) \to p \in CL$, where p is a fixed propositional variable.

In other words, for every formula A

either $A \in CL$ or A is refutable in the Łukasiewicz refutation system:

Refutation axiom: p
Refutation rules:
(*reverse substitution*) $s(A)/A$, where $s(A)$ is a substitution instance of A
(*reverse modus ponens*) B/A, where $A \to B \in CL$

Thus, if this refutation system is complete for CL, then CL is Post complete. On the other hand, it can be shown that if CL is Post complete, then the system is complete for CL. This fact suggests a close relationship between Post completeness and syntactic refutability.

Post completeness is such a nice property. However, although there are infinitely many Post complete logics (see [4], [8]), it seems that there is no interesting standard non-classical logic that is Post complete. What happens if the term logic is construed more generally? There are two possibilities.

(1) A *logic* is a consequence relation $\vdash$ between finite sets of formulas and formulas. Then the set $L_\vdash = \{A : \vdash A\}$ of is theorems is a logical theory. We now say that a logic $\vdash$ is Post complete iff $\vdash$ is consistent and has no consistent proper extension. However, it turns out that $\vdash$ is Post complete iff $L_\vdash$ is Post complete (see e.g. [6]), so that in this case nothing happens.

(2) A *logic* is a consequence relation $\vdash\dashv$ between finite sets of formulas (or a multiple-conclusion consequence relation). Again, we say that $\vdash\dashv$ is Post complete iff $\vdash\dashv$ is consistent and has no consistent proper extension. Then the relation $\vdash = \{X/A : X \vdash\dashv A\}$ (where X is a set of formulas and A is a formula) is a logic in the sense defined in (1). Interestingly, the situation is now dramatically different. For example, consider any non-classical logic L that is not Post complete. Let $\vdash\dashv^L = \{X/Y : \text{If } X \subseteq L \text{ then } Y \cap L \neq \emptyset\}$ (so $\vdash\dashv^L$ is the set of multiple-conclusion inferences preserving L). Then $\vdash\dashv^L$ is Post complete (cf. Corollary 3.4).

In this paper we give a necessary and sufficient condition (in terms of proof/refutation systems) for a multiple-conclusion consequence relation to be Post complete.

2 Preliminaries

Let FOR be the set of formulas generated from the propositional variables $p, p_1, p_2, ...$ by standard connectives. By an *inference* we mean a pair X/A, where X is a finite set of formulas and $A \in FOR$. And by a *rule* $\mathcal{R}$ we mean a set of inferences. Note that if $\mathcal{R}, \mathcal{R}'$ are rules, then so is $\mathcal{R} \cup \mathcal{R}'$, so that a number of rules can be presented as a single rule. An inference system is a pair (AX, RU), where AX is a set of formulas (axioms) and RU is a set of rules. Since axioms A can be regarded as rules $\emptyset/A$, any inference system can be presented as a rule.

A *consequence relation* is a relation $\vdash$ between finite sets of formulas and formulas satisfying the following conditions.

$A \vdash A$.
If $X \vdash A$ then $X' \vdash A$, where $X \subseteq X'$.
If $X \vdash A$ and $X, A \vdash B$, then $X \vdash B$.

Every rule $\mathcal{R}$ determines a consequence relation $\vdash_{\mathcal{R}}$ defined as follows.

$X \vdash_{\mathcal{R}} A$ iff A is derivable from X by $\mathcal{R}$, that is, there is a sequence $A_1, ..., A_n$ such that $A_n = A$ and each A_i is in X or is obtained by $\mathcal{R}$.

We say that a set Z of formulas is closed under a rule $\mathcal{R}$ (or $\mathcal{R}$ preserves Z) iff for all $X/A \in \mathcal{R}$, if $X \subseteq Z$ then $A \in Z$. We remark that if $\mathcal{R}$ preserves Z, then so does $\vdash_{\mathcal{R}}$.

A *multiple-conclusion inference* (or a sequent) is a pair X/Y, where X, Y are finite sets of formulas. And a *multiple-conclusion rule* (or a sequent rule) is a set of sequents.

A *multiple-conclusion consequence relation* is a relation $\vdash\!\dashv$ between finite sets of formulas satisfying the flowing conditons.

$X \vdash\!\dashv Y$ if $X \cap Y \neq \emptyset$.
If $X \vdash\!\dashv Y$ then $X' \vdash\!\dashv Y'$, where $X \subseteq X', Y \subseteq Y'$.
If $X \vdash\!\dashv A, Y$ and $X, A \vdash\!\dashv Y$, then $X \vdash\!\dashv Y$.

We say that $\vdash\!\dashv$ is *consistent* iff for no A both $\vdash\!\dashv A$ and $A \vdash\!\dashv$. And we say that $\vdash\!\dashv$ is *complete* iff for all A, either $\vdash\!\dashv A$ or $A \vdash\!\dashv$. Since $FOR \neq \emptyset$, we have: $\vdash\!\dashv$ is consistent iff $\emptyset \not\vdash\!\dashv \emptyset$.

By an *extension* (proper extension) of a consequence relation $\vdash\!\dashv$ we mean a consequence relation $\vdash\!\dashv'$ such that $\vdash\!\dashv \subseteq \vdash\!\dashv'$ ($\vdash\!\dashv \subset \vdash\!\dashv'$).

Every sequent rule Σ determines a multiple-conclusion consequence relation $\vdash\!\dashv_\Sigma$ defined as follows. $X \vdash\!\dashv_\Sigma Y$ iff there is a finite sequence $\alpha_1, .., \alpha_n$ of sequents such that $\alpha_n = X/Y$ and each α_i is in Σ or is obtained from preceding sequents by one of the following rules.

$$(R) \quad \frac{}{X/Y} \quad \text{where } X \cap Y \neq \emptyset$$

$$(M) \quad \frac{X/Y}{X'/Y'} \quad \text{where } X \subseteq X', Y \subseteq Y'$$

$$(C) \quad \frac{X/A, Y \quad X, A/Y}{X/Y}$$

3 Consequence Relations

Lemma 3.1 (Scott [7]) *If a consequence relation $\vdash\!\dashv$ is Post complete, then it is complete.*

PROOF Assume that $\vdash\!\dashv$ is Post complete, but it is not complete. Since $\vdash\!\dashv$ is consistent, there are finite sets $X_0, Y_0 \subseteq FOR$ such that $X_0 \not\vdash\!\dashv Y_0$. And since $\vdash\!\dashv$ is not complete, there is $A \in FOR$ such that both $\not\vdash\!\dashv A$ and $A \not\vdash\!\dashv$. We now use the technique introduced by Scott in the proof of Proposition 1.3 [7]. We define $\vdash\!\dashv_0, \vdash\!\dashv_1$ as follows.

$X \vdash\!\dashv_0 Y$ iff $X \vdash\!\dashv A, Y$

$X \vdash\!\dashv_1 Y$ iff $X, A \vdash\!\dashv Y$

for all finite $X, Y \subseteq FOR$. It is easy to check that $\vdash\!\dashv_0, \vdash\!\dashv_1$ are consequence relations. Also, both $\vdash\!\dashv_0$ and $\vdash\!\dashv_1$ are proper consistent extensions of $\vdash\!\dashv$. Hence $X_0 \vdash\!\dashv_0 Y_0$ and $X_0 \vdash\!\dashv_1 Y_0$ (because $\vdash\!\dashv$ is Post complete). But then $X_0 \vdash\!\dashv Y_0$ (by C), which is a contradiction. QED

Lemma 3.2 *Let $\vdash\!\dashv$ be a consistent consequence relation.*

(i) If both $\vdash\!\dashv A$ for all $A \in X$ and $B \vdash\!\dashv$ for all $B \in Y$, then $X \not\vdash\!\dashv Y$.

(ii) If $\vdash\!\dashv$ is complete and $X \not\vdash\!\dashv Y$, then both $\vdash\!\dashv A$ for all $A \in X$ and $B \vdash\!\dashv$ for all $B \in Y$.

PROOF (i) By induction on the number $|X \cup Y|$ of formulas in $X \cup Y$.

(1) $|X \cup Y| = 0$. Then $X = Y = \emptyset$, so $X \not\vdash\!\dashv Y$, and this is true.

(2) $|X \cup Y| > 0$. Then, say, $X \neq \emptyset$. (If $Y \neq \emptyset$, the argument is similar.) So there is a formula $D \in X$. Suppose that $\vdash\!\dashv A$ for all $A \in X$ and $B \vdash\!\dashv$ for all $B \in Y$, but $X \vdash\!\dashv Y$. Then, in particular, $\vdash\!\dashv D$, so $X^- \vdash\!\dashv D, Y$ (by M),

where $X^- = X - \{D\}$. Also $X^-, D \vdash\dashv Y$. Hence $X^- \vdash\dashv Y$ (by C). Since $|X^- \cup Y| < |X \cup Y|$, by the induction hypothesis, we get $X^- \not\vdash\dashv Y$, which is a contradiction.

(ii) Assume that $\vdash\dashv$ is complete and $X \not\vdash\dashv Y$, but either $\not\vdash\dashv A$ for some $A \in X$ or $B \not\vdash\dashv$ for some $B \in Y$. Then either $A \vdash\dashv$ for some $A \in X$ or $\vdash\dashv B$ for some $B \in Y$. Hence $X \vdash\dashv Y$ (by M), which is a contradiction. QED

Lemma 3.3 *Let $\vdash\dashv$ be a consequence relation. If $\vdash\dashv$ is consistent and complete, then it is Post complete.*

PROOF Assume that $\vdash\dashv$ is consistent and complete, but it is not Post complete. then there is a consistent relation $\vdash\dashv'$ such that $\vdash\dashv \subset \vdash\dashv'$, so there are finite sets $X, Y \subseteq FOR$ such that $X \vdash\dashv' Y$ and $X \not\vdash\dashv Y$. Since $\vdash\dashv$ is complete (by Lemma 3.2 (ii)), we have: $\vdash\dashv A$ for all $A \in X$ and $B \vdash\dashv$ for all $B \in Y$, so

$\vdash\dashv' A$ for all $A \in X$ and $B \vdash\dashv'$ for all $B \in Y$ (because $\vdash\dashv \subseteq \vdash\dashv'$).
Hence, by Lemma 3.2 (i), $X \not\vdash\dashv' Y$, which is a contradiction. QED

Corollary 3.4 *A consequence relation $\vdash\dashv$ is Post complete iff it is consistent and complete.*

PROOF From Lemma 3.1 and Lemma 3.3. QED

The following simple facts will be useful in Section 4.

Proposition 3.5 *Let $\vdash\dashv$ be a consequence relation.*

(i) If $X \vdash\dashv A_i$ for all $1 \le i \le n$ and $X, A_1, ..., A_n \vdash\dashv Y$, then $X \vdash\dashv Y$.

(ii) If $X \vdash\dashv A_i$ for all $1 \le i \le n$ and $A_1, ..., A_n \vdash\dashv Y$, then $X \vdash\dashv Y$.

PROOF (i) By induction on n.

(1) $n = 1$. Assume $X \vdash\dashv A_1$ and $X, A_1 \vdash\dashv Y$. Then $X \vdash\dashv A_1, Y$ (by M), so $X \vdash\dashv Y$ (by C).

(2) $n > 1$. Assume $X \vdash\dashv A_i$ for all $1 \le i \le n$ and $X, A_1, A_2, ..., A_n \vdash\dashv Y$. Then $X, A_2, ..., A_n \vdash\dashv A_1, Y$ (by M), so $X, A_2, ..., A_n \vdash\dashv Y$ (by C). Also $X \vdash\dashv A_i$ for all $2 \le i \le n$. Hence, by the induction hypothesis, we get $X \vdash\dashv Y$.

(ii) Assume $X \vdash\dashv A_i$ for all $1 \le i \le n$ and $A_1, ..., A_n \vdash\dashv Y$. Then $X, A_1, ..., A_n \vdash\dashv Y$ (by M), so $X \vdash\dashv Y$ (by (i)). QED

Proposition 3.6 *Let $\vdash\!\dashv$ be a consequence relation.*

(i) If $X \vdash\!\dashv Y, A_1, ..., A_n$ and $X, A_i \vdash\!\dashv Y$ for all $1 \leq i \leq n$, then $X \vdash\!\dashv Y$.

(ii) If $X \vdash\!\dashv A_1, ..., A_n$ and $A_i \vdash\!\dashv Y$ for all $1 \leq i \leq n$, then $X \vdash\!\dashv Y$.

PROOF (i) By induction on n.

(1) $n = 1$. Then this holds by C.

(2) $n > 1$. Assume $X \vdash\!\dashv Y, A_1, ..., A_n$ and $X, A_i \vdash\!\dashv Y$ for all $1 \leq i \leq n$. Then $X, A_1 \vdash\!\dashv Y, A_2, ..., A_n$ and $X \vdash\!\dashv A_1, Y, A_2, ..., A_n$ (by M). Hence $X \vdash\!\dashv Y, A_2, ..., A_n$ (by C). Also $X, B \vdash\!\dashv Y$ for all $B \in Y$ (by R). Therefore, by the induction hypothesis, $X \vdash\!\dashv Y$.

(ii) From (i). QED

Proposition 3.7 *Let $\mathcal{R}$ be an inference rule.*

(i) If $X \vdash_{\mathcal{R}} A$ then $X \vdash\!\dashv_{\mathcal{R}} A$.

(ii) If $X \vdash_{\mathcal{R}} A$ and $\mathcal{R}' = \{B/Y : Y/B \in \mathcal{R}\}$, then $A \vdash\!\dashv_{\mathcal{R}'} X$.

PROOF (i) Assume that $X \vdash_{\mathcal{R}} A$. Then there is a derivation of A from X by $\mathcal{R}$, that is, a sequence $A_1, ..., A_n$ such that $A_n = A$ and each A_i is in X or is obtained by $\mathcal{R}$. We show, by induction on n, that there is a sequence $\alpha_1, ..., \alpha_m$ such that $\alpha_m = X/A$ and each α_i is in $\mathcal{R}$ or is obtained by R, M, C (so that $X \vdash\!\dashv_{\mathcal{R}} A$).

(1) $n = 1$. Then $A_1 = A$.

(Case 1) $A \in X$. Then $\alpha_1 = X/A$ is obtained by R.

(Case 2) A is obtained by $\mathcal{R}$. Then $\emptyset/A \in \mathcal{R}$ and $\alpha_1 = \emptyset/A \in \mathcal{R}$.

(2) $n > 1$ and we assume that every shorter derivation of B from X by $\mathcal{R}$ has a sequent derivation of X/B from $\mathcal{R}$ by R, M, C. We may also assume that $A \notin X$ (otherwise see (1)). Then A is obtained from some $B_1, ..., B_k \in \{A_1, ..., A_{n-1}\}$ and $B_1, ..., B_k/A \in \mathcal{R}$. By the induction hypothesis, for every $1 \leq i \leq k$, X/B_i is derivable from $\mathcal{R}$ by R, M, C. Hence, by Proposition 3.5 (ii), so is X/A, so that $X \vdash\!\dashv_{\mathcal{R}} A$.

(ii) The proof is similar to that of (i). The important modification is the following.

(Case 2) A is obtained from some $B_1, ..., B_k \in \{A_1, ..., A_{n-1}\}$ and $B_1, ..., B_k/A \in \mathcal{R}$. We have $A/B_1, ..., B_k \in \mathcal{R}'$. By the induction hypothesis, for every $1 \leq i \leq k$, B_i/X is derivable from $\mathcal{R}'$ by R, M, C. Hence, by Proposition 3.6 (ii), so is A/X, so that $A \vdash\!\dashv_{\mathcal{R}'} X$. QED

4 Proof/Refutation Systems

Definition (i) A *proof/refutation system for* a set $L \subseteq FOR$ is a pair

$$\mathbf{S} = (\mathcal{T}, \mathcal{F})$$

of inference rules such that $\mathcal{T}$ preserves L and $\mathcal{F}$ preserves $-L$.

(ii) $\mathbf{S}$ is *complete for* L iff for every $A \in FOR$ we have

$$\vdash_{\mathcal{T}} A \text{ or } \vdash_{\mathcal{F}} A$$

(iii) The *multiple-conclusion consequence relation determined by* $\mathbf{S}$ is the relation $\vdash\!\dashv_{\Sigma}$, where

$$\Sigma = \mathcal{T} \cup \{A/X : X/A \in \mathcal{F}\}$$

Lemma 4.1 *Let* $\mathbf{S}$ *be a complete proof/refutation system for* L.

(i) $A \in L$ *iff* $\vdash_{\mathcal{T}} A$*, and* $A \notin L$ *if* $\vdash_{\mathcal{F}} A$.

(ii) $\vdash\!\dashv_{\Sigma}$ *preserves* L.

(iii) $\vdash\!\dashv_{\Sigma}$ *is consistent.*

(iv) $A \in L$ *iff* $\vdash\!\dashv_{\Sigma} A$*, and* $A \notin L$ *iff* $A \vdash\!\dashv_{\Sigma}$.

(v) $\vdash\!\dashv_{\Sigma}$ *is complete.*

PROOF (i) Since $\mathcal{T}$ preserves L and $\mathcal{F}$ preserves $-L$, we have:

If $\vdash_{\mathcal{T}} A$ then $A \in L$, and if $\vdash_{\mathcal{F}} A$ then $A \notin L$.

We show that if $A \in L$ then $\vdash_{\mathcal{T}} A$, and if $A \notin L$ then $\vdash_{\mathcal{F}} A$.

Suppose that $A \in L$ ($A \notin L$) but $\nvdash_{\mathcal{T}} A$ ($\nvdash_{\mathcal{F}} A$). Since $\mathbf{S}$ is complete, this gives $\vdash_{\mathcal{F}} A$ ($\vdash_{\mathcal{T}} A$), so $A \notin L$ ($A \in L$), which is a contradiction.

(ii) This follows from the fact that Σ preserves L and this property is preserved by R, M, C. (For example, to check M, assume that X, Y preserves L. If $X' \subseteq L$, then $X \subseteq L$, so $Y \cap L \neq \emptyset$, so $Y' \cap L \neq \emptyset$, which means that X'/Y' preserves L.)

(iii) Suppose that $\vdash\!\dashv_{\Sigma} A$ and $A \vdash\!\dashv_{\Sigma}$ for some $A \in FOR$. Then $\emptyset \vdash\!\dashv \emptyset$, which contradicts (ii).

(iv) We first show that

($*$) if $A \in L$ then $\vdash\!\dashv_{\Sigma} A$, and if $A \notin L$ then $A \vdash\!\dashv_{\Sigma}$.

Assume that $A \in L$ ($A \notin L$). Then, by (i), we have $\vdash_{\mathcal{T}} A$ ($\vdash_{\mathcal{F}} A$). Hence, by Proposition 3.7, we get $\vdash\!\dashv_{\mathcal{T}} A$ ($A \vdash\!\dashv_{\mathcal{F}'}$), so $\vdash\!\dashv_{\Sigma} A$ ($A \vdash\!\dashv_{\Sigma}$), as required.

Now suppose that $\vdash\!\dashv_\Sigma A$ ($A \vdash\!\dashv_\Sigma$) but $A \notin L$ ($A \in L$). Then, by ($*$), $A \vdash\!\dashv_\Sigma$ ($\vdash\!\dashv_\Sigma A$), which contradicts (iii).

(v) For every $A \in FOR$ we have either $A \in L$ or $A \notin L$. Hence, by (iv), either $\vdash\!\dashv_\Sigma A$ or $A \vdash\!\dashv_\Sigma$. QED

Corollary 4.2 *If* $\mathbf{S}$ *is a proof/refutation system that is consistent and complete, then* $\vdash\!\dashv_\Sigma$ *is Post complete.*

PROOF From Lemma 3.3 and Lemma 4.1 (iii,v).

Lemma 4.3 *If* $\vdash\!\dashv$ *is Post complete, then* $\vdash\!\dashv = \vdash\!\dashv_\Sigma$ *for some complete consistent proof/refutation system* $\mathbf{S}$.

PROOF Assume that $\vdash\!\dashv$ is Post complete. Then $\vdash\!\dashv$ is complete (by Lemma 3.1). Let $\mathbf{S} = (\mathcal{T}, \mathcal{F})$, where

$\mathcal{T} = \{\emptyset/A : \vdash\!\dashv A\}$ and $\mathcal{F} = \{\emptyset/B : B \vdash\!\dashv\}$.

We show that $\vdash\!\dashv = \vdash\!\dashv_\Sigma$.

(1) $\vdash\!\dashv_\Sigma \subseteq \vdash\!\dashv$.

Indeed, by Lemma 3.2, we have:

($\star$) $X \vdash\!\dashv Y$ iff either $A \vdash\!\dashv$ for some $A \in X$ or $\vdash\!\dashv B$ for some $B \in Y$.

Note that Σ satisfies ($\star$) and the rules R, M, C preserve ($\star$). For example, to check C, assume that both $X, A \vdash\!\dashv Y$ and $X \vdash\!\dashv A, Y$ satisfy ($\star$), but $X \vdash\!\dashv Y$ does not. Then $\vdash\!\dashv B$ for all $B \in X$ and $D \vdash\!\dashv$ for all $D \in Y$. Hence $A \vdash\!\dashv$ and $\vdash\!\dashv A$, which is impossible for $\vdash\!\dashv$ is consistent.

(2) $\vdash\!\dashv \subseteq \vdash\!\dashv_\Sigma$.

Indeed, Let $X \vdash\!\dashv Y$. Then $A \vdash\!\dashv$ for some $A \in X$ or $\vdash\!\dashv B$ for some $B \in Y$. Thus, we have $A \vdash\!\dashv_\Sigma$ or $\vdash\!\dashv_\Sigma B$. Hence $X \vdash\!\dashv_\Sigma Y$ (by M), as required. QED

Theorem *A consequence relation* $\vdash\!\dashv$ *is Post complete iff* $\vdash\!\dashv$ *is determined by a proof/refutation system that is both consistent and complete.*

PROOF From Corollary 4.2 and Lemma 4.3.

Remark Thus, Post complete logics and complete proof/refutation systems are two sides of one coin. Of course, there are various kinds of proof/refutation systems. The one given in the proof of Lemma 4.3 is rather trivial. Genuine proof systems (enabling proof search) for standard logics are well-known. And genuine refutation systems (enabling refutation search) are possible. For example, such systems for standard modal logics (and for Classical Logic) are given in [10, 12].

References

[1] Fiorentini, C.: Terminating sequent calculi for proving and refuting formulas in S4. Journal of Logic and Computation (2012) doi: 10.1093/logcom/exs053

[2] Goré, R. and L. Postniece. Combining derivations and refutations for cut-free completeness in bi-intuitionistic logic, Journal of Logic and Computation **20** (2010), 233-260

[3] Łukasiewicz, J.: Aristotle's Syllogistic from the Standpoint of Modern Formal Logic. Oxford (1951)

[4] McKinsey, J.C.C.: On the number of complete extensions of the Lewis systems of sentential calculus. Journal of Symbolic Logic **9** (1944), 42-45

[5] Pinto, L. and R. Dyckhoff: Loop-free construction of counter-models for intuitionistic propositional logic. In: Behara, Fritsch, and Lintz (eds.) Symposia Gaussiana (1995), 225-232

[6] Pogorzelski, W. and P. Wojtylak: Elements of the Theory of Completeness in Propositional Logic. Katowice (1982)

[7] Scott, D.: Completeness and axiomatizability in many-valued logic. In: Proceedings of the Tarski Symposium, American Mathematical Society, Providence (1974)

[8] Segerberg, K.: Post completeness in modal logic. Journal of Symbolic Logic **37** (1972), 711-715

[9] Skura, T.: A refutation theory. Logica Universalis **3** (2009), 293-302

[10] Skura, T.: On refutation rules. Logica Universalis **5** (2011), 249-254

[11] Skura, T.: Refutation systems in propositional logic. In: Gabbay, D. and F. Guenthner (eds.) Handbook of Philosophical Logic, vol. 16, 115-157 (2011)

[12] Skura, T.: Refutation Methods in Modal Propositional Logic. Wydawnictwo Naukowe Semper, Warszawa (2013), forthcoming

e-mail: T.Skura@ifil.uz.zgora.pl