

Aristotle's particularisation

The Achilles' heel of Hilbertian and Brouwerian perspectives of classical logic

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Abstract

We argue that the defining beliefs of both the Hilbertian and the Brouwerian perspectives of classical logic have fatal vulnerabilities—due to their uncritical acceptance in the first case, and their uncritical denial in the second, of Aristotle's particularisation. This is the postulation that an existentially quantified formula—such as $[(\exists x)P(x)]$ —of a first order language S can be assumed to always interpret as the proposition, 'There exists some s in the domain D of the interpretation such that $P^*(s)$ holds in D ', without inviting inconsistency. We show that if the first order Peano Arithmetic PA is consistent, then the postulation is false. However, we cannot conclude from this that the Law of the Excluded Middle, too, is false.

1 Introduction

Paul J. Cohen's proof of the independence of the Axiom of Choice raises an interesting philosophical issue. Cohen's argument—in common with the arguments of many important theorems in standard texts on the foundations of mathematics and logic—appeals to a putative object when interpreting the existential axioms of ZF (or statements about ZF ordinals)¹:

“When we try to construct a model for a collection of sentences, each time we encounter a statement of the form $(\exists x)B(x)$ we must invent a symbol \bar{x} and adjoin the statement $B(\bar{x})$ when faced with $(\exists x)B(x)$, we should choose to have it false, unless we have already invented a symbol \bar{x} for which we have strong reason to insist that $B(\bar{x})$ be true.”

Two questions arises:

Query 1 *Can we always introduce a symbol \bar{x} into a formal language—in Cohen's sense—for which we can insist that $B(\bar{x})$ is true under an interpretation without inviting inconsistency?*

¹[Co66], p.19.

Query 2 *If so, what are the necessary and sufficient conditions that would always permit the introduction of a symbol \bar{x} into a formal language—in Cohen’s sense—for which we can insist that $B(\bar{x})$ is true under an interpretation without inviting inconsistency?*

We note that Query 1 essentially questions the introduction of Cantor’s first transfinite ordinal ω into the formal language of Set Theory, purely on the debatable basis that we can ‘insist’ upon the self-evident nature of the existential Axiom of Infinity².

The question highlights the persisting difference between what are commonly perceived as the Classical (implicitly Platonic, Hilbertian) and Intuitionist (explicitly constructive, Brouwerian) perspectives of those first order languages that appeal formally to the standard first order predicate calculus.

The difference has so far eluded a satisfactory consensus. Thus, Intuitionist systems not only do not subscribe to the Classical belief that Query 1 can be answered in the affirmative without inviting contradiction, but go further in denying legitimacy to standard first order classical logic itself by denying the Law of the Excluded Middle³; whereas, from a Classical perspective, Query 2 appears meaningful.

1.1 The significance of Aristotle’s particularisation for the Classical (Hilbertian) perspective of the standard first order predicate calculus

A formal approach to Query 2 was offered by David Hilbert in a 1925 address⁴. He showed that the axiomatisation \mathcal{L}_ε of classical Aristotelean predicate logic proposed by him as a first order ε -predicate calculus—in which he used a primitive choice-function symbol, ‘ ε ’, for defining the quantifiers ‘ \forall ’ and ‘ \exists ’—would adequately express, and yield under a sound interpretation, Aristotle’s logic of predicates⁵ if the ε -function was interpreted so as to yield an unspecified object (as required by arguments such as Cohen’s)⁶.

- *Soundness:* We define an interpretation of a formal language S as sound if, and only if, the axioms of S are true under the interpretation and the rules of inference preserve such truth.

However, whilst conventional wisdom refrains from appealing to Hilbert’s ε -predicate calculus formally⁷, it does reflect Hilbert’s intent by implicitly accepting that Query 2 is meaningful and—equally implicitly—holding that the concepts of semantic truth and syntactic provability are bound in first order languages—at the very least—by the following thesis.

- *Notation:* We use square brackets to indicate that the contents represent a symbol or a formula—of a formal theory—generally assumed to be well-formed unless otherwise

²[Me64], Axiom I, p.169.

³Which is a Theorem of classical first order logic.

⁴[Hi25], pp.382-383.

⁵Which subsumes Aristotle’s particularisation.

⁶[Hi27], pp.465-466.

⁷Even though Query 1 would necessarily be answered in the affirmative under any sound interpretation of a formal language that appeals to the ε -calculus.

indicated by the context.⁸ We use an asterisk to indicate that the associated expression is to be interpreted semantically with respect to some well-defined interpretation.

Meta-thesis 1 (*Aristotle’s particularisation*) *If the formula $[(\exists x)F(x)]$ of a first order language S interprets as true under a sound interpretation of S , then we may always conclude that there must be some object s in the domain D of the interpretation such that, if the formula $[F(x)]$ interprets as the unary relation $F^*(x)$ in D , then the proposition $F^*(s)$ is true under the interpretation.*

In this investigation we shall show that Aristotle’s particularisation is false; hence Query 1 cannot be answered in the affirmative.

1.2 The significance of Aristotle’s particularisation for the Brouwerian perspective of first order predicate calculus

We begin by noting that, in a first order language, the formula $[(\exists x)P(x)]$ is an abbreviation for the formula $[\neg(\forall x)\neg P(x)]$:⁹

The commonly accepted interpretation of this formula appeals—generally tacitly, but sometimes explicitly¹⁰—to Aristotle’s particularisation. This is a fundamental tenet of the Hilbertian perspective of classical logic; and one which continues to be unrestrictedly adopted as intuitively obvious by standard literature¹¹ that seeks to build upon the first order predicate calculus.

However, L. E. J. Brouwer had noted in his seminal 1908 paper¹²—on the unreliability of some prevailing logical principles—that the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer’s argument was essentially that¹³, even supposing the formula $[P(x)]$ of a formal Arithmetical language interprets as an arithmetical relation that is denoted in the interpretation by $P^*(x)$, and the formula $[\neg(\forall x)\neg P(x)]$ as the arithmetical proposition denoted by $\neg(\forall x)\neg P^*(x)$, the formula $[(\exists x)P(x)]$ cannot be assumed to always interpret as the arithmetical proposition ‘There exists some object a in the domain of the interpretation such that $P^*(a)$ ’, which is denoted by the usual abbreviation $(\exists x)P^*(x)$.

Brouwer held that such postulation is invalid as a general logical principle in the absence of a finitary means for constructing some putative object a for which the proposition $P^*(a)$ holds in the domain of the interpretation.

⁸In other words, expressions inside the square brackets are to be only viewed syntactically as juxtaposition of symbols that are to be formed and manipulated upon strictly in accordance with specific rules for such formation and manipulation—in the manner of a mechanical or electronic device—without any regards to what the symbolism might represent semantically under an interpretation that gives them meaning.

⁹[Me64], p.47.

¹⁰e.g., [Me64], para (ii), p.52.

¹¹e.g., [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], p.52(ii); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [BBJ03], p.102.

¹²[Br08].

¹³cf. [Me64], p.4-5.

Since we shall show that Aristotle’s particularisation is false, Brouwer’s objection is valid.

However, although we could validly argue that if Aristotle’s particularisation is true then any first order system that appeals to the standard first order predicate calculus must have a sound interpretation in which the Law of the Excluded Middle holds, the falsity of Aristotle’s particularisation would not justify Brouwer’s further rejection of the Law of the Excluded Middle¹⁴ as a necessary condition for the consistency of such first order systems.

We shall hold the view, rather, that a first order system which appeals to the standard first order predicate calculus can be consistent¹⁵ even if Aristotle’s particularisation is false.

1.3 Semantic truth and syntactic ω -consistency

Now, in order to avoid appealing to the semantic concept of classical arithmetical truth under an interpretation¹⁶, Kurt Gödel explicitly introduced the syntactic property of ω -consistency as a putative ‘weaker’ assumption in his seminal 1931 paper on formally undecidable arithmetical propositions¹⁷.

Definition 1 (*ω -consistency*) *A first order language S is ω -consistent if, and only if, there is no S -formula $[F(x)]$ such that:*

- (i) $[\neg(\forall x)F(x)]$ is S -provable;
- (ii) $[F(a)]$ is S -provable for any given S -term $[a]$.

Gödel’s introductory remarks¹⁸ suggest that his intent was to substitute the semantic assumption of the truth of provable arithmetic formulas under an interpretation—in a classical logic which appeals to Aristotle’s particularisation—by what he believed to be “a purely formal and much weaker assumption” of ω -consistency; introduction of which, presumably, would not invite inconsistency in any formal reasoning that appeals only syntactically to consistent sets of formulas of systems that subsume the axioms of the first order predicate calculus.

However, we note that Gödel’s remarks¹⁹ suggest that Gödel simultaneously embraced Aristotle’s particularisation, since he implicitly appealed to Aristotle’s particularisation in his reasoning when needed²⁰.

Now, although we could validly argue that if Aristotle’s particularisation is true, then any first order system that appeals to the standard first order predicate calculus must be ω -consistent, the falsity of Aristotle’s particularisation would suggest a need to justify afresh the assumption that a first order Peano Arithmetic can be ω -consistent²¹.

¹⁴Which, when expressed formally, is a Theorem of classical first order predicate calculus.

¹⁵cf. [An12].

¹⁶Which is based on an intuitionistically objectionable logic that implicitly embraces Aristotle’s particularisation.

¹⁷[Go31].

¹⁸On p.9 of [Go31].

¹⁹In his statement of Theorem IX and in his footnote #55 on p.32 of [Go31].

²⁰[Go31], Theorem VII, p.29; Theorem X, p.34; Theorem XI, footnote 63, p.36.

²¹As required for Gödel’s Theorems VI and XI in [Go31] to be mathematically significant.

We therefore now examine whether ω -consistency can—as intended by Gödel²²—be accepted as “a purely formal and much weaker assumption” than the semantic assumption of the truth of provable arithmetical formulas under an interpretation (in a classical logic) which appeals to Aristotle’s particularisation.

1.4 The significance of Aristotle’s particularisation for the first order Peano Arithmetic *PA*

We note first that²³:

Lemma 1 *If PA were to have two sound interpretations over two denumerable domains under the assumption of Aristotle’s particularisation, then the two interpretations would be isomorphic.*

Proof: Let $[G(x)]$ denote the PA-formula²⁴:

$$[x = 0 \vee (\exists y)(x = y')]$$

If we assume Aristotle’s particularisation, then this translates under every sound interpretation of PA as:

If x denotes an element in the domain of a sound interpretation of PA, either x is 0, or x is a ‘successor’.

Further, in every such interpretation of PA, if $G^*(x)$ denotes the interpretation of $[G(x)]$, then:

- (a) $G^*(0)$ is true;
- (b) If $G^*(x)$ is true, then $G^*(x')$ is true.

Hence, by Gödel’s completeness theorem:

- (c) PA proves $[G(0)]$;
- (d) PA proves $[G(x) \rightarrow G(x')]$.

- *Gödel’s Completeness Theorem*²⁵: In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (*i. e. those that are true in every model of the calculus*).

Further, by Generalisation:

- (e) PA proves $[(\forall x)(G(x) \rightarrow G(x'))]$;
- *Generalisation in PA*²⁶: $[(\forall x)A]$ follows from $[A]$.

²²We note that although [An12] gives a finitary proof of consistency for *PA* that does not depend on the truth or falsity of Aristotle’s particularisation, it also echoes Gödel’s belief and naively claims—without proof—that “It is straightforward to show that the two concepts are meta-mathematically equivalent in the sense that, if PA is consistent, then PA is ω -consistent if, and only if, Aristotle’s particularisation holds under the standard interpretation of PA over N ”. The proof is not straightforward.

²³cf. [Me64], Proposition 2.33, p.90.

²⁴Where $[n']$ denotes the ‘successor’ of the numeral $[n]$.

²⁵[Me64], Corollary 2.14, p.68.

²⁶[Me64], p.57.

Hence, by Induction:

(f) $[(\forall x)G(x)]$ is provable in PA.

- *Induction Axiom of PA*²⁷: For any formula $[F(x)]$ of PA:
 $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x')) \rightarrow (\forall x)F(x))]$

In other words, except 0, every element in the domain of any sound interpretation of PA is a ‘successor’. Further, x can only be a ‘successor’ of a unique element in any such interpretation of PA.

It follows that²⁸:

- (i) Every element in the domain of any sound interpretation of PA is either a ‘non-successor’ or a ‘successor’;
- (ii) There can only be a unique ‘non-successor’ in the domain of any sound interpretation of PA;
- (iii) Each ‘successor’ has a unique ‘predecessor’ in the domain of any sound interpretation of PA.

Hence if PA has two sound interpretations over two denumerable domains, then the two interpretations would be isomorphic since they have the same structure²⁹; i.e., the axioms and rules of inference would be common, and we would have the 1-1 correspondence between the members of the two domains:

unique non-successor \leftrightarrow unique non-successor
 successor \leftrightarrow successor.

The lemma follows. □

2 If PA is consistent then Aristotle’s particularisation is false

We note next that:

Lemma 2 *If the first order Peano Arithmetic PA is consistent but not ω -consistent, then Aristotle’s particularisation is false.* □

Proof: If PA is consistent, it has a sound interpretation $\mathcal{I}_{PA(N, Sound)}$ over the domain N of the natural numbers. Further, if PA is not ω -consistent, there is some PA-formula $[R(x)]$ such that:

- (i) $[\neg(\forall x)R(x)]$ is PA-provable, and therefore true in N under $\mathcal{I}_{PA(N, Sound)}$;
- (ii) For any numeral $[n]$ of PA, the formula $[R(n)]$ is PA-provable, and therefore true in N under $\mathcal{I}_{PA(N, Sound)}$.

²⁷[Me64], p.103 (S9).

²⁸cf. [An08], p.7-8.

²⁹cf. [Me64], p.90.

Hence Aristotle's particularisation is false. The Lemma follows. \square

We note further that:

Lemma 3 *If PA is consistent and ω -consistent, then Aristotle's particularisation is false.* \square

Proof: By Gödel's reasoning³⁰, if PA is ω -consistent then there is a formula $[R(x)]$ such that both $[(\forall x)R(x)]$ and $[\neg(\forall x)R(x)]$ are not provable in PA , but $[R(n)]$ is provable for any numeral $[n]$ of PA .

- In his paper Gödel introduces—and thereafter refers to—the corresponding formula in his system P of Peano Arithmetic only by its Gödel number r ³¹.

Hence the formal system PA^+ obtained by adjoining $[\neg(\forall x)R(x)]$ as an axiom to the axioms of PA is consistent, but not ω -consistent³².

By the (downward) Löwenheim-Skolem Theorem, PA^+ has a sound interpretation $\mathcal{I}_{PA^+(D, Sound)}$ over a denumerable domain D . Ipso facto $\mathcal{I}_{PA^+(D, Sound)}$ is also a sound interpretation of PA .

- (Downwards) Löwenheim-Skolem Theorem³³: If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.

Now, if we assume that Aristotle's particularisation is true then, by Lemma 1, the denumerable numerals of PA would correspond 1-1 to the denumerable members of the domain D under the interpretation $\mathcal{I}_{PA^+(D, Sound)}$.

This would imply that $[\neg(\forall x)R(x)]$ is true in D under $\mathcal{I}_{PA^+(D, Sound)}$ whilst, for any numeral $[n]$, the formula $[R(n)]$ always interprets as true in D under $\mathcal{I}_{PA^+(D, Sound)}$. Since there is a 1-1 correspondence between the PA^+ numerals and the members of the domain D , this contradicts the assumption that Aristotle's particularisation is true.

Hence Aristotle's particularisation is false if PA is consistent and ω -consistent. \square

Lemmas 2 and 3 immediately yield that:

Theorem 1 *If PA is consistent then Aristotle's particularisation is false.* \square

³⁰In [Go31].

³¹[Go31], eqn. (12), p.25.

³²[Go31], p.27.

³³[Lo15], p.245, Theorem 6; [Sk22], p.293; [Me64], Corollary 2.16, p.69.

2.1 Is PA consistent?

The question arises: Is PA consistent?

We note that the question is addressed by the following arguments:

- (i) PA is consistent since the standard interpretation of PA ³⁴ is intuitively acceptable as sound;

Under the standard interpretation of PA , an existentially quantified PA formula such as $'[(\exists x)P(x)]'$ is assumed³⁵ to always interpret as the arithmetical proposition 'There exists some natural number n such that $P^(n)$ holds in N '.*

- (ii) PA is consistent since transfinite induction in Gerhard Gentzen's proof³⁶ is intuitively acceptable as logically valid.

The accepted interpretation of the Axiom of Infinity of ZF appeals implicitly to Aristotle's particularisation. Transfinite induction appeals to the the accepted interpretation of the Axiom of Infinity for the existence of transfinite ordinals below ε_0 ³⁷.

- (iii) PA is consistent since it has a sound algorithmic interpretation over N ³⁸.

This is a finitary proof of consistency for PA that does not appeal to Aristotle's particularisation.

Although both (i) and (ii) are denied legitimacy by Theorem 1 since they appeal critically to Aristotle's particularisation, we conclude from (iii) and Theorem 1 that:

Theorem 2 *Aristotle's particularisation is false.* □

3 Conclusion

We have formally defined Aristotle's particularisation, which is the postulation that an existentially quantified formula of a first order language S , such as $'[(\exists x)P(x)]'$, can be assumed to always interpret as the proposition, 'There exists some s in the domain D of the interpretation such that $P^*(s)$ holds in D ', without inviting inconsistency.

We have then shown that:

- (a) If PA is consistent but not ω -consistent, then Aristotle's particularisation is false.

- (b) If PA is consistent and ω -consistent, then Aristotle's particularisation is false.

We concluded from the above and the finitary proof of consistency for PA ³⁹ that:

³⁴[Me64], p.107.

³⁵cf. [Me64], p.52(ii).

³⁶cf. [Me64], p.258.

³⁷cf. [Me64], p.270.

³⁸[An12].

³⁹In [An12].

(c) Aristotle's particularisation is false.

We leave for future consideration the query: Is PA ω -consistent?

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