On the scope of the completeness theorem for first-order predicate logic*

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1 Introduction

The problem of completeness of classical first-order predicate logic was formulated, for the first time in precise mathematical terms, in 1928 by Hilbert and Ackermann in [16] and solved positively by Gödel in his Ph.D. thesis one year later [8, 9]. In 1947 Henkin [14, 15] presented an alternative simpler proof that has become standard in logic textbooks. The main advantage of Henkin's proof is that, building on the Linbenbaum–Tarski method for propositional logic, it shows how to construct a model to invalidate a derivation in the calculus. It was adapted to other first-order systems based on non-classical logics with a corresponding algebraic semantics, as e.g. in the presentation of intuitionistic first-order predicate logic by Rasiowa and Sikorski [20] where, following the ideas of Mostowski [18], they used the order relation on Heyting algebras to interpret the existential (resp. universal) quantification of a formula as the supremum (resp. infimum) of the values of its instances. Another such case was the infinitary standard Łukasiewicz first-order logic axiomatized by Hay [13] in 1963, where its semantics based on the real unit interval $[0, 1]$ was perfectly suited for the interpretation of quantifiers and suprema and infima. Yet another example was provided by Horn [17] in 1969 with first-order superintuitionistic logic of linearly ordered Heyting algebras (built on the propositional calculus studied by Dummett [6] and inspired by some works of Gödel on intuitionistic logic [10], and for these reasons, usually called *Gödel–Dummett logic*). It was shown that this logic was also complete with respect to a standard $[0, 1]$-valued model.

It was hence natural to wonder what is the scope of the completeness theorem in the context of non-classical first-order logics. A crucial contribution in this matter was that of Rasiowa in her monograph [19] published in 1974 when she generalized her approach from intuitionism to a rather wide class of logics which she called *implicative logics* (logics with an implication connective satisfying identity, transitivity, *modus ponens*, congruence w.r.t. all other connectives, and weakening). These systems have an algebraic semantics ordered by the implication which, as in the case of intuitionism, does not always ensure the existence of suprema and infima for the interpretation of quantifiers. For this reason Rasiowa had to deal with the possibility of leaving some formulae with an undefined truth-value in some particular models, but nevertheless obtaining a completeness theorem with respect to those where all formulae can be interpreted.

In finitary *propositional* logics one can always refine the completeness theorem by restricting to subdirectly irreducible algebras.[1] This result is not preserved at the first-order level. Actually, an interesting difference between first-order classical and intuitionistic logics is that, while in the former one can re-

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[1]For logics whose semantics is not closed under quotients, one has to consider *relatively* sudirectly irreducible algebras.

strict the algebraic semantics from the class of all Boolean algebras to its (unique) subdirectly irreducible member $\boldsymbol{B}_2$ (the two-element Boolean algebra), subdirectly irreducible Heyting algebras do not provide a complete semantics for the latter. In the case of Gödel–Dummett logic, the aforementioned Heyting algebra defined on $[0,1]$ is actually *finitely* sudirectly irreducible. The same happens with the standard $[0,1]$-valued algebra for Łukasiewicz logic. In fact, (relatively) finitely sudirectly irreducible algebras provide a more meaningful semantics for some logics. For instance, in many so-called *fuzzy logics* (a rather big class of logics which has Gödel–Dummett and Łukasiewicz logics as prominent examples) these algebras are exactly those whose underlying order forms a chain, and are seen as an intended semantics for both propositional and first-order formalisms. For the latter, Rasiowa's axiomatization describing the (intuitionistic) behaviour of quantifiers is not enough and one needs an extra axiom, already used by Horn [17], for universal-quantifier shift over disjunctions:[2] $(\forall 3)\colon (\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi$, where x is not free in χ. This kind of axiomatization became standard for fuzzy logics when Petr Hájek started using it systematically for first-order systems in his book [11]. Following Rasiowa's footsteps, he also had to handle first-order structures not rich enough to interpret the values of all quantified formulae (he called *safe models* those with all necessary suprema and infima).[3] The direct predecessor of our work is the paper [12], where the proof of completeness (w.r.t. models over linearly ordered algebras) was not only generalized to arbitrary languages but performed uniformly for the class of *($\triangle$-)core fuzzy logics*, identifying the crucial rôle of disjunction (not only in the axiom $(\forall 3)$) and distinguishing two forms of *Henkin theories* in the process.

Our goal is to show that all the systematic approaches mentioned so far [11, 12, 19] assumed unnecessary conditions on propositional logics. We want to demonstrate that the Mostowski–Rasiowa–Sikorski–Hájek approach to first-order logics (as based on logics with an implication defining an order that allows to interpret quantifiers as suprema and infima) is a very useful one that can be stretched much further.

The first steps towards this goal were taken in [4] where we still assumed completeness w.r.t. linearly ordered algebras, a condition we drop in this presentation of our approach. Due to the space restrictions we will not present our approach in its full generality but we restrict ourselves to a certain natural class of logics. Namely, we study first-order logics over finitary algebraizable (in the sense of [2]) propositional logics with a very simple truth definition and suitable connectives of implication $\to$, constant $\overline{1}$, and disjunction $\vee$ (see the details in the next section). This framework is still wide enough to encompass most logics referred to in the literature as *substructural* and *fuzzy* logics. For each such propositional logic L we present the axiomatizations for its corresponding minimal first-order logic $\mathrm{L}\forall^{\mathrm{m}}$ and its extension $\mathrm{L}\forall$ and prove that the former is complete w.r.t. to all models (hence the name 'minimal' first-order logic), while the latter is complete w.r.t. models over relatively finitely subdirectly irreducible algebras.[4] The weakness of assumptions on the propositional side helps to illuminate the 'essentially first-order' steps in the classical Henkin proof. As a byproduct we also manage to extend the proof of completeness in order to obtain (a form of) Skolemization for the logics $\mathrm{L}\forall$ (thus vastly generalizing previous studies of Skolemization in non-classical logics, e.g. in [1]).

2 Setting the framework

This section presents the basic definitions and notational conventions for the paper (for further information on Algebraic Logic notions see [5, 7]). The definitions of a propositional language $\mathcal{L}$, the free term algebra $\boldsymbol{Fm}_{\mathcal{L}}$ over a denumerable set of generators (propositional variables), and finitary Hilbert-style proof systems are as usual. Let us introduce the notion of propositional logic that we use in this paper.

Convention 2.1. *Let $\mathcal{L}$ be a language containing at least a truth constant $\overline{1}$ and a binary connective $\to$, and let τ be a term in one variable. In this paper a propositional logic* L *in $\mathcal{L}$ is a finitary algebraically*

[2] Also known as the *axiom constant domains* of intuitionistic logic.

[3] The reader might wonder why Hájek did not restrict to semantics over completely ordered algebras (such as $[0,1]$-valued algebras) for first-order fuzzy logics, where the truth-values for quantified formulae would always be defined. The reason is that this would lead to non-axiomatizable logics as shown e.g. in [21].

[4] It is worth mentioning that the proof of completeness $\mathrm{L}\forall^{\mathrm{m}}$ works for all protoalgebraic logics (i.e. assuming neither finitarity nor algebraizability, with a general notion of implication as in [3], and with no need for $\overline{1}$ and disjunction); however the proof of completeness of $\mathrm{L}\forall$, does requires finitarity of L and finitely-defined implication and disjunction (though not necessarily by a single connective as we assume here; also algebraizability is not needed).

implicative logic with a truth definition given by the single equation $\tau(x) \approx \overline{1}$ (as studied in [3]). In more details, this means that L *is identified with the provability relation* $\vdash_{\mathrm{L}}$ *on* $\boldsymbol{Fm}_{\mathcal{L}}$ *given by a finitary Hilbert-style system such that:*[5]

$$\vdash_{\mathrm{L}} \varphi \to \varphi \qquad \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi \qquad \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi \qquad \varphi \dashv\vdash_{\mathrm{L}} \overline{1} \to \varphi \qquad \varphi \dashv\vdash_{\mathrm{L}} \tau(\varphi) \leftrightarrow \overline{1}$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \circ(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow \circ(\chi_1, \dots \chi_i, \psi, \dots, \chi_n) \quad \textit{for every } n\textit{-ary } \circ \in \mathcal{L} \textit{ and } i < n.$$

We recall now the basics of semantics. Let us fix from now on a logic L in a language $\mathcal{L}$. $\mathcal{L}$-algebras are algebras with signature $\mathcal{L}$; homomorphisms from $\boldsymbol{Fm}_{\mathcal{L}}$ to an $\mathcal{L}$-algebra $\boldsymbol{A}$ are called $\boldsymbol{A}$-evaluations.

Definition 2.2. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra, we define a relation $\leq_{\boldsymbol{A}}$ and a set $F_{\boldsymbol{A}}$ as*

$$x \leq_{\boldsymbol{A}} y \quad \textit{iff} \quad \tau^{\boldsymbol{A}}(x \to^{\boldsymbol{A}} y) = \overline{1}^{\boldsymbol{A}} \qquad F_{\boldsymbol{A}} = \{x \mid \tau^{\boldsymbol{A}}(x) = \overline{1}^{\boldsymbol{A}}\} = \{x \mid \overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} x\}.$$

$\boldsymbol{A}$ is an L*-algebra, in symbols: $\boldsymbol{A} \in \mathbb{L}$, if for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and $x, y \in A$ hold:*

1. *$\Gamma \vdash_{\mathrm{L}} \varphi$ implies that for each $\boldsymbol{A}$-evaluation e we have $e(\varphi) \in F_{\boldsymbol{A}}$ whenever $e[\Gamma] \subseteq F_{\boldsymbol{A}}$,*
2. *$x \leq_{\boldsymbol{A}} y$ and $y \leq_{\boldsymbol{A}} x$ implies $x = y$.*[6]

$\mathbb{L}$ is in fact a quasivariety and it is the equivalent algebraic semantics of L in the sense of [2]. A non-trivial L-algebra $\boldsymbol{A}$ is *(finitely) subdirectly irreducible relative to* $\mathbb{L}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{L}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. $\mathbb{L}_{\mathrm{R(F)SI}}$ denotes the class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{L}$. Of course $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{\mathrm{RFSI}}$.

Theorem 2.3. *Let* L *be a logic. Then* $\vdash_{\mathrm{L}} = \models_{\mathbb{L}_{\mathrm{RSI}}} = \models_{\mathbb{L}_{\mathrm{RFSI}}}$.

Definition 2.4. *A logic* L *is called* disjunctive *if it has a (primitive or definable) connective $\vee$, called a* disjunction*, such that $\varphi \vdash_{\mathrm{L}} \varphi \vee \psi$, $\psi \vdash_{\mathrm{L}} \varphi \vee \psi$ and it satisfies the* Proof by Cases Property *(*PCP *for short):*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$$

The notion of disjunction is intrinsic for a given logic, i.e., for any pair $\vee, \vee'$ of disjunctions in L we have $\varphi \vee \psi \dashv\vdash_{\mathrm{L}} \varphi \vee' \psi$. In many prominent cases, such as classical or intuitionistic logic, the lattice connective $\vee$ is itself a disjunction in the sense just defined. But this is not always the case (e.g. in linear logic $\vee$ is not a disjunction, but the logic is still disjunctive with the defined connective $(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$). In some other logics, like the full Lambek logic FL, one would need a higher level of complexity and take a connective defined by an infinite parameterized set of formulae (see [4]). Finally, we list a few properties of disjunctive logics that will be needed in the upcoming text.

Proposition 2.5. *Let* L *be a logic with a disjunction $\vee$ and $\boldsymbol{A}$ an* L*-algebra. Then:*[7]

- $(\mathrm{C}_\vee)$ $\varphi \vee \psi \vdash_{\mathrm{L}} \psi \vee \varphi$ $\qquad (\mathrm{I}_\vee)$ $\varphi \vee \varphi \vdash_{\mathrm{L}} \varphi$ $\qquad (\mathrm{A}_\vee)$ $\varphi \vee (\psi \vee \chi) \dashv\vdash_{\mathrm{L}} (\varphi \vee \psi) \vee \chi$.
- *$\Gamma \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ whenever $\Gamma \vdash_{\mathrm{L}} \varphi$.*
- *$\boldsymbol{A} \in \mathbb{L}_{\mathrm{RFSI}}$ iff for each $a, b \in A$ we have $a \in F_{\boldsymbol{A}}$ or $b \in F_{\boldsymbol{A}}$ whenever $a \vee^{\boldsymbol{A}} b \subseteq F_{\boldsymbol{A}}$.*

3 First-order logic

§3.1 Basic syntactic and semantic notions Let us fix a logic L in a propositional language $\mathcal{L}$. As usual, a *predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{ar}$ is a function assigning to each predicate and function symbol a natural number called the *arity* of the symbol. The functions f for which $\mathbf{ar}(f) = 0$ are called *object constants*.

[5] We write '$\varphi \leftrightarrow \psi$' for '$\{\varphi \to \psi, \psi \to \varphi\}$', '$T \vdash S$' for '$T \vdash \varphi$ for each $\varphi \in S$', and '$T \dashv\vdash S$' for '$T \vdash S$ and $S \vdash T$'.

[6] Note that in each L-algebra $\boldsymbol{A}$, $\leq_{\boldsymbol{A}}$ is an order and $F_{\boldsymbol{A}}$ is an upper set w.r.t. $\leq_{\boldsymbol{A}}$.

[7] Given sets $\Phi, \Psi \subseteq Fm_{\mathcal{L}}$, $\Phi \vee \Psi$ denotes the set $\{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\}$.

Let us further fix a predicate language $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ and a denumerable set V whose elements are called *object variables*. The sets of *$\mathcal{P}$-terms*, *atomic $\mathcal{P}$-formulae*, and *$\langle \mathcal{L}, \mathcal{P} \rangle$-formulae* are defined as in classical logic. We omit the symbols for propositional or predicate languages when clear from the context (analogously any other notion parameterized by propositional or predicate languages). The notions of bound and free variables, closed terms, sentences, and substitutability are also defined in the standard way. Instead of $\xi_1, \ldots, \xi_n$ (where ξ_i's are terms or formulae and n is arbitrary or fixed by the context) we shall sometimes write just $\vec{\xi}$. Unless stated otherwise, by the notation $\varphi(\vec{z})$ we signify that *all* free variables of φ are among those in the vector of pairwise different object variables $\vec{z}$. If $\varphi(x_1, \ldots, x_n, \vec{z})$ is a formula and we replace all free occurrences of x_i's in φ by terms t_i, we denote the resulting formula in the context simply by $\varphi(t_1, \ldots, t_n, \vec{z})$. A theory T is a pair $\langle \mathcal{P}, \Gamma \rangle$, where $\mathcal{P}$ is a predicate language and Γ is a set of $\mathcal{P}$-formulae. For convenience we sometimes identify the theory T and its set of formulae Γ and say that T is a $\mathcal{P}$-theory to indicate that its language is $\mathcal{P}$.

Definition 3.1 (Structure)**.** *A $\mathcal{P}$-*structure *$\mathfrak{S}$ is a pair $\langle \boldsymbol{A}, \mathbf{S} \rangle$ where $\boldsymbol{A} \in \mathbb{L}$ and $\mathbf{S} = \langle S, \langle P_{\mathbf{S}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{S}} \rangle_{f \in \mathbf{F}} \rangle$, where S is a non-empty domain; $P_{\mathbf{S}}$ is a function $S^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$; and $f_{\mathbf{S}}$ is a function $S^n \to S$ for each n-ary function symbol $f \in \mathbf{F}$.*

An $\mathfrak{S}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to S$; by $\mathrm{v}[x{\to}a]$ we denote the $\mathfrak{S}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$.

Definition 3.2 (Truth definition)**.** *Let $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ be a $\mathcal{P}$-structure and v an $\mathfrak{S}$-evaluation. We define the* values *of the terms and the* truth values *of the formulae in $\mathfrak{S}$ for an evaluation v as:*

$$
\begin{array}{rcll}
\|x\|^{\mathfrak{S}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
\|f(t_1, \ldots, t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & f_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } f \in \mathbf{F} \\
\|P(t_1, \ldots, t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & P_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } P \in \mathbf{P} \\
\|\circ(\varphi_1, \ldots, \varphi_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|\varphi_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \textit{for } \circ \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \inf_{\leq_A}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}, & \\
\|(\exists x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \sup_{\leq_A}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}. &
\end{array}
$$

If the infimum or supremum does not exist, we take the corresponding value as undefined. We say that $\mathfrak{S}$ is safe *iff $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula φ and each $\mathfrak{S}$-evaluation v. Finally, we write $\mathfrak{S} \models \varphi[\mathrm{v}]$ if $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}} \in F^{\boldsymbol{A}}$.*

Definition 3.3 (Model)**.** *Let T be a $\mathcal{P}$-theory and $\mathbb{K} \subseteq \mathbb{L}$. A $\mathcal{P}$-structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ is called a $\mathbb{K}$-model of T, denoted as $\mathfrak{M} \models T$, if it is* safe*, $\boldsymbol{A} \in \mathbb{K}$, and $\mathfrak{S} \models \varphi[\mathrm{v}]$ for each $\varphi \in T$ and each $\mathfrak{S}$-evaluation v.*

We speak of '$\boldsymbol{A}$-model' instead of '$\{\boldsymbol{A}\}$-model' and we also use this term for safe structures over $\boldsymbol{A}$; we also write $\mathfrak{M} \models \varphi$ instead of $\mathfrak{M} \models \{\varphi\}$. Notice that, since each theory comes with a fixed predicate language, we need not to specify the language of $\mathfrak{M}$ when we say that it is a model of a theory T. By a slight abuse of language we will use the term 'model' instead of 'safe $\mathcal{P}$-structure' when $\mathcal{P}$ is clear from the context.

Definition 3.4 (Consequence relation)**.** *Let $\mathbb{K} \subseteq \mathbb{L}$. A $\mathcal{P}$-formula φ is a semantical (sentential) consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$, in symbols $T \models_{\mathbb{K}} \varphi$, if for each $\mathbb{K}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.*

Note that both in the definition of model and semantical consequence, the language of the theory T plays a minor rôle; basically they could be formulated just for sets of formulae. Indeed we can prove that $\langle \mathcal{P}, \Gamma \rangle \models_{\mathbb{K}} \varphi$ iff $\langle \mathcal{P}', \Gamma \rangle \models_{\mathbb{K}} \varphi$ for all $\mathcal{P}' \supseteq \mathcal{P}$ iff $\langle \mathcal{P}', \Gamma \rangle \models_{\mathbb{K}} \varphi$ for some $\mathcal{P}' \supseteq \mathcal{P}$ (actually, due to the safeness restriction, this is not as trivial to prove as in classical predicate logic).

Let us give a series of examples. The first demonstrates the need for unit in the language for the validity of the well-known generalization rule; the other two show that in first-order logics, unlike in the propositional case (Theorem 2.3), the consequence relations $\models_{\mathbb{L}_{\mathrm{RFSI}}}$ and $\models_{\mathbb{L}}$ need not coincide.

Example 3.5. We show that for any L, $\varphi \models_{\mathbb{L}} (\forall x)\varphi$. Consider an $\boldsymbol{A}$-model $\mathfrak{M}$ of φ and an $\mathfrak{M}$-evaluation e. We know that $\overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} \|\varphi\|^{\mathfrak{M}}_{e[x\to a]}$ for each $a \in M$. Thus $\overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} \inf^{\boldsymbol{A}}\{\|\varphi\|^{\mathfrak{M}}_{e[x\to a]} \mid a \in M\}$.

Example 3.6. We show that for any disjunctive logic L, $\varphi \vee \psi \models_{\mathbb{L}_{\mathrm{RFSI}}} ((\forall x)\varphi) \vee \psi$ whenever x is not free in ψ. Consider an $\mathbb{L}_{\mathrm{RFSI}}$-model $\mathfrak{M}$ of $\varphi \vee \psi$ and an $\mathfrak{M}$-evaluation e. If $\mathfrak{M} \models \psi[e]$ we are done. Assume that $\mathfrak{M} \not\models \psi[e]$, then also $\mathfrak{M} \not\models \psi[e[x\to a]]$ for each $a \in M$ (because x is not free in ψ). Using the characterization of $\mathbb{L}_{\mathrm{RFSI}}$ from Proposition 2.5 we know that $\mathfrak{M} \models \varphi[e[x\to a]]$; the rest works as in Example 3.5.

Example 3.7. Let $\mathbb{HA}$ be the class of Heyting algebras and we show that $\varphi \vee \psi \not\models_{\mathbb{HA}} ((\forall x)\varphi) \vee \psi$. Consider $\varphi = P(x)$ for a unary predicate P and $\psi = c$ a truth constant. Take the lattice (actually a frame) whose domain is $\{0, \alpha\} \cup \{1/n \mid n \in \mathsf{N}\}$, elements different from α are ordered as usual, $0 \leq \alpha \leq 1$, and α is incomparable with all the other elements. Let $\boldsymbol{A}$ be the Heyting algebra over this lattice. Now take N as the domain of a first-order structure $\mathbf{S}$ and interpret $c_{\mathbf{S}} = \alpha$ and $P_{\mathbf{S}}(n) = 1/n$, and we have the desired counterexample.

§3.2 Axiomatic systems

The goal of this subsection is to propose axiomatizations for the two natural semantical consequence relations (shown to be different in the previous two examples) we have introduced and show their basic properties.

Definition 3.8. *Let* L *be a logic in $\mathcal{L}$ presented by an axiomatic system $\mathcal{AS}$. The* minimal predicate logic over L *(in a predicate language $\mathcal{P}$), denoted as* $\mathrm{L}\forall^{\mathrm{m}}$*, is given by the following axiomatic system:*[8]

- *(P)* *the axioms and rules resulting from those of $\mathcal{AS}$ by substituting variables by $\langle\mathcal{L},\mathcal{P}\rangle$-formulae*
- $(\forall 1)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x,\vec{z}) \to \varphi(t,\vec{z})$*, where t is substitutable for x in φ*
- $(\exists 1)$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi(t,\vec{z}) \to (\exists x)\varphi(x,\vec{z})$*, where t is substitutable for x in φ*
- $(\forall 2)$ $\chi \to \varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \chi \to (\forall x)\varphi$*, where x is not free in χ*
- $(\exists 2)$ $\varphi \to \chi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \to \chi$*, where x is not free in χ.*

If L *is disjunctive, we also define a stronger predicate logic over* L *(in a predicate language $\mathcal{P}$), denoted here as* $\mathrm{L}\forall$*,*[9] *as the extension of* $\mathrm{L}\forall^{\mathrm{m}}$ *by:*

- $(\forall 2)^{\vee}$ $(\chi \to \varphi) \vee \psi \vdash_{\mathrm{L}\forall} (\chi \to (\forall x)\varphi) \vee \psi$*, where x is neither free in χ nor in ψ*
- $(\exists 2)^{\vee}$ $(\varphi \to \chi) \vee \psi \vdash_{\mathrm{L}\forall} ((\exists x)\varphi \to \chi) \vee \psi$*, where x is neither free in χ nor in ψ.*

Let us list some theorems and derivable rules to demonstrate that the quantification theory is not too different from the classical one (we assume that x is not free in χ, and x' does not occur in $\varphi(x,\vec{z})$):

$(\forall 0)$	$\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$	$(\forall 0)^{\vee}$	$\varphi \vee \chi \vdash_{\mathrm{L}\forall} (\forall x)\varphi \vee \chi$
$(T1)$	$\varphi \to \psi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi \to (\forall x)\psi$	$(T2)$	$\varphi \to \psi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \to (\exists x)\psi$
$(T3)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \chi \leftrightarrow (\forall x)\chi$	$(T4)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \leftrightarrow \chi$
$(T5)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x,\vec{z}) \leftrightarrow (\forall x')\varphi(x',\vec{z})$	$(T6)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi(x,\vec{z}) \leftrightarrow (\exists x')\varphi(x',\vec{z})$
$(T7)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$	$(T8)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi.$

The following three theorems state crucial properties of our first-order logics. The first one is easily proved by induction using (P), $(T1)$, and $(T2)$. The second one is the usual Constants Theorem (almost proved as in classical logic) showing that free variables behave as constants naming arbitrary elements. The third one requires a more elaborated proof and shows the rôle of our notion of disjunction.

Theorem 3.9 (Congruence Property)**.** *Let φ, ψ, δ be sentences, χ a formula, and $\hat{\chi}$ a formula obtained from χ by replacing some occurrences φ by ψ. Then for $\vdash \in \{\vdash_{\mathrm{L}\forall^{\mathrm{m}}}, \vdash_{\mathrm{L}\forall}\}$:*

$$\vdash \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi. \qquad \varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \hat{\chi}.$$

Theorem 3.10 (Constants Theorem)**.** *Let $\vdash \in \{\vdash_{\mathrm{L}\forall^{\mathrm{m}}}, \vdash_{\mathrm{L}\forall}\}$, $T \cup \{\varphi(x,\vec{z})\}$ be a theory, and c a constant not occurring there. Then $\Sigma \vdash \varphi(c,\vec{z})$ iff $\Sigma \vdash \varphi(x,\vec{z})$*

[8] Note that we have omitted the propositional language $\mathcal{L}$ in the symbol $\mathrm{L}\forall^{\mathrm{m}}$ for it is always that of L. Omitting the symbol for the predicate language could be more confusing. In order to avoid possible problems, we first define the notion of proof from a $\mathcal{P}$-theory T in the (minimal) predicate logic over L in a predicate language $\mathcal{P}$ in the same way we did it in the propositional case, denoting it by means of $\vdash$. As in the semantical case, the language of the theory T plays a little rôle in the notion of the proof.

[9] Observe that there is no need to mention the used disjunction in the symbol for $\mathrm{L}\forall$ (as all disjunctions are mutually derivable) and that axioms $(\forall 2)$ and $(\exists 2)$ are redundant in the axiomatization of $\mathrm{L}\forall$.

Theorem 3.11 (Proof by Cases Property). *For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:*

$$\frac{T, \varphi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \vdash_{\mathrm{L}\forall} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}\forall} \chi} \tag{PCP}$$

Proof. First we prove that for each set of formulae $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash_{\mathrm{L}\forall} \varphi$ we have $\Gamma \vee \psi \vdash_{\mathrm{L}\forall} \varphi \vee \psi$ for any sentence ψ. We show $\Gamma \vee \psi \vdash_{\mathrm{L}\forall} \delta \vee \psi$ for each δ appearing in the proof of φ from Γ. If $\delta \in \Gamma$ or is an axiom, the proof is trivial. Now assume that $\Gamma' \vdash_{\mathrm{L}\forall} \delta$ is the rule used to obtain δ. By hypothesis $\Gamma \vee \psi \vdash_{\mathrm{L}\forall} \Gamma' \vee \psi$. Since $\Gamma' \vee \psi \vdash_{\mathrm{L}\forall} \delta \vee \psi$ (for (P) due to the second item of Proposition 2.5, for $(\forall 2)$ and $(\exists 2)$ it is due resp. to $(\forall 2)^{\vee}$ and $(\exists 2)^{\vee}$, and for the latter it is due to $(\mathrm{A}_\vee)$), the proof of this claim is done.

Now, from $T, \varphi \vdash_{\mathrm{L}\forall} \chi$ and $T, \psi \vdash_{\mathrm{L}\forall} \chi$, using the claim we have just proved, we obtain $T, \varphi \vee \psi \vdash_{\mathrm{L}\forall} \chi \vee \psi$ and $T, \psi \vee \chi \vdash_{\mathrm{L}\forall} \chi \vee \chi$. Using $(\mathrm{C}_\vee)$ and $(\mathrm{I}_\vee)$ completes the proof. □

§3.3 Completeness theorem In this subsection we show that the axiomatic systems $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ are respectively presentations of the semantically defined first-order logics $\models_{\mathbb{L}}$ and $\models_{\mathbb{L}_{\mathrm{RFSI}}}$. The proofs of soundness (i.e., $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbb{L}}$ and $\vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbb{L}_{\mathrm{RFSI}}}$) are easy. To prove the reverse inclusions we need the notions of prime and $\forall$-Henkin theory; unless said otherwise $\vdash$ stands for either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$.

Definition 3.12 (Prime and $\forall$-Henkin theories). *Let $\mathcal{P}$ be a predicate language. A $\mathcal{P}$-theory T is*

- Prime *if for each pair of $\mathcal{P}$-sentences φ, ψ we have $T \vdash_{\mathrm{L}\forall} \varphi$ or $T \vdash_{\mathrm{L}\forall} \psi$ whenever $T \vdash_{\mathrm{L}\forall} \varphi \vee \psi$.*
- $\forall$-Henkin *(in $\vdash$) if for each $\mathcal{P}$-formula ψ such that $T \nvdash (\forall x)\psi(x)$ there is a constant c in $\mathcal{P}$ such that $T \nvdash \psi(c)$.*

The next definition is sound thanks to the congruence property of $\leftrightarrow$ stated in Theorem 3.9.

Definition 3.13 (Lindenbaum–Tarski algebra). *Let φ be a $\mathcal{P}$-sentence and T a $\mathcal{P}$-theory. We define*

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

The Lindenbaum–Tarski algebra of T (in $\vdash$), denoted by $\mathbf{LindT}^{\vdash}_T$, has the domain $L_T = \{[\varphi]_T \mid \varphi \text{ a } \mathcal{P}\text{-sentence}\}$, and operations (for each n-ary connective c of L and each $\mathcal{P}$-sentences $\varphi_1, \dots, \varphi_n$):

$$\circ^{\mathbf{LindT}^{\vdash}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [\circ(\varphi_1, \dots, \varphi_n)]_T$$

Proposition 3.14. *Let T be a $\mathcal{P}$-theory. Then $\mathbf{LindT}^{\vdash}_T \in \mathbb{L}$ and $\mathbf{LindT}^{\vdash_{\mathrm{L}\forall}}_T \in \mathbb{L}_{\mathrm{RFSI}}$ if, and only if, T is prime.*

Proof. Observe that $[\varphi]_T \in F_{\mathbf{LindT}^{\vdash}_T}$ iff $\tau^{\mathbf{LindT}^{\vdash}_T}([\varphi]_T) = \overline{1}^{\mathbf{LindT}^{\vdash}_T}$ iff $[\tau(\varphi)]_T = [\overline{1}]_T$ iff $T \vdash \overline{1} \leftrightarrow \tau(\varphi)$ iff $T \vdash \varphi$. Thus $[\varphi]_T \leq_{\mathbf{LindT}^{\vdash}_T} [\psi]_T$ iff $[\varphi]_T \to^{\mathbf{LindT}^{\vdash}_T} [\psi]_T \in F_{\mathbf{LindT}^{\vdash}_T}$ iff $T \vdash \varphi \to \psi$.

Using the latter observation we obtain the second condition from Definition 2.2. Now we show the first one; assume that $\Gamma \vdash_{\mathrm{L}} \psi$ and let us fix a $\mathbf{LindT}^{\vdash}_T$-evaluation e such that $e[\Gamma] \subseteq F_{\mathbf{LindT}^{\vdash}_T}$. Let us inductively define a mapping σ from propositional formulae to $\langle \mathcal{L}, \mathcal{P}\rangle$-sentences: $\sigma(v) \in e(v)$ (arbitrarily for each propositional variable v) and $\sigma(\circ(\varphi_1, \dots, \varphi_n)) = \circ(\sigma\varphi_1, \dots, \sigma\varphi_n)$ for each n-ary connective $\circ$. Now we show by induction that for each propositional formula φ, $[\sigma\varphi]_T = e(\varphi)$. For variables it is clear; if $\circ$ is a connective, we have $[\sigma\circ(\varphi_1, \dots, \varphi_n)]_T = [\circ(\sigma\varphi_1, \dots, \sigma\varphi_n)]_T = \circ^{\mathbf{LindT}^{\vdash}_T}([\sigma\varphi_1]_T, \dots, [\sigma\varphi_n]_T) = \circ^{\mathbf{LindT}^{\vdash}_T}(e(\varphi_1), \dots, e(\varphi_n)) = e(\circ(\varphi_1, \dots, \varphi_n))$. Since $e[\Gamma] \subseteq F_{\mathbf{LindT}^{\vdash}_T}$, we have $T \vdash \sigma[\Gamma]$. From $\Gamma \vdash_{\mathrm{L}} \psi$ we obtain $\sigma[\Gamma] \vdash \sigma\psi$ (due to (P)). Taken together, we have $T \vdash \sigma\psi$ and so $e(\psi) = [\sigma(\psi)]_T \in F_{\mathbf{LindT}^{\vdash}_T}$.

The second part easily follows from the first observation, primality of T, and Proposition 2.5. □

Lemma 3.15. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory and $\mathbf{C}$ the set of all closed $\mathcal{P}$-terms. Then for any $\mathcal{P}$-formula φ with only one free variable x holds:*

$$[(\forall x)\varphi]_T = \inf_{\leq_{\mathbf{LindT}^{\vdash}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\} \qquad [(\exists x)\varphi]_T = \sup_{\leq_{\mathbf{LindT}^{\vdash}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\},$$

Proof. We prove only the first claim (the second one is completely analogous). Recall that $[\varphi]_T \leq_{\mathbf{LindT}_T^{\vdash}} [\psi]_T$ iff $T \vdash \varphi \to \psi$. From this and ($\forall$1) we obtain that $[(\forall x)\varphi]_T$ is a lower bound.

Assume that $[\chi]_T \not\leq_{\mathbf{LindT}_T^{\vdash}} [(\forall x)\varphi]_T$. Without loss of generality we assume that x is not free in χ (because we know that $[(\forall x)\varphi]_T = [(\forall y)\varphi]_T$ if y does not occur in $\varphi(x)$). Thus $T \nvdash \chi \to (\forall x)\varphi$ and so $T \nvdash \chi \to \varphi(x)$ (by rule ($\forall$2)) and $T \nvdash (\forall x)(\chi \to \varphi(x))$ (by rule ($\forall$0)). By the $\forall$-Henkin property of T we obtain a constant $d \in \mathbf{C}$ such that $T \nvdash \chi \to \varphi(d)$. Thus finally $[\chi]_T \not\leq_{\mathbf{LindT}_T^{\vdash}} [\varphi(d)]_T$, i.e. $[\chi]_T$ is not a lower bound of $\{[\varphi(c)]_T \mid c \in \mathbf{C}\}$. □

Definition 3.16 (Canonical model)**.** *Let T be a $\forall$-Henkin $\mathcal{P}$-theory. The* canonical model *of T (in $\vdash$), denoted by $\mathfrak{CM}_T^{\vdash}$, is the $\mathcal{P}$-structure $\langle \mathbf{LindT}_T^{\vdash}, \mathbf{S}\rangle$ where the domain of $\mathbf{S}$ consists of the closed $\mathcal{P}$-terms,*

- $f_{\mathbf{S}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ *for each n-ary function symbol $f \in \mathcal{P}$, and*
- $P_{\mathbf{S}}(t_1, \dots, t_n) = [P(t_1, \dots, t_n)]_T$ *for each n-ary predicate symbol $P \in \mathcal{P}$.*

Now we can easily prove the following proposition which shows that $\mathfrak{CM}_T^{\vdash}$ is indeed a $\mathcal{P}$-model of T:

Proposition 3.17. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T^{\vdash}} = [\varphi]_T$ and so $\mathfrak{CM}_T^{\vdash} \models \varphi$ if, and only if, $T \vdash \varphi$.*

The following result, actually a first-order version of Lindenbaum lemma, gives the final ingredient to obtain completeness. For now, we give the proof for $\mathrm{L}\forall^{\mathrm{m}}$ only. The case of $\mathrm{L}\forall$ is more involved and it will follow from Theorem 3.23.

Theorem 3.18. *Let $\mathcal{P}$ be a predicate language and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$ (or $T \nvdash_{\mathrm{L}\forall} \varphi$ resp.). Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a (prime) $\forall$-Henkin $\mathcal{P}'$-theory in* $\mathrm{L}\forall^{\mathrm{m}}$ *(in* $\mathrm{L}\forall$ *resp.) $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$ ($T' \nvdash_{\mathrm{L}\forall} \varphi$ resp.).*

Proof. Let $\mathcal{P}'$ be an expansion of $\mathcal{P}$ by countably many new object constants, and take $T' = \langle \mathcal{P}', T\rangle$. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\psi(x)$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x)$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(c)$ for some c not occurring in $T' \cup \{\psi\}$ (since T' contains just $\mathcal{P}$-formulae and ψ is a finite object, there is always such $c \in \mathcal{P}'$ and so we can use Constants Theorem). □

Theorem 3.19 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$)**.** *Let* L *be a logic and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then*

$$T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi \qquad\qquad \textit{and} \qquad\qquad T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}_{\mathrm{RFSI}}} \varphi.$$

A natural question is whether one could obtain axiomatizations and corresponding completeness theorems for other meaningful semantics. Certainly, the class of models over completely ordered algebras is very a very natural one, because there one needs not bother with safe models, but unfortunately, as mentioned before (Footnote 3), the corresponding logics may fail to be recursively enumerable. However, in some cases it is possible to obtain such result by *regular completion* methods, i.e. by finding embeddings of the algebras into completely ordered ones (respecting infinite suprema and infima).

§3.4 $\exists$-Henkin theories and Skolemization Next we show that logics $\mathrm{L}\forall$ admit a form of Skolemization (the one which allows to erase existential quantifiers in a formula by conservatively introducing new functional symbols) restricted to a certain class Σ of formulae (of arbitrary language). However, we need to assume that Σ is *term-closed* (i.e. for each $\varphi(x, \vec{y}) \in \Sigma$, each language $\mathcal{P}$, and each sequence of closed $\mathcal{P}$-terms $\vec{t}$, we have $\varphi(x, \vec{t}) \in \Sigma$) and the logic $\mathrm{L}\forall$ enjoys “Skolemization for constants”, formally defined as:[10]

Definition 3.20. *We say that* $\mathrm{L}\forall$ *is Σ-preSkolem if $T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ and any constant $c \notin \mathcal{P}$.*

For example, every logic is trivially $\emptyset$-preSkolem; intuitionistic and most substructural logics are Σ-preSkolem for Σ being the class of all formulae; some other logics (e.g. fuzzy logics expanded by the Monteiro–Baaz Δ connective) are Σ-preSkolem for Σ being the class of provably classical formulae.

[10] In this subsection we are only working in $\mathrm{L}\forall$ and so $\vdash$ always stands for $\vdash_{\mathrm{L}\forall}$.

Now we are almost ready to prove the fundamental lemma, but first we observe why it needs to be formulated in such complex fashion. In the process of extending a theory T into a $\forall$-Henkin extension T' we obtain formula φ unprovable in T we want to keep unprovable in T'. In classical logic we just add $\neg\varphi$ to T and proceed from there. In our non-classical setting the situation is not that simple and so we need to "store" the formulae we want to keep unprovable in a special set Ψ. A set of $\mathcal{P}$-formulae Ψ is *deductively directed* if for each $\varphi, \psi \in \Psi$ there is $\delta \in \Psi$ (called an *upper bound* of φ and ψ) such that $\varphi \vdash \delta$ and $\psi \vdash \delta$. We write $T \nVdash \Psi$ whenever $T \nvdash \psi$ for each $\psi \in \Psi$.

Definition 3.21. *Let $\mathcal{P} \subseteq \mathcal{P}'$ be predicate languages. We say that a $\mathcal{P}'$-theory T is:*

- $\mathcal{P}$-$\forall$-Henkin *if for each $\mathcal{P}$-formula $\varphi(x)$ such that $T \nvdash (\forall x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \nvdash \varphi(c)$.*[11]
- Σ-$\mathcal{P}$-$\exists$-Henkin *if for each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ such that $T \vdash (\exists x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \vdash \varphi(c)$.*
- Σ-Henkin *if it is $\mathcal{P}'$-$\forall$-Henkin and Σ-$\mathcal{P}'$-$\exists$-Henkin.*

Lemma 3.22 (Fundamental Lemma)**.** *Let T be a $\mathcal{P}$-theory and Ψ a deductively directed set of $\mathcal{P}$-sentences such that $T \nVdash \Psi$. Then the following hold:*

1. *There exist $\mathcal{P}' \supseteq \mathcal{P}$, a $\mathcal{P}'$-theory $T' \supseteq T$, and a deductively directed set of $\mathcal{P}'$-sentences $\Psi' \supseteq \Psi$, such that $T' \nVdash \Psi'$ and each theory $S \supseteq T'$ in arbitrary language is $\mathcal{P}$-$\forall$-Henkin whenever $S \nVdash \Psi'$.*
2. *If* $\mathrm{L}\forall$ *is Σ-preSkolem, then there exist $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nVdash \Psi$ and each theory $S \supseteq T'$ in arbitrary language is Σ-$\mathcal{P}$-$\exists$-Henkin whenever $S \nVdash \Psi$.*
3. *There is a prime $\mathcal{P}$-theory $T' \supseteq T$ such that $T' \nVdash \Psi$.*

Proof. 1. We construct the extensions by transfinite recursion. Let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by new constants $\{c_\nu \mid \nu < \|\mathcal{P}\|\}$. We enumerate all $\mathcal{P}$-formulae with one free variable by ordinals as χ_μ for $\mu < \|\mathcal{P}\|$, we construct $\mathcal{P}'$-theories T_μ and sets of $\mathcal{P}'$-sentences Ψ_μ such that $T_\mu \subseteq T_\nu$ and $\Psi_\mu \subseteq \Psi_\nu$ for each $\mu \leq \nu$, $T_\mu \nVdash \Psi_\mu$, and Ψ_μ is deductively directed. Take $T_0 = T$ and $\Psi_0 = \Psi$, which fulfil our conditions.

For each $\mu \leq \|\mathcal{P}\|$ we define: $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$ and $\Psi_{<\mu} = \bigcup_{\nu<\mu} \Psi_\nu$. Notice that, by the induction assumption, we have $T_{<\mu} \nVdash \Psi_{<\mu}$ (by finitarity) and $\Psi_{<\mu}$ is deductively directed. We distinguish two possibilities:

(H1) If $T_{<\mu} \vdash \varphi \vee (\forall x)\chi_\mu(x)$ for some $\varphi \in \Psi_{<\mu}$, then we define $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\}$ and $\Psi_\mu = \Psi_{<\mu}$.

(H2) Otherwise we define $T_\mu = T_{<\mu}$ and $\Psi_\mu = \Psi_{<\mu} \cup (\Psi_{<\mu} \vee \chi_\mu(c_\mu))$.

We show that our conditions are met no matter which possibility occurred.

(H1) Ψ_μ is obviously deductively directed. Assume, for a contradiction, that $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\} \vdash \psi$ for some $\psi \in \Psi_\mu$. Take an upper bound δ of φ and ψ and notice that $T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\} \vdash \delta$ and $T_{<\mu} \cup \{\varphi\} \vdash \delta$. Thus by Theorem 3.11 we obtain $T_{<\mu} \cup \{\varphi \vee (\forall x)\chi_\mu(x)\} \vdash \delta$ and so $T_{<\mu} \vdash \delta$. Since $\delta \in \Psi_{<\mu}$ we have a contradiction with $T_{<\mu} \nVdash \Psi_{<\mu}$.

(H2) Assume that $T_\mu = T_{<\mu} \vdash \varphi$ for some $\varphi \in \Psi_\mu$. From the induction assumption we know that $T_{<\mu} \nvdash \varphi$ for each $\varphi \in \Psi_{<\mu}$ and so φ has to be of the form $\psi \vee \chi_\mu(c_\mu)$ for some $\psi \in \Psi_{<\mu}$. Since c_μ does not appear in $T_{<\mu} \cup \Psi_{<\mu}$, we can use Constants Theorem to obtain $T_\mu \vdash \psi \vee \chi_\mu(x)$, and, by $(\forall 0)^\vee$, $T_\mu \vdash \psi \vee (\forall x)\chi_\mu(x)$, a contradiction with the fact that we are in the case (H2). To show that Ψ_μ is deductively directed we distinguish four cases: first if both $\varphi, \psi \in \Psi_{<\mu}$ then they have an upper bound already in $\Psi_{<\mu}$. Second assume that $\varphi \in \Psi_{<\mu}$ and $\psi = \chi \vee \chi_\mu(c_\mu)$ for some $\chi \in \Psi_{<\mu}$. Let $\delta \in \Psi_{<\mu}$ be the upper bound of φ and χ. Thus $\delta \vee \chi_\mu(c_\mu) \in \Psi_\mu$ is an upper bound of φ (trivially) and ψ (by the PCP and the trivial fact that $\chi_\mu(c_\mu) \vdash \chi \vee \chi_\mu(c_\mu)$). The final two cases are analogous.

[11] Notice that when $\mathcal{P}' = \mathcal{P}$ we obtain the already defined (without the prefix '$\mathcal{P}$') notion of $\forall$-Henkin theory.

Now take $T' = T_{<\|\mathcal{P}\|}$ and $\Psi' = \Psi_{<\|\mathcal{P}\|}$. Thus by the induction assumption $T' \nVdash \Psi'$. Let now S be any theory such that $T' \subseteq S$ and $S \nVdash \Psi'$. We show that S is $\mathcal{P}$-$\forall$-Henkin. Clearly for each $\mu < \|\mathcal{P}\|$ if $S \nvdash (\forall x)\chi_\mu(x)$, then we must have used case (H2) (otherwise $T_\mu \vdash (\forall x)\chi_\mu(x)$ and so $S \vdash (\forall x)\chi_\mu(x)$). If $S \vdash \chi_\mu(c_\mu)$, then $S \vdash \varphi \vee \chi_\mu(c_\mu)$ for any $\varphi \in \Psi_{<\mu}$. Since we have used case (H2), we know that $\varphi \vee \chi_\mu(c_\mu) \in \Psi_\mu$—a contradiction with $S \nVdash \Psi'$.

2. We proceed by transfinite recursion as in 1. Let $\bar{\Sigma}$ be the set of all $\mathcal{P}$-formulae of the form $\varphi(x) \in \Sigma$. We expand $\mathcal{P}$ with new constants $\{c_\nu \mid \nu < \|\bar{\Sigma}\|\}$ and enumerate all formulae from $\bar{\Sigma}$ by ordinals as $\chi_\mu(x)$.

We construct theories T_μ such that $T_\mu \subseteq T_\nu$ for $\mu \leq \nu$ and $T_\mu \nVdash \Psi$. Let $T_0 = T$ and observe that it fulfils our condition. For each μ we define the set $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$. Notice that from the induction assumption and finitarity we obtain that $T_{<\mu} \nVdash \Psi$. We distinguish two possibilities:

(W1) If $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \nVdash \Psi$, we define $T_\mu = T_{<\mu} \cup \{\chi_\mu(c_\mu)\}$.

(W2) Otherwise we define $T_\mu = T_{<\mu}$.

In the case (W1) we use the fact that $T_{<\mu} \cup \{\chi_\mu(c_\mu)\}$ is a conservative expansion of $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\}$ (because L$\forall$ is Σ-preSkolem) to obtain $T_\mu \nVdash \Psi$. In the case (W2) we obtain it trivially.

Take $T' = T_{<\|\bar{\Sigma}\|}$ and observe that $T' \nVdash \Psi$. Let S be an arbitrary theory such that $T' \subseteq S$ and $S \nVdash \Psi$. We show that S is Σ-$\mathcal{P}$-$\exists$-Henkin. If $S \vdash (\exists x)\chi_\mu(x)$ then we used case (W1) (from $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \vdash \varphi$ for some $\varphi \in \Psi$ we would obtain $S \vdash \varphi$, a contradiction). Thus $T_\mu \vdash \chi_\mu(c_\mu)$ and so $S \vdash \chi_\mu(c_\mu)$.

3. We say that T is maximally consistent w.r.t. Ψ if $T \nVdash \Psi$ and for each $\varphi \notin T$ there is $\chi \in \Psi$ such that $T, \varphi \vdash \chi$. By Zorn's Lemma be obtain a theory $T' \supseteq T$ which is maximally consistent w.r.t. Ψ. Let us check that T' is prime. Assume that $\varphi \notin T'$ and $\psi \notin T'$. Thus there are $\chi_\varphi, \chi_\psi \in \Psi$ such that $T', \varphi \vdash \chi_\varphi$ and $T', \psi \vdash \chi_\psi$; take an upper bound δ of χ_φ and χ_ψ and using the PCP we obtain that $T', \varphi \vee \psi \vdash \delta$ and so $T' \nvdash \varphi \vee \psi$. □

Besides proving the Skolemization, the next theorem serves another purpose: as any logic is $\emptyset$-preSkolem and $\emptyset$-Henkin theories are just $\forall$-Henkin, it yields the promised proof of the second part of Theorem 3.18.

Theorem 3.23. *Let Σ be a term-closed class of formulae. Then the following are equivalent:*

1. L$\forall$ *is Σ-preSkolem.*

2. *For each $\mathcal{P}$-theory $T \cup \{\varphi\}$ such that $T \nvdash \varphi$ there is $\mathcal{P}' \supseteq \mathcal{P}$ and a prime Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash \varphi$.*

3. *$T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ is a conservative expansion of $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x, \vec{y}) \in \Sigma$ and any functional symbol $f_\varphi \notin \mathcal{P}$ of the proper arity.*

Proof. 3 implies 1 is straightforward. We show first that 1 implies 2. Assume that $T \nvdash \varphi$ for some $\mathcal{P}$-formulae $T \cup \{\varphi\}$. We proceed by induction over N. Take $T_0 = T$ and $\Psi_0 = \{\varphi\}$, $\mathcal{P}_0 = \mathcal{P}$. We construct predicate languages $\mathcal{P}_i$, $\mathcal{P}_i$-theories T_i, and directed sets Ψ_i of $\mathcal{P}_i$-sentences such that $T_i \nVdash \Psi_i$ and $\mathcal{P}_i \subseteq \mathcal{P}_j$, $T_i \subseteq T_j$, and $\Psi_i \subseteq \Psi_j$ for $i \leq j$. Observe that the theory T_0, the set Ψ_0 and the language $\mathcal{P}_0$ fulfil these conditions. The induction step:

- If i is odd: use part 1 of Lemma 3.22 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ'_i.

- If i is even: use part 2 of Lemma 3.22 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ_i.

Now we define $\mathcal{P}' = \bigcup\{\mathcal{P}_i \mid i \in \mathsf{N}\}$, $\hat{T} = \bigcup\{T_i \mid i \in \mathsf{N}\}$, and $\Psi' = \bigcup\{\Psi_i \mid i \in \mathsf{N}\}$. Finally, we use part 3 of Lemma 3.22 for $\mathcal{P}'$, $\hat{T}$, and Ψ' and define T' as $\hat{T}'$.

Obviously T' is prime, $T_i \subseteq T'$, and $T' \nVdash \Psi_i$ for each i. Thus from parts 1 and 2 of Lemma 3.22 and the definition of $\mathcal{P}'$ we obtain that T' is Σ-Henkin.

Next we prove that the second claim implies the third one. We denote $T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ as T_1 and $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ as T_2. We show that $T_2 \nvdash \chi$ implies $T_1 \nvdash \chi$ for each formula χ. We know that there is $\mathcal{P}' \supseteq \mathcal{P}$ and a Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T_2$ such that $T' \nvdash \chi$, and hence $\mathfrak{CM}_{T'}^{\vdash} \not\models \chi$. For each sequence

$\vec{t}$ of closed $\mathcal{P}'$-terms $T' \vdash (\exists x)\varphi(x, \vec{t})$ (by ($\forall$1)) and hence there is a $\mathcal{P}'$-constant $c_{\vec{t}}$ such that $T' \vdash \varphi(c_{\vec{t}}, \vec{t})$ (we know that $\varphi(x, \vec{t}) \in \Sigma$ because Σ is term-closed). Since $c_{\vec{t}}$ is an element of the domain of $\mathfrak{CM}'^{\vdash}_{T}$, we can define a model $\mathfrak{M}$ by expanding $\mathfrak{CM}'^{\vdash}_{T}$ with one functional symbol defined as: $(f_\varphi)_{\mathfrak{M}}(\vec{t}) = c_{\vec{t}}$. Since, for each $\mathcal{P}'$-formula, obviously, $\mathfrak{M} \models \psi$ iff $\mathfrak{CM}'^{\vdash}_{T} \models \psi$, we obtain: $\mathfrak{M}$ is a model of T and $\mathfrak{M} \not\models \chi$. Also clearly $\mathfrak{M} \models (\forall y)\varphi(f_\varphi(\vec{y}), \vec{y})$, and thus the proof is done.

□

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