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Actualizing Dugundji's Theorem

Abstract

In 1940 Dugundji proved that no system between **S1** and **S5** can be characterized by finite matrices. Dugundji's result forced the development of alternative semantics, in particular Kripke's relational semantic. The success of this semantics allowed the creation of a huge family of modal systems. With few adaptations, this semantics can characterize almost the totality of the modal systems developed in the last five decades.

This semantics however has some limits. Two results of incompleteness (for the systems **KH** and **KVB**)[1] showed that not every modal logic can be characterized by Kripke semantics. Besides, the creation of non-classical modal logics puts the problem of characterization of finite matrices very far away from the original scope of Dugundji's result.

In this sense, we will show how to actualize Dugundji's result in order to precise the scope and the limits of many-valued matrices as semantic of modal systems.

1 Introduction

The birth of symbolic modal logic seems to have a date: in general it is postulated[2] that Lewis inaugurated in 1918 this large family of logics. Aiming to create a new implication, the *strict implication*, the author proposes a hierarchy of five systems: **S1**-**S5**.

Shortly thereafter, in 1920, Łukasiewickz presents a set of matrices for a 3-valued logic $\mathbf{Ł}_3$ in order to modelize the new modal concept of *possibly* true.[3]. Thus, a question arises: is it possible that many-valued matrices characterize the systems **S1**-**S5**?

This question was resolved by Dugundji twenty years later: it was shown that not only Łukasiewickz's matrices but no finite matrix can be a complete semantic for any system between **S1** and **S5**.

[1]Those results can be found in [2] and [6].
[2]According to [1], p. 38
[3]See [13].

Although the theorem of Dugundji seemed to have given the final strike to any modal logic characterized by finite matrices, it is not clear that the argument holds for modal logics of which its propositional fragment is not classical (such as implicative, positive, paraconsistent or paracomplete modal logics).

Among all these fragments, perhaps one of the most interesting of them was proposed by Henkin[4] in 1949. Henkin's system has only the implication $\supset$ as operator, which preserves convenient properties such as the Deduction Metatheorem. We will see that the most part of modal systems whose propositional fragment is between Henkin's system and Propositional Classical Logic cannot be characterized by finite matrices.

There is also a large list of important modal systems based on Propositional Classical Logic that are also outside the scope of Dugundji's result. Among them it is worth mentioning: **K**, **D**, **T**, **B**, **KGL**, **KVB**, **KH**, **S0.5**, and others.

What we demonstrate here[5] is that the original result of Dugundji can be extended in two different senses: by embracing many modal systems developed from forties until today, on the one hand, and by considering some modal logics whose propositional fragment is not classical, on the other.

2 Axiomatics

Consider the following schemes of axioms and rules of inference, where α and β are variables ranging over formulas:

(A1) $\alpha \supset (\beta \supset \alpha)$

(A2) $(\alpha \supset \beta) \supset ((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \gamma))$

(A3) $(\alpha \supset \gamma) \supset (((\alpha \supset \beta) \supset \gamma) \supset \gamma)$

(A4) $\alpha \supset (\beta \supset (\alpha \wedge \beta))$

(A5) $(\alpha \wedge \beta) \supset \alpha$

(A6) $(\alpha \wedge \beta) \supset \beta$

(A7) $(\neg\alpha \supset \neg\beta) \supset ((\neg\alpha \supset \beta) \supset \alpha)$

(K) $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$

[4]In [10]

[5]Although it was considered the original Dugundji's article [8], we preferred to follow the clearer proof of it given in [5].

(T) $\Box\alpha \supset \alpha$

(4) $\Box\alpha \supset \Box\Box\alpha$

(5) $\Diamond\alpha \supset \Box\Diamond\alpha$

(GL) $\Box(\Box \supset \alpha) \supset \Box\alpha$

(Lem$_0$) $\Box((\alpha \wedge \Box\alpha) \supset \beta) \vee \Box((\beta \wedge \Box\beta) \supset \alpha)$

(MP) if $\vdash \alpha$ and $\vdash \alpha \supset \beta$ then $\vdash \beta$

(N) if $\vdash \alpha$ then $\vdash \Box\alpha$

(N') if $\vdash \alpha$ and α is a $\mathbf{PC}^{\supset}$-tautology, then $\vdash \Box\alpha$

(N*) if $\vdash \alpha \supset \beta$ then $\vdash \Box\alpha \supset \Box\beta$

Definition 2.1

(i) $\mathbf{PC}^{\supset} = \{(A1), (A2), (A3), (MP)\}$

(i) $\mathbf{PC}^{\supset,\wedge} = \mathbf{PC}^{\supset} \cup \{(A4),(A5),(A6)\}$

(iii) $\mathbf{PC} = \mathbf{PC}^{\supset} \cup \{(A7)\}$

(iv) $\mathbf{S0.5}^{0,\supset} = \mathbf{PC}^{\supset} \cup \{(K),(N')\}$

(v) $\mathbf{C2}^{0,\supset} = \mathbf{PC}^{\supset} \cup \{(K),(N^*)\}$[6]

(vi) $\mathbf{K}^{\supset} = \mathbf{PC}^{\supset} \cup \{(K),(N)\}$

(vii) $\mathbf{K}^{\supset,\wedge} = \mathbf{K}^{\supset} \cup \{(A3)\}$

(viii) $\mathbf{K4.3W} = \mathbf{K} \cup \{(4), (GL), (Lem_0)\}$

(ix) $\mathbf{S5} = \mathbf{PC} \cup \{(K),(T),(4),(5),(N)\}$

Consider, now, the following definition:

$$\alpha \vee \beta \equiv_{\mathbf{PC}^{\supset}} (\alpha \supset \beta) \supset \beta$$

[6]The systems **S0.5** and **C2** were considered by Lemmon in [12] and [11] as being modally minimal. $\mathbf{S0.5}^0$ is obtained by removing the axiom **(T)** from **S0.5**, check [7] p. 207. Following the same notation, it will be used $\mathbf{C2}^0$ to mean the exclusion of **(T)** in **C2**. As the reader may have noticed, the notation $^{\supset}$ and $^{\supset,\wedge}$ means that we are in the propositional context of $\mathbf{PC}^{\supset}$ and $\mathbf{PC}^{\supset,\wedge}$, respectively.

Theorem 2.2

(i) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset \alpha$

(ii) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset (\alpha \vee \beta)$

(iii) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset (\beta \vee \alpha)$

(iv) $\vdash_{\mathbf{PC}^{\supset,\wedge}} \alpha \supset (\alpha \wedge \alpha)$

Proof: See [4], p. 27-28 and p. 30-31. ■

Will will prove that any modal logic between $\mathbf{S0.5}^0$ and $\mathbf{S5}$ or between $\mathbf{C2}^0$ and $\mathbf{S5}$ whose propositional fragment is between $\mathbf{PC}^{\supset}$ and $\mathbf{PC}$ cannot be characterized by finite matrices. Then, we will show the same for systems between $\mathbf{K}$ and $\mathbf{K4.3W}$ whose propositional fragment is between $\mathbf{PC}^{\supset,\wedge}$ and $\mathbf{PC}$.

3 Generalizing Dugundji's Theorem

Definition 3.1 *A matrix $\mathcal{M}$ is a triple $\mathcal{M} = \langle M, D, O\rangle$ in which:*

(i) $M \neq \emptyset$

(ii) $D \subseteq M$ is a set of distinguished values

(iii) O is a set of operations over M □

Definition 3.2 *A matrix $\mathcal{M}$ characterizes a logical system* $\mathbf{S}$ *if all theorems of* $\mathbf{S}$ *and only them receive distinguished values when* $\mathbf{S}$ *in interpreted in $\mathcal{M}$. A matrix $\mathcal{M}$ is a model of a logical system* $\mathbf{S}$ *if all theorems of* $\mathbf{S}$ *(but not necessarily only them) receive distinguished values when* $\mathbf{S}$ *in interpreted in $\mathcal{M}$* □

Definition 3.3 *For each natural number n, the adapted Dugundji's formula D'_n is defined in the following way:*

$$D'_n \equiv_{def} \bigvee_{i \neq j} (p_i \asymp p_j)$$

in which $1 \leq i, j \leq n+1$ and $p_i \asymp p_j$ means $\Box(p_i \supset p_j) \vee \Box(p_j \supset p_i)$. □

Proposition 3.4 *Any finite matrix with n truth-values that is a model of* $\mathbf{S0.5}^{0,\supset}$ *validates D'_n.*

Proof:

Suppose that there is $\mathcal{M}$ with n truth-values that characterizes $\mathbf{S0.5}^{0,\supset}$, and let v be a valuation over $\mathcal{M}$. Since in D'_n we have n truth-values for $n+1$ variables, there will be $i \neq j$ such that the values assigned by v to p_i and p_j coincide. Then, the values assigned by v to $(p_i \asymp p_i)$ and $(p_i \asymp p_j)$ coincide. But, by **Theorem** 2.2 (i) and (N') we have that $\Box(p_i \supset p_i)$ is a theorem of $\mathbf{S0.5}^{0,\supset}$ and by **Theorem** 2.2 (ii), $(p_i \asymp p_i)$ is also a theorem, and so it is valid in $\mathcal{M}$. Then, the value assigned by v to $(p_i \asymp p_i)$ (and so, to $(p_i \asymp p_j)$) is distinguished. Therefore, by **Theorem** 2.2 (ii) the values assigned by v to $(p_i \asymp p_j) \vee \alpha$ and $\alpha \vee (p_i \asymp p_j)$ are distinguished, for every α, and so the valued assigned by v to the formula D'_n is distinguished. This shows that matrix $\mathcal{M}$ validates the formula D'_n. ■

Proposition 3.5 *There is a infinite matrix $\mathcal{M}_\infty$ that is a model of* **S5***.*

Proof: Define the following matrix $\mathcal{M}_\infty$

- $M = \wp(\mathbb{N})$, that is, the powerset of the set of natural numbers;
- $D = \{\mathbb{N}\}$;
- $O = \{\cup, \cap, -, \triangledown, \vartriangle\}$, in which $\cup, \cap$ and $-$ are the usual set-theoretic operations, while $\triangledown$ and $\vartriangle$ are defined in the following way:

$$\vartriangle X = \begin{cases} \mathbb{N} & \text{if } X = \mathbb{N} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\triangledown X = \begin{cases} \mathbb{N} & \text{if } X \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Consider valuations v over M as functions that assign to each formula of **S5** an element of $\wp(\mathbb{N})$ in the following way (using the notation $\overline{X}$ for $\mathbb{N} \setminus X$):

- $v(\neg\alpha) = \overline{v(\alpha)}$
- $v(\alpha \supset \beta) = \overline{v(\alpha)} \cup v(\beta)$
- $v(\Box\alpha) = \vartriangle(v(\alpha))$
- $v(\Diamond\alpha) = \triangledown(v(\alpha))$

It is enough to prove that all the theorems of **S5** receive from any valuation v the distinguished value $\mathbb{N}$, and the inference rules preserve this value. For the axioms of **PC** and **(MP)**, it is a consequence of the algebraic completeness of **PC**. For the modal axioms:

(K) $v(\Box(p \to q) \to (\Box p \to \Box q)) = \overline{\triangle(\overline{v(p)} \cup v(q))} \cup \overline{\triangle v(p)} \cup \triangle v(q)$

- if $v(p) \neq \mathbb{N}$, then $\overline{\triangle v(p)} = \mathbb{N}$
- if $v(p) = \mathbb{N}$, then
  - if $v(q) = \mathbb{N}$, then $\triangle v(q) = \mathbb{N}$
  - if $v(q) \neq \mathbb{N}$, then $\overline{v(p)} \cup v(q) \neq \mathbb{N}$ and $\overline{\triangle(\overline{v(p)} \cup v(q))} = \mathbb{N}$

(T) $v(\Box p \supset p) = \overline{\triangle v(p)} \cup v(p)$

- if $v(p) = \mathbb{N}$, then $\overline{\triangle v(p)} \cup v(p) = \mathbb{N}$
- if $v(p) \neq \mathbb{N}$, then $\overline{\triangle v(p)} = \mathbb{N}$

(4) $v(\Box p \supset \Box\Box p) = \overline{\triangle v(p)} \cup \triangle\triangle v(p)$

- if $v(p) = \mathbb{N}$, then $\triangle v(p) = \mathbb{N}$ and $\triangle\triangle v(p) = \mathbb{N}$
- if $v(p) \neq \mathbb{N}$, then $\triangle v(p) = \emptyset$ and $\overline{\triangle v(p)} = \mathbb{N}$

(5) $v(\Diamond p \supset \Box\Diamond p) = \overline{\triangledown v(p)} \cup \triangle\triangledown v(p)$

- if $v(p) = \emptyset$, then $\triangledown v(p) = \emptyset$ and $\overline{\triangledown v(p)} = \mathbb{N}$
- if $v(p) \neq \emptyset$, then $\triangledown v(p) = \mathbb{N}$ and $\triangle\triangledown v(p) = \mathbb{N}$

Finally, for (N) note that if α is a theorem, then $v(\alpha) = \mathbb{N}$ and $\triangle v(\alpha) = \mathbb{N}$. ∎

Theorem 3.6 *No modal system* **S** *between* **S0.5**^{0} *and* **S5** *whose propositional fragment is between* **PC**$^{\supset}$ *and* **PC** *can be characterized by finite matrices.*

Proof: Take the following valuation v of $\mathcal{M}_\infty$: to each propositional variable p_k we associate the set $\{k\} \subset \mathbb{N}$. Note that for two distinct p and q, we have $\overline{P} \neq \mathbb{N}$ and $\overline{Q} \neq \mathbb{N}$, where P and Q are the values assigned by v to p and q, respectively. Additionally, we have $\overline{P} \cup Q \neq \mathbb{N}$ and $\overline{Q} \cup P \neq \mathbb{N}$ and then:

$$v(p \asymp q) = \triangle(\overline{P} \cup Q) \cup \triangle(\overline{Q} \cup P) = \emptyset \cup \emptyset = \emptyset$$

Therefore, the value assigned by v in the matrix $\mathcal{M}_\infty$ to each adapted Dugundji's formula D'_n is $\emptyset$. By Proposition 3.5, no D'_n is a theorem of **S5**.

Now, by Proposition 3.4, if **S** could be characterized by a finite matrix, then D'_n would be a theorem of **S** and therefore a theorem of **S5**, an absurd. ■

Theorem 3.7 *No modal system* **S** *between* $\mathbf{C2}^0$ *and* **S5** *whose propositional fragment is between* $\mathbf{PC}^{\supset}$ *and* **PC** *can be characterized by a finite matrix.*

Proof: Take another adapted Dugundji formula D^*_n such that the only difference with respect to D'_n is that the formula $p_i \asymp p_j$ is now an abbreviation for $\Box(p_i \supset p_i) \supset \Box(p_j \supset p_i)$. Suppose that $\mathcal{M}$ is a finite matrix with n truth-values that is a model of $\mathbf{C2}^{0,\supset}$. Observe that, by the same reasons presented in Proposition 3.4, for every valuation v over $\mathcal{M}$ there will be $i \neq j$ such that the values assigned by v to $p_i \asymp p_j$ and $p_i \asymp p_i$ will be the same. By Theorem 2.2 (i) we have that $p_i \asymp p_i = \Box(p_i \supset p_i) \supset \Box(p_i \supset p_i)$ is a theorem of $\mathbf{C2}^{0,\supset}$ and so v assigns to it (and then, to $p_i \asymp p_j$) a distinguished value. Therefore, v assigns to $(p_i \asymp p_i) \vee \alpha$ and $\alpha \vee (p_i \asymp p_i)$ a distinguished value, for every formula α. Then v assigns to the formula D^*_n a distinguished value. This means that D^*_n is valid in $\mathcal{M}$.

Consider again the matrix $\mathcal{M}_\infty$ of Proposition 3.5 and the valuation v of Proposition 3.6. In this way, we have:

$$v(p \asymp q) = \overline{\triangle(\overline{P} \cup P)} \cup \triangle(\overline{Q} \cup P) = \overline{\triangle\mathbb{N}} \cup \triangle(\overline{Q}) = \emptyset \cup \emptyset = \emptyset.$$

The rest of the proof is identical to that of Theorem 3.6. ■

Definition 3.8 *The original Dugundji formula is defined, for each natural number* n*, as follows:*

$$D_n \equiv_{def} \bigvee_{i \neq j} (p_i \asymp p_j)$$

in which $1 \leq i, j \leq n+1$ *and* $p_i \asymp p_j$ *means* $\Box(p_i \supset p_j) \wedge \Box(p_j \supset p_i)$. □

Proposition 3.9 *Any finite matrix of* n *truth-values that is a model of* $\mathbf{K}^{\wedge,\supset}$ *validates also the original Dugundji formula.*

Proof: The argument is analogous to that of Proposition 3.4, using also Theorem 2.2 (iv). ■

Proposition 3.10 *There is an infinite matrix* $\mathcal{M}'_\infty$ *that is a model of* **K4.3W**.

Proof: Define the following matrix $\mathcal{M}'_\infty$

- $M = \wp(\mathbb{N})$
- $D = \{\mathbb{N}\}$
- $O = \{\cup, \cap, -, \boxtimes\}$, in which $\cup, \cap$ and $-$ are usual Boolean operations, while we define $\boxtimes$[7] as:

$$\boxtimes X = \begin{cases} \mathbb{N} & \text{if X is cofinite} \\ \mathbb{N} - \{\emptyset\} & \text{if X is finite} \end{cases}$$

Consider valuations over $\mathcal{M}'_\infty$ as mappings v which assign an element of $\wp(\mathbb{N})$ to each formula of **K4.3W** in the following way:

- $v(\neg\alpha) = \overline{v(\alpha)}$;
- $v(\alpha \to \beta) = \overline{v(\alpha)} \cup v(\beta)$;
- $v(\Box\alpha) = \boxtimes(v(\alpha))$.

Let us see that every modal axiom is valid:

(K) $v(\Box(p \to q) \to (\Box p \to \Box q)) = \overline{\boxtimes(\overline{v(p)} \cup v(q))} \cup \overline{\boxtimes v(p)} \cup \boxtimes v(q)$

- if $v(q)$ is cofinite, then $\boxtimes v(q) = \mathbb{N}$ and $v(K) = \mathbb{N}$
- if $v(q)$ is finite, then $\boxtimes v(q) = \mathbb{N} - \{\emptyset\}$
  - if $v(p)$ is cofinite, then $\overline{v(p)}$ is finite and $\overline{v(p)} \cup v(q)$ is finite. So, $\boxtimes(\overline{v(p)} \cup v(q))) = \mathbb{N}$ and $v(K) = \mathbb{N}$.
  - if $v(p)$ is finite, then $\boxtimes v(p) = \mathbb{N} - \{\emptyset\}$ and $\overline{\boxtimes v(p)} = \{\emptyset\}$ So, $\boxtimes v(q) \cup \overline{\boxtimes v(p)} = \mathbb{N}$ and $v(K) = \mathbb{N}$

(4) $v(\Box p \to \Box\Box p) = \overline{\boxtimes v(p)} \cup \boxtimes\boxtimes v(p)$

- if $v(p)$ is cofinite, then $\boxtimes v(p) = \mathbb{N}$, $\boxtimes\boxtimes v(p) = \mathbb{N}$ and $v(4) = \mathbb{N}$
- if $v(p)$ is finite, then $\boxtimes v(p) = \mathbb{N} - \{\emptyset\}$ but $\overline{\boxtimes v(p)} = \{\emptyset\}$ So, $\boxtimes\boxtimes v(p) = \mathbb{N} - \{\emptyset\}$ and $v(4) = \mathbb{N}$

(GL) $v(\Box(\Box p \to p) \to \Box p) = \overline{\boxtimes(\overline{\boxtimes v(p)} \cup v(p))} \cup \boxtimes v(p)$

- if $v(p)$ is cofinite, then $\boxtimes v(p) = \mathbb{N}$ and $V(GL) = \mathbb{N}$

[7] The function that calculates $\boxtimes$ was inspired in [15], as an example of a modal operator of diagonalizable algebras.

- if $v(p)$ is finite, then $\boxtimes v(p) = \mathbb{N} - \{\emptyset\}$ and $\overline{\boxtimes v(p)} = \{\emptyset\}$. Then $\overline{\boxtimes v(p)} \cup v(p)$ is finite. So, $\boxtimes(\overline{\boxtimes v(p)} \cup v(p)) = \mathbb{N} - \{\emptyset\}$, that is, $\overline{\boxtimes(\overline{\boxtimes v(p)} \cup v(p))} = \{\emptyset\}$ and $v(GL) = \mathbb{N}$

(Lem$_0$) $v(\Box((p \wedge \Box p) \to q) \vee \Box((q \wedge \Box q) \to p)) = \boxtimes(\overline{(v(p) \cap \boxtimes v(p))} \cup v(q)) \cup \boxtimes(\overline{(v(q) \cap \boxtimes v(q))} \cup v(p))$

- if $v(p)$ is cofinite, then $\overline{(v(q) \cap \boxtimes v(q))} \cup v(p)$ is cofinite. So, $\boxtimes(\overline{(v(q) \cap \boxtimes v(q))} \cup v(p)) = \mathbb{N}$ an therefore $v(Lem_0) = \mathbb{N}$.
- if $v(p)$ if finite
  * if $v(q)$ is cofinite, then $\overline{(v(p) \cap \boxtimes v(p))} \cup v(q)$ is cofinite. So, $\boxtimes(\overline{(v(p) \cap \boxtimes v(p))} \cup v(q)) = \mathbb{N}$ an therefore $v(Lem_0) = \mathbb{N}$.
  * if $v(q)$ is finite, then $\boxtimes v(q) = \mathbb{N} - \{\emptyset\}$ and $v(q) \cap \boxtimes v(q)$ is finite. So $\overline{(v(q) \cap \boxtimes v(q))} \cup v(p)$ is cofinite, and than $\boxtimes(\overline{(v(q) \cap \boxtimes v(q))} \cup v(p)) = \mathbb{N}$. Therefore $v(Lem_0) = \mathbb{N}$.

Finally, if $v(p) = \mathbb{N}$, then $v(p)$ is cofinite and $\boxtimes v(p) = \mathbb{N}$. So, $\mathcal{M}'_\infty$ preserves (Nec) and all the **K4.3W** axioms. ■

Theorem 3.11 *No system between* **K** *and* **K4.3W** *whose propositional fragment is between* $\mathbf{PC}^{\wedge,\supset}$ *and* **PC** *can be characterized by finite matrices*

Proof. Consider the Dugundji's formula D_n. For each propositional variable p_i we associate the set

$$X_i = \{x : x = n.k + (i-1) \text{ for some } k \in \mathbb{N}\}$$

for $i > 0$. Then, $\overline{X_i} \cup X_j = \overline{X_i}$ and $\overline{X_i}$ is not cofinite. Besides, $\overline{X_j} \cup X_i = \overline{X_j}$ and $\overline{X_j}$ is not cofinite neither. Then,

$$\boxtimes(\overline{X_i} \cup X_j) \cap \boxtimes(\overline{X_j} \cup X_i) = \boxtimes\overline{X_i} \cap \boxtimes\overline{X_j} = \overline{\{\emptyset\}} \cap \overline{\{\emptyset\}} = \overline{\{\emptyset\}}.$$

Therefore D_n takes the non-distinguished value $\mathbb{N} - \{\emptyset\}$. If there is an n-valued matrix $\mathcal{M}_n$ that characterizes **K4.3W** then by Proposition 3.9 D_n would be a theorem, so by Proposition 3.10 D_n would receive the distinguished value $\mathbb{N}$ in $\mathcal{M}'_\infty$, an absurd. ■

4 Conclusion

We show in this paper that Dugundji's Theorem can be generalized not only to systems modal systens **S1 - S5**, but for almost any modal system known which fragment is between Henkin's implicative calculus and Propositional Calculus.

A parallel result to Dugundji's one was proven by Scroggs in 1951[8] showing that all extension of **S5** can be characterized by finite matrices. Scroggs' theorem was generalized by L. Esakia and V. Meskhi,[9] showing that there are five extensions of **S4** (among them,**S5** itself) such that all of their extensions can also be characterized by finite matrices.

Our result in this sense is parallel to the L. Esakia and V. Meskhi one, updating an old result on finite matrices and modal logic.

We hope that our results together with the above mentioned can contribute to the current resumption of multi-valued semantics for modal logic that was marginalized after the incredible success of Kripke semantics.[10]

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