There is No Paradox of Logical Validity

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1 Introduction

A number of arguments, including (Whittle 2004), (Field 2008), (Shapiro 2012), and (Beall & Murzi 2012), have been put forward for the claim that logical validity is plagued by a paradox (or paradoxes) similar to the more well-known paradoxes that afflict notions such as truth, knowledge, or set. These claims are incorrect. As I will show here, there is no paradox of logical validity. In addition, I will demonstrate that a recent argument (Ketland 2012) against the existence of a paradox of logical validity, although promoting the right conclusion, nevertheless provides a partially mistaken picture of what goes wrong in the arguments for the existence of such a paradox. Further, and perhaps most importantly, this analysis of the purported paradox of logical validity brings with it important lessons regarding the kinds of inferences that can be taken to be logically valid (versus, e.g., merely truth-preserving).

There is general agreement amongst those who have put forward one or another version of the paradox that the stakes are high. Of course, almost no one in this group agrees on exactly what these high stakes are: Whittle argues that the paradox shows that a dialethiest, such as (Priest 2002), will need to appeal to a Tarski-like hierarchy of logical validity predicates; Field uses the paradox to (among other things) suggest that validity does not preserve truth; and Shapiro and Beall & Murzi argue (independently) that the paradox forces us to give up the structural rule of contraction.[1] Thus, the conclusions drawn from the purported paradox of validity are varied, and in the latter case, at least, the paradox is not the only evidence marshalled for the conclusion in question. Thus, I will not attempt to examine any of these further claims directly. Instead, I will be content to show that there is no paradox of logical validity. As a result, whatever other arguments one might have, this supposed paradox provides no additional support for the views just listed.

[1] There are subtle differences between (Shapiro 2012)'s rejection of contraction and (Beall & Murzi 2012)'s rejection of contraction, but these are irrelevant to the task at hand.

Before examining the purported paradox itself, it is worth clarifying exactly what notion is, or notions are, at issue. The paradox of logical validity (if there is such) is a paradox that arises via the mobilization, in some sense or another, of a logical validity predicate – that is, a predicate that applies to a premise and a conclusion (or to codes of, or other 'representatives' of, that premise and that conclusion) if and only if the argument from premise to conclusion is logically valid. Validity, strictly speaking, is thus a relation holding of formulas within some logical calculus, but validity is intended to capture the logical consequence relation in natural language. Hence, with appropriate clauses in place regarding appropriate translations from formal language to natural language[2], the thought is that an premise/conclusion pair in the formal language will be valid (and hence the validity predicate will hold of that pair) if and only if the natural language translation of the conclusion of the formal argument is a logical consequence of the natural language translation of the premise of the formal argument.

Further, I take it that the notion of logical consequence in question is, at least roughly, the notion identified by Tarski in his seminal work on logical consequence:

> Consider any class Δ of sentences and a sentence Φ which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class Δ consists only of true sentences and the sentence Φ is false. Moreover, since we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds... the consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects. ((Tarski 1936): 414 – 415)

This notion can be (again, roughly) captured by the slogan that logical consequence is necessary preservation of truth in virtue of logical form. Of course, almost every notion involved in this slogan – necessity, truth, and formality – is the subject of lively philosophical debate. Fortunately, for our purposes here we require only a single, simple observation regarding the nature of logical consequence.

Logical consequence in natural language, at least as understood by Tarski (and, as we shall, following Tarski, understand it here), is *formal*. Thus, logical validity in formal languages must be formal as well. The are numerous competing precise characterizations of exactly what the formality of logical validity amounts to – typically involving permutation invariance or similar combinatorial notions – but nearly all agree that the formality of logical

[2] It is worth noting that these clauses are non-trivial, since the details will depend on, among other things, how one draws the logical/non-logical vocabulary divide.

validity entails that the validity relation must be closed under arbitrary substitution of like expressions for like expressions. More carefully, if it is to be formal, then logical validity must meet the following constraint:

Logical Substitutivity:

For any formulas Φ_1, Φ_2, primitive non-logical expression Ψ, and (possibly complex) expression Ω of the same logical type as Ψ, if:

$$\Phi_1 \vdash \Phi_2$$

is a valid argument, then:

$$\Phi_1[\Psi/\Omega] \vdash \Phi_2[\Psi/\Omega]$$

is a valid argument.[3]

Of course, there are other validity notions that are of philosophical interest – arithmetic validity, metaphysical validity, analytic validity, etc. – and these notions need not satisfy the substitutivity requirement.[4] Logical validity, however, must be formal in some sense of formality that underwrites the substitutivity requirement. As we shall see, the formality of logical validity will provide an important piece of the story regarding why the purported paradox of logical validity is no paradox at all.

2 The Beall-Murzi Validity Paradox

Along lines that will be familiar to anyone who has worked with formal versions of other paradoxes involving semantic notions, such as the Liar, Curry, or Yablo paradox, the paradox of logical validity arises when we ask the following question: What happens when we add a validity predicate to Peano Arithmetic (hereafter PA)?

Before answering this question, however, it is worth noting that the puzzle can, and has, arisen due to asking a slightly different question: What happens when we add a validity connective – that is, a connective that holds between two sentences just in case the argument from the first to the second

[3] $\Phi[\Psi/\Omega]$ is the result of uniformly replacing all occurrences of Ψ in Φ with Ω. Here and below I represent the formal logical validity relation using the single turnstile "$\vdash$". Since the definitions, arguments, models, etc. discussed below apply to classical first-order logic (and classical first-order arithmetical theories), the completeness theorem guarantees nothing critical (of a technical nature) hinges on whether we use "$\vdash$" or the double turnstile "$\models$".

[4] Further (as I shall suggest in passing below), *these* notions might be susceptible to genuine paradoxes similar in structure to the construction involving logical validity examined here.

is logically valid – to PA? This is the approach taken in (Whittle 2004) and (Shapiro 2012). In order to construct a version of the paradox using a validity connective, the language in question also needs to contain a truth predicate – else it will not support the required fixed-point construction necessary to carry out the proof to contradiction. Since this complicates the presentation of the paradox, and since this method is equally susceptible to versions of the criticisms developed below for the predicate version of the paradox, I set aside discussion of the validity connective version of the puzzle until §5 below.

Fleshing things out a bit more (and following the presentation in (Beall & Murzi 2012) rather closely), what we are investigating is the behavior of a predicate "$Val(x, y)$" that holds of the Gödel code $<\Phi>$ of Φ and the Gödel code $<\Psi>$ of Ψ (in that order) if and only if the argument whose sole premise is Φ and whose conclusion is Ψ is logically valid.[5] Of course, merely adding such a predicate to the language of arithmetic causes no more problems than merely adding a new predicate "$T(x)$" for truth does. Problems, or at least apparent problems, *do* arise once we supplement the axioms and rules of arithmetic with plausible rules for "$Val(x, y)$". First, we have an 'introduction rule' for "$Val(x, y)$":[6]

$$\begin{aligned} VS_1 : \ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \text{If} : \Phi \vdash \Psi \\ & \text{Then} : \varnothing \vdash Val(<\Phi>, <\Psi>) \end{aligned}$$

In short, VS_1 codifies the natural thought that if we have a proof of Ψ from Φ, then the argument with Φ as premise and Ψ as conclusion is valid. That *some* version of VS_1 holds of the validity predicate is obvious, although as we shall see below, the correct formulation of this rule turns out to be more subtle than it initially appears. Nevertheless, with introductions out of the way, we also need something akin to an 'elimination rule' for "$Val(x, y)$", and this is provided by:

$$\begin{aligned} VS_2 : \ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \varnothing \vdash Val(<\Phi>, <\Psi>) \rightarrow (\Phi \rightarrow \Psi) \end{aligned}$$

In short, VS_2 codifies the very natural (and very Tarskian) thought that validity preserves truth.

[5] Of course, validity is, more generally, a relation that holds between a (possibly infinite) set of premises and a single conclusion (or, sometimes, a set of conclusions as in the sequent calculus). So long as the logic in question is compact, however, we can mimic the more general notion by replacing a set of premises with the conjunction of (some finite subset of) those premises (and, in the sequent calculus, a set of conclusions with the disjunction of some subset of those conclusions).

[6] "VS_1" stands for *validity schema one*, etc.

These are not the only possible, or only plausible, rules for the validity predicate. For example, if our task were to construct a powerful theory of logical validity within PA, rather than to determine whether such a theory is susceptible to paradoxes, then the following additional rule would be worth consideration:[7]

$$\begin{aligned} VS_3 :\ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \varnothing \vdash Val(<\Phi>, <\Psi>) \rightarrow Val(<\Sigma>, <Val(<\Phi>, <\Psi>)>) \end{aligned}$$

For our purposes here, however, VS_1 and VS_2 suffice.

That being said, it is obvious that some rules are unacceptable. One such unacceptable rule – one that will play a role in the analysis that follows, is the following *Unacceptable Rule*:

$$\begin{aligned} UR :\ & \text{For any formulas } \Phi \text{ and } \Psi \text{ and set of formulas } \Delta : \\ & \text{If} : \Delta, \Phi \vdash \Psi \\ & \text{Then} : \Delta \vdash Val(<\Phi>, <\Psi>) \end{aligned}$$

In short, just because we can prove that Ψ follows from Φ plus some additional set of premises Δ, it does not follow that Δ entails that "$Val(x, y)$" holds of (the codes of) Φ and Ψ. Obvious counterexamples to the acceptability of UR are not hard to come by. For example, let:[8]

$$\Phi^n = \text{"There are exactly } n \text{ objects."}$$

Clearly, the claim that there are exactly four objects and the claim that there are exactly five objects entails a contradiction:

$$\Phi^4, \Phi^5 \vdash \bot$$

It does not follow, however, that the claim that there are exactly four objects entails that the argument whose premise is that there are exactly five objects and whose conclusion is a contradiction is valid – that is:

$$\Phi^4 \nvdash Val(<\Phi^5>, <\bot>)$$

Hence, we must (as the name suggests) reject the unacceptable rule UR.

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[7] Note that VS_1 entails that if:

$$Val(<\Phi>, <\Psi>)$$

is a theorem then:

$$Val(<\Sigma>, <Val(<\Phi>, <\Psi>)>)$$

is also a theorem.

[8] For any finite n, Φ^n is definable in the language of first-order logic with identity.

We are now in a position to formulate the purported paradox of logical validity. First, we apply the Gödelian diagonalization lemma to the predicate:

$$Val(x, <\perp>)$$

to obtain a sentence Π such that:

$$\Pi \leftrightarrow Val(<\Pi>, <\perp>)$$

is a theorem. We can then, using arithmetic, VS_1, and VS_2, derive a paradox along lines similar to the reasoning underlying the Curry paradox:

1	$\quad \vert\ \Pi$	Assumption for application of VS_1.
2	$\quad \vert\ Val(<\Pi>, <\perp>)$	1, diagonalization.
3	$\quad \vert\ \Pi \rightarrow \perp$	2, VS_2.
4	$\quad \vert\ \perp$	1, 3, *modus ponens.*
5	$Val(<\Pi>, <\perp>)$	1 – 4, VS_1.
6	$\Pi \rightarrow \perp$	5, VS_2.
7	Π	5, diagonalization.
8	$\perp$	6, 7, *modus ponens.*

This is the paradox described in (Beall & Murzi 2012), and (modulo replacement of the primitive validity predicate "$Val(x, y)$" with a complex predicate constructed from a validity connective and the truth predicate – see §5 below) it is essentially that found in (Whittle 2004) and (Shapiro 2012). The problem, however, is that if "$Val(x, y)$" is meant to capture logical validity, then the argument given above is fallacious.

To see why, note that the equivalence between Π and "$Val(<\Pi>, <\perp>)$" is not a logical truth, but rather a truth of PA. Spelling out the reasoning above a bit more carefully, we should have noted that when we apply the Gödelian diagonalization lemma we obtain a Π such that:

$$\Pi \leftrightarrow Val(<\Pi>, <\perp>)$$

is a theorem *of arithmetic* (note the emphasis!) Hence, the inference from line 1 to line 2 depends on arithmetic. As a result, line 5, which is labeled as an application of VS_1, is no such thing. Lines 1 through 4 do not constitute a proof that a contradiction logically follows from Π, but rather that PA entails that a contradiction follows from Π – that is, lines 1 through 4 show that:

$$PA, \Pi \vdash \perp$$

As a result, in concluding "$Val(<\Pi>, <\perp>)$" at line 5, we did not actually apply VS_1, but instead applied the unacceptable rule UR. Reasoning more carefully, we *can* conclude at line 5 that:[9]

$$PA \vdash Bew_{PA}(<\Pi>, <\perp>)$$

But provability in PA and logical validity are, of course, different things.

We should note, however, that the application of arithmetic in passing from lines 1 to 2 is not the only questionable step in the sub-proof terminating with the application of VS_1 at line 5. In addition, the application of VS_2 at line 3 is also of questionable legitimacy. After all, even if the invalidity of the rules VS_1 and VS_2 is not as obvious as the invalidity of the axioms and rules of PA,[10] it is surely not straightforwardly obvious that VS_1 and VS_2 are, in fact, logically valid. In a recent examination of the Beall-Murzi version of the paradox, Jeffrey Ketland concludes, in effect, that we should disallow both arithmetic and the validity rules themselves in sub-proofs terminating in an application of VS_1:[11]

> ...the fact that Φ is a theorem of V-logic does *not* imply that Φ is itself valid or logically true. For example, $Val(<0 = 0>)$ and $Val(<0 = 1>) \rightarrow 0 = 1$ are theorems of V-logic, but neither formula is *valid*. Perhaps an analogy is that the fact that one can derive certain results using axioms/rules for the natural numbers (e.g., the induction scheme, or an induction rule, or an ω-rule) does not imply that such results are valid. In the usual (current) setting of first-order logic, the validity of Φ is equivalent to Φ's being logically derivable; and this is why the introduction rule (V-Intro) is restricted. So, one may infer $Val(< \Phi >)$ only if Φ has been derived using logic. If extra, non-logical principles have been used to prove Φ, then Φ might not be valid. ((Ketland 2012): 423)

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[9]For any recursively axiomatizable theory T:

$$Bew_T(x, y)$$

is the arithmetic predicate codifying T-entailment. That is, "$Bew_T(<\Phi>, <\Psi>)$" is true if and only if there is a proof of Ψ from Φ in T. Note that, for any recursively axiomatizable theory T, $Bew_T(x, y)$ is definable in PA.

[10]I am allowing a bit of terminological sloppiness here. Strictly speaking, we should talk of rules being valid and formulas being logical truths – or, at the very least, we ought to distinguish the sense in which rules might be valid from the sense in which axioms might be valid. Using the term 'valid' interchangably for both notions simplifies the discussion, however, and is harmless in the present context.

[11]Ketland considers a one-place validity predicate $Val(x)$ holding of (the codes of) logical truths. His V-Intro is the a one-place analogue of the rule called VS_1^L below. The upshot is the same, however.

Thus, according to Ketland, both the move from 1 to 2 in the derivation above, and the move from 2 to 3, are illegitimate in a sub-proof leading to an application of VS_1 (his V-Intro), since the former involves arithmetic and the latter involves the rules for the validity predicate itself, neither of which are logically valid (even though both are truth-preserving).

As we shall see, Ketland is right – a proper formulation of the validity predicate will disallow applications of both arithmetic and the rules for the validity predicate in sub-proofs terminating in an application of VS_1. But his analysis leaves two critical questions unanswered: First, even if both arithmetic and the validity rules VS_1 and VS_2 fail to be valid, it does not follow that both PA and the validity rules are *responsible* for the apparent paradox that results from (incorrectly) assuming that they are valid. As we shall see, even though neither arithmetic nor the validity rules are themselves valid, and hence neither should be used within a sub-proof terminating in an application of VS_1, it is demonstrably the validity rules, and not arithmetic, that lies at the root of the paradox. Second, although Ketland is thus clearly correct in asserting that the validity rules are not valid, he supports this claim with nothing more than intuition:

> Perhaps one might try to reply. . . that the *theory* of validity is itself valid, in some more general sense of 'valid'. But it seems to me that here there is a genuine disanalogy with, e.g., the notions of truth and necessity. ((Ketland 2012): 427)

Unfortunately, Ketland does not specify what, exactly, this disanalogy is. And in fact this is just as well, since there is no disanalogy – at least, not between logical validity and truth. The rules for the logical validity predicate, as we shall see, are not themselves logically valid for exactly the same reasons that the rules for the truth predicate (the T-schema) are not logically valid. This *does* point, however, to a deeper disanalogy, not between truth and logical validity, but between the paradoxes involving truth (such as the Liar) and the purported paradoxes involving logical validity.

3 Validity, Rules, and Paradox

Reflecting on the fallacy detected in the Beall-Murzi proof, the issue is this: What resources are, and are not, allowed in a sub-proof of Ψ from Φ if we are to apply VS_1 to that sub-proof and conclude that "$Val(<\Phi>, <\Psi>)$" is true? This question is equivalent to asking: Which axioms and rules of inference, of those currently at issue, are logically valid?

The discussion above suggests that arithmetic is not an allowable resource: Any ineliminable use of PA in a sub-proof should prevent application of VS_1 to that sub-proof, since the derivation in question will as a result not be logically valid (even if arithmetically sound). In addition, we have reasons for at least some doubt with regard to whether the validity rules VS_1 and VS_2 should be allowed in such sub-proofs. Summing all this up, there are three possible resources that we need to consider:

$$\begin{aligned} L &= \text{Pure First-Order Logic.} \\ PA &= \text{Peano Arithmetic.} \\ V &= VS_1 + VS_2. \end{aligned}$$

I take it to be obvious that the resources of pure first-order logic should be allowable in such sub-proofs – after all, if first-order logic doesn't preserve logical validity, then it is not clear that anything does.[12] Additionally, and more substantially, I will assume in what follows that VS_1 and VS_2 stand or fall together – that is, either both of VS_1 and VS_2 are allowed in sub-proofs that can be terminated with an application of VS_1, or neither are.[13] This leaves us with four possibilities:

- L is logically valid, but VS_1, VS_2, and PA are not.
- L and PA are logically valid, but VS_1 and VS_2 are not.
- L, VS_1, and VS_2 are logically valid, but PA is not.
- L, VS_1, VS_2, and PA are all logically valid.

Of course, we have already ruled out the second and fourth option, since the axioms and rules of PA are not logically valid. Examining systems that allow the use of PA in such sub-proofs will turn out to be illuminating nevertheless. Given these four distinct possible answers to our question, we obtain four distinct versions of VS_1:

$$\begin{aligned} VS_1^L : \ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \text{If} : \Phi \vdash_L \Psi \\ & \text{Then} : \varnothing \vdash_{L+PA+V} Val(<\Phi>, <\Psi>) \end{aligned}$$

[12]Further, I will assume for the sake of argument, and for the sake of actually engaging with those I am criticizing, that first-order classical logic can be used in such sub-proofs. For the reader whose is (like myself) sympathetic to intuitionistic (or other non-classical) logics, however, it is worth noting that the derivation of a contradiction from VS_1 and VS_2 given in the last section is intuitionistically valid, and that both the derivation above and the results to follow can be easily adapted to many non-classical contexts. In particular, the substitutivity requirement introduced in §1, which will play a central role in §4 below, holds not only of classical logic but also of the vast majority of non-classical logics defended in the literature.

[13]Treating these rules separately, considering systems that allow VS_1 but not VS_2 to be allowed in such sub-proofs (or *vice versa*), would double the number of cases we need to consider, with no additional philosophical insight. The methods of the next section can be generalized, however, to settle these additional cases. Doing so is left to the interested reader.

$$VS_1^{L+PA}: \text{ For any formulas } \Phi \text{ and } \Psi: \\ \text{If}: \Phi \vdash_{L+PA} \Psi \\ \text{Then}: \varnothing \vdash_{L+PA+V} Val(<\Phi>, <\Psi>)$$

$$VS_1^{L+V}: \text{ For any formulas } \Phi \text{ and } \Psi: \\ \text{If}: \Phi \vdash_{L+V} \Psi \\ \text{Then}: \varnothing \vdash_{L+PA+V} Val(<\Phi>, <\Psi>)$$

$$VS_1^{L+PA+V}: \text{ For any formulas } \Phi \text{ and } \Psi: \\ \text{If}: \Phi \vdash_{L+PA+V} \Psi \\ \text{Then}: \varnothing \vdash_{L+PA+V} Val(<\Phi>, <\Psi>)$$

VS_1^L states that, if we have a sub-proof of Ψ from Φ that uses only the resources of first-order logic, then we can apply (this version of) VS_1 and conclude that "$Val(<\Phi>, <\Psi>)$" is true. VS_1^{L+PA} states that, if we have a sub-proof of Ψ from Φ that uses only the resources of first-order logic and Peano arithmetic, then we can apply (this version of) VS_1 and conclude that "$Val(<\Phi>, <\Psi>)$" is true. VS_1^{L+V} states that, if we have a sub-proof of Ψ from Φ that uses only the resources of first-order logic, (this version of) VS_1, and VS_2, then we can apply (this version of) VS_1 and conclude that "$Val(<\Phi>, <\Psi>)$" is true. And finally, VS_1^{L+PA+V} states that, if we have a sub-proof of Ψ from Φ that uses only the resources of first-order logic, Peano arithmetic, (this version of) VS_1, and VS_2, then we can apply (this version of) VS_1 and conclude that "$Val(<\Phi>, <\Psi>)$" is true.

The obvious next step is to investigate the consistency of the systems that result from extending PA with VS_2 and one of these four variants of VS_1. As we shall see, the results are somewhat surprising, and suggest that Ketland's analysis of the paradox of logical validity misses something important. The following theorems settle the consistency question for the four systems in question:

Theorem 3.1. *The system that results from adding VS_1^{L+PA+V} and VS_2 to PA is inconsistent.*

Proof. This is settled by the Beall-Murzi derivation given above. □

Theorem 3.2. *The system that results from adding VS_1^{L+PA} and VS_2 to PA is consistent.*

Proof. Let:

$$PA^H = PA + \{Bew_{PA}(<\Phi>, <\Psi>) \rightarrow (\Phi \rightarrow \Psi) : \Phi, \Psi \in L_{PA}\}.$$

Then VS_1^{L+PA} and VS_2 are interpretable in PA^H: Let:

$$Val(x, y) = Bew_{PA}(x, y).$$

For VS_1, note that, if:

$$\Phi \vdash_{L+PA} \Psi$$

then:

$$\varnothing \vdash_{L+PA} Bew_{PA}(<\Phi>, <\Psi>)$$

hence:

$$\varnothing \vdash_{L+PA_H} Bew_{PA}(<\Phi>, <\Psi>)$$

since PA^H extends PA. VS_2 is easy. Hence, if $VS_1^{L+PA} + VS_2 + PA$ were inconsistent, then PA^H would be inconsistent. But PA^H is true on the standard model of arithmetic.[14] □

Corollary 3.3. *(Ketland 2012):The system that results from adding VS_1^L and VS_2 to PA is consistent.*

Proof. Immediate consequence of Theorem 3.2.[15] □

Theorem 3.4. *The system that results from adding VS_1^{L+V} and VS_2 to PA is inconsistent.*

Proof. Let $\mathfrak{A}$ be the conjunction of any finitely axiomatizable theory of arithmetic strong enough to support diagonalization (e.g. Robinson arithmetic). Let $\Pi(x, y)$ be the function that maps any pair of Gödel codes of sentences t_1, t_2, onto the code of their conjunction (in that order). Note that $\Pi(x, y)$ is primitive recursive. By diagonalization we obtain a statement Λ such that:

$$\Lambda \leftrightarrow Val(\Pi(<\Lambda>, <\mathfrak{A}>), <\perp>)$$

[14] The following technical clarification might be helpful: The provability predicate for PA, "$Bew_{PA}(x, y)$", obeys the rules VS_1^{L+PA} and VS_2 in the stronger system PA^H, but not in the weaker system PA. This is merely another way of stating the well-known fact that an 'arithmetic validity' predicate for PA can consistently be *added* to PA (if PA is itself consistent), but cannot be defined in PA itself.

[15] (Ketland 2012) actually proves (something equivalent to the claim) that the result of adding VS_1^L and VS_2 to PA is a conservative extension of PA (Theorem 1, p. 426). This result is non-trivial, depending on the *essential reflexivity* of PA, and does not generalize to VS_1^{L+PA}. More generally, since the system investigated in (Ketland 2012) – essentially the result of adding VS_1^L and VS_2 to PA – is, as we shall see, the *correct* formalization of a theory of the logical validity predicate, and since Ketland's paper is a good bit more technically sophisticated than the present essay, the reader interested in further details regarding how the logical validity predicate behaves within PA is strongly encouraged to read (Ketland 2012).

and hence:[16]

$$\Lambda \leftrightarrow Val(<\Lambda \wedge \mathfrak{A}>), <\perp>)$$

are theorems of $\mathfrak{A}$ (and hence of PA). Note that, a bit loosely, Λ says something like:

The argument with this sentence and $\mathfrak{A}$ as premises,
and $\perp$ as conclusion, is valid.

We now reason as follows:

1	│ $\Lambda \wedge \mathfrak{A}$	Assumption for application of VS_1^{L+V}.
2	│ $\mathfrak{A}$	1, logic.
3	│ Λ	1, logic.
4	│ $Val(<\Lambda \wedge \mathfrak{A}>), <\perp>)$	2, 3, logic.[17]
5	│ $\perp$	1, 4, VS_2.
6	$Val(<\Lambda \wedge \mathfrak{A}>), <\perp>)$	1 – 5, VS_1^{L+V}.
7	Λ	6, arithmetic.
8	$\Lambda \wedge \mathfrak{A}$	7, arithmetic.
9	$\perp$	6, 8, VS_2.

□

The situation, viewed from a purely technical perspective, is summarized in the following table:

	PA allowed	*PA disallowed*
VS_1, VS_2 *allowed*	*Inconsistent*	*Insconsistent*
VS_1, VS_2 *disallowed*	*Consistent*	*Consistent*

In short, whether or not our theory of the logical validity predicate is consistent co-varies with whether or not we allow the rules for the logical validity predicate themselves to appear in sub-proofs terminating with an application of VS_1 – that is, with whether or not we treat the rules for the validity

[16]Note that, for all Φ and Ψ:

$$<\Phi \wedge \Psi> = \Pi(<\Phi>, <\Psi>)$$

is a theorem of PA (or of $\mathfrak{A}$).

[17]This line represents the 'trick' in the proof: Unlike the Beall-Murzi derivation, in this case we get that the validity claim "$Val(<\Lambda \wedge \mathfrak{A}>)$" follows from previous lines in the sub-proof, as a matter of logic, rather than as a theorem of PA, since the conjunction of the axioms of (a finite but sufficiently strong subsystem of) arithmetic just *is* one of the previous lines in the sub-proof.

predicate as being logically valid themselves. Whether or not we allow arithmetic within such sub-proofs turns out to be completely irrelevant to the *consistency* status of the resulting systems, however, strongly suggesting that the logical status of arithmetic and its use within such sub-proofs is orthogonal to a correct assessment of whether there truly is a paradox of logical validity. Thus, we have answered the first of the two questions left open by Ketland's discussion: Our consistency proofs demonstrate that no paradoxes will arise from assuming that the axioms of PA are logical truths, (although as we have seen, *falsity* will arise from such an assumption, since the axioms of PA are not, in fact, logical truths!) Rather, it is the assumption that the validity rules VS_1 and VS_2 are logically valid that leads to paradox.

As a result, if we want to know whether or not there really is a genuine paradox of logical validity, we need to determine whether the rules for the logical validity predicate are themselves logically valid rules. Of course, we could, like (Ketland 2012), just stipulate that these rules are not logically valid, on the grounds that assuming otherwise leads to contradictions. But this approach seems no more illuminating than merely abandoning the T-schema in light of the Liar paradox. What is needed is some account that explains *why* the validity predicate is not a logical operator, and hence explains why the validity rules VS_1 and VS_2 are not logically valid.

4 Validity and Substitutivity

The results of the previous section have left us with two options: either we can conclude that the addition of a logical validity predicate to PA is paradoxical, in much the same way that the addition of an unrestricted truth predicate to PA is paradoxical, or we can conclude that the rules VS_1 and VS_2 are not logically valid, and hence cannot be applied in sub-proofs terminating in an application of VS_1. In short, we need to decide whether VS_1^{L+V} or VS_1^{L} is the right 'introduction rule' for the logical validity predicate.

Before defending the second option, it would be remiss not to note that the first option is tempting. After all, at first glance, it would seem that accepting that the rules for the logical validity predicate can be applied in sub-proofs terminating in an application of VS_1, and the paradox that arises as a result, in many respects parallels accepting the unrestricted T-schema for the truth predicate and the paradoxes that arise as a result. There is a critical disanalogy, however. Deriving the Liar paradox (or any other variant of semantic paradox) from the T-schema requires only that the relevant instances of the T-schema are *true* (or, equivalently, that the analogous inference rules for the truth-predicate are *truth-preserving*). No similar paradoxes arise from the assumption that the validity predicate rules

VS_1 and VS_2 are merely truth-preserving. Paradox only arises if we assume that VS_1 and VS_2 are not only truth-preserving but logically valid, since only then are they eligible for application in sub-proofs terminating in an application of VS_1. This is the crucial disanalogy between truth and logical validity mentioned in §2 above: If (as will be argued below) the rules for both the truth predicate and the validity predicate are truth-preserving but not logically valid, then there is a genuine paradox involving truth yet no paradox involving logical validity.[18]

Regardless of this disanalogy with truth, the intuition that VS_1 and VS_2 are not only truth-preserving but also logically valid remains, admittedly, a strong one. After all, why wouldn't predicates codifying something like 'necessary entailment in virtue of logical form' be true of particular sentences (or their names, or codes, or whatever) necessarily, and in virtue of form, when they *are* true of those sentences? The negative answer to this question has to do not only with what such predicates encode (logical validity), but *how* they encode this information (Gödel codes or other non-logical naming devices).

The argument is simple, involving a straightforward application of the logical substitutivity constraint discussed in §1 above:

> *Logical Substitutivity*:
>
> For any formulas Φ_1, Φ_2, primitive non-logical expression Ψ, and (possibly complex) expression Ω of the same logical type as Ψ, if:
>
> $$\Phi_1 \vdash \Phi_2$$
>
> is a valid argument, then:
>
> $$\Phi_1[\Psi/\Omega] \vdash \Phi_2[\Psi/\Omega]$$
>
> is a valid argument.

If we assume that VS_1 and VS_2 are logically valid, and that logical substitutivity holds, we can prove clearly absurd claims. For example:[19]

Pseudo-theorem 4.1. *If VS_1 and VS_2 are logically valid, and the logical substitutivity constraint holds, then there are sentences Φ_1 and Φ_2 such that, for any n:*

$$Val(<\Phi_1> +n, <\Phi_2> +n)$$

[18] This of course suggests a more general phenomenon: Are there other significant semantic notions Φ such that a paradox arises if, but only if, we assume, loosely speaking, that the rules for a Φ-predicate are themselves Φ (or Φ-preserving, etc.)? Unfortunately, space considerations preclude exploring this topic in depth in the present essay.

[19] I have labelled this result a *pseudo-theorem* since it is hoped that, by the end of the paper, the antecedent of the claim will be so clearly false as to render the result uninteresting!

is a logical truth.

Proof. Let Φ_1 and Φ_2 be any formulas such that the former logically entails the latter:

$$\Phi_1 \vdash \Phi_2$$

Since VS_1 is valid, we have:

$$\varnothing \vdash Val(<\Phi_1>, <\Phi_2>)$$

Note that the above is of the form:

$$\vdash Val(s(s(s(\ldots s(0) \ldots))), s(s(s(\ldots s(0) \ldots))))$$

Then, by logical substitutivity, we may replace the simple non-logical term "0" with:

$$n = s(s(s(\ldots s(0) \ldots)))$$

(for arbitrary n). Since n arbitrary, it follows that, for any n:

$$\varnothing \vdash Val(<\Phi_1> +n, <\Phi_2> +n)$$

that is, all instances of:

$$Val(<\Phi_1> +n, <\Phi_2> +n)$$

are logical truths.[20] □

This 'pseudo-theorem' clearly fails to hold of many easily constructed recursive coding functions (including most 'intuitive' ones used in standard textbooks such as (Boolos, Burgess, & Jeffrey 2007)).[21]

Of course, one could perhaps 'cook up' a particular coding function for which the result just proved is not absurd. Such computational shenanigans will not avoid the problem, however. Surely whether or not the rules for the logical validity predicate are logically valid should not depend on the arithmetic properties of the particular coding function we choose (other than the fact that the coding function is recursive, etc.) Thus, if logical validity must be closed under the logical substitutivity constraint, then VS_1 and VS_2 are not logically valid.

Further, if the logical validity predicate is a logical operator, and the rules for this operator are logically valid, then this fact ought to be independent

[20] Note that this proof actually only assumed that VS_1 is logically valid. Note further that, if the "$Val(x, y)$" only holds of x and y if x and y are both codes of sentences, then an easy corollary of this theorem is that all but finitely many natural numbers are codes of sentences. This also fails for most standard coding functions.

[21] In addition, it is not difficult to 'cook up' particular coding functions such that the 'pseudo-theorem' implies that obviously invalid arguments are valid, and obviously false claims are logical truths.

of the particular naming device used and of the nature of those names – that is, its legitimacy ought to be independent of whether we use Gödel codes, or primitive names of some sort, or quotation names, etc. As a result, we can produce even simpler absurdities. Just extend the language with three additional, primitive names t_1, t_2, and t_3, where t_1 and t_2 are names for two logical truths Φ_1 and Φ_2 respectively and t_3 is the name of some contradiction Φ_3. Then clearly:

$$\Phi_1 \vdash \Phi_2$$

So, by VS_1 we obtain:

$$\varnothing \vdash Val(t_1, t_2)$$

Hence, by logical substitutivity:

$$\varnothing \vdash Val(t_1, t_3)$$

This last is obviously absurd, however, since no logical truth entails any contradiction. Further, a final application of VS_2 at this point provides:

$$\Phi_1 \vdash \Phi_3$$

rendering the system trivial.

Thus, either we give up the idea that logical validity is formal, and hence must satisfy the logical substitutivity constraint, or we give up the idea that the rules for the logical validity predicate are themselves logically valid. Since giving up on the formality of logical validity would seem to be giving up on the intended and intuitive notion of logical validity altogether (robbing the claim that VS_1 and VS_2 are logically valid of most of its interest!), it seems that our only viable option is to abandon the idea that VS_1 and VS_2 are logically valid. As a result, the proper formulation of VS_1 is its weakest formulation: VS_1^L.

It should be noted that the arguments above do not, on their own, show that the logical validity predicate is not a logical operator. On the contrary, these arguments are compatible with the claim that the logical validity predicate is a logical operator, and that there are logical truths that involve the validity predicate in an essential manner. For example, it might be the case that:

$$(\forall x)(\exists y)(Val(x, y))$$

is, in fact, a logical truth.[22] What *has* been shown, however, is that the rules VS_1 and VS_2, in particular, are not logically valid. But that is enough for our purposes: If VS_1 and VS_2 are not logically valid, they cannot be used in sub-proofs terminating with an application of VS_1, and there is no paradox of logical validity.[23]

[22] Of course, no particular claim of the form $Val(n, m)$, for particular numerals n and m, can be a logical truth, for the reasons already given.

[23] In fact, I do not think that the logical validity predicate is a logical operator, nor do

5 Logical Validity and the Truth Predicate

As noted in §1 above, (Whittle 2004) and (Shapiro 2012) provide a slightly different version of the paradox of logical validity – one relying on a logical validity *connective* instead of a logical validity predicate. As a result, in order to construct the fixed-point sentence generating the paradox, they also make use of an unrestricted truth predicate. Since their constructions of the paradox are, in essentials, extremely similar from a technical perspective (even if they use the supposed paradox to argue for very different conclusions), I will focus on Shapiro's construction. Similar comments apply to Whittle's version of the paradox.[24] Shapiro's version of the paradox proceeds as follows:

> Let K be a sentence equivalent to $T(<K>) \Rightarrow P$, where $\Rightarrow$ is an entailment connective. (To simplify my presentation, *I shall pretend that these are not just equivalent sentences, but the same sentence.*) Currys paradox can now be formulated as the following. . . :
>
> | | | |
> |---|---|---|
> | 1 | $\quad\vert\ T(<K>)$ | Assumption. |
> | 2 | $\quad\vert\ T(<K>) \Rightarrow P$ | 1, T-Elim. |
> | 3 | $\quad\vert\ P$ | 1, 2, *modus ponens.* |
> | 4 | $T(<K>) \Rightarrow P$ | 1 – 3, Conditional Proof. |
> | 5 | $T(<K>)$ | 4, T-Intro. |
> | 6 | P | 4, 5, *modus ponens.* |
>
> (2010: 17, emphasis added.[25])

Shapiro's presentation clings more closely to the traditional formulation of the Curry paradox than that found in (Beall & Murzi 2012), demonstrating that we can, via the paradoxical reasoning, prove any statement P whatsoever. Plugging "$\perp$" in for P, however, provides the explicit contradiction.

I think that there are any logical truths that involve occurrences of the logical validity predicate in an essential manner. Arguments for these additional claims would take us too far afield, however, and are not needed for the purposes of the primary task undertaken in this paper – to demonstrate that there is nothing paradoxical about a logical validity predicate.

[24]With one striking exception: (Whittle 2004) does not mistakenly assume that the T-schema is a logical truth, but rather notes, in his criticism of (Priest 2002), that the latter explicitly asserts that the T-schema is a logical truth. Thus, the mistake in this case is Priest's, not Whittle's (for more discussion, see (Cook 2012)).

[25]I have freely adapted the derivation found in (Shapiro 2012), aligning the notation to match that used elsewhere in this essay.

It should be clear by now that this derivation no more provides a genuine paradox of logical validity than the construction given in (Beall & Murzi 2012). First (and most obviously, given the helpful emphasis I have provided), there is the simplifying assumption that K and "$T(<K>) \Rightarrow P$" are not equivalent sentences, but identical sentences. If, however, the sentence K is obtained through diagonalization (it need not be, but the admission that these sentences are really merely equivalent suggests this) then this pretense hides the illegitimate use of PA in moving from lines 1 to 2 (it also hides the fact that if the fixed point is obtained via diagonalization, then K and "$T(<K>) \Rightarrow P$" are, on most standard codings, provably *not* the same sentence!) Given our discovery above that it is not the use of PA, but the illegitimate use of the rules for the validity operator, that lie at the root of the purported paradox, however, let us set this aside. As we shall see, there are other reasons for objecting to Shapiro's derivation.

For our purposes, it is the sort of inference applied at line 4 that is of real interest. Shapiro labels line 4 as an application of conditional proof, but it is not an instance of the standard conditional proof rule for the material conditional. Rather, it is an *analogue* of this rule for the logical validity connective "$\Rightarrow$". Likewise, the inference from 1 and 2 to 3 is not, contrary to Shapiro's labeling, an instance of the standard *modus ponens* rule for the material conditional, but is an analogue of this rule for "$\Rightarrow$" (as is the inference from 4 and 5 to 6). In short, Shapiro is utilizing connective versions of VS_1 and VS_2, which we can represent as follows:

$$\begin{aligned} \Rightarrow\!S_1 :\ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \text{If} : \Phi \vdash \Psi \\ & \text{Then} : \varnothing \vdash \Phi \Rightarrow \Psi \end{aligned}$$

$$\begin{aligned} \Rightarrow\!S_2 :\ & \text{For any formulas } \Phi \text{ and } \Psi : \\ & \varnothing \vdash (\Phi \Rightarrow \Psi) \rightarrow (\Phi \rightarrow \Psi) \end{aligned}$$

(where "$\rightarrow$" remains the everyday material conditional). Interestingly, although we have seen that VS_1 and VS_2 are not logically valid rules, the argument given above does not generalize to $\Rightarrow S_1$ and $\Rightarrow S_2$. Since the latter do not involve predicates but operators, we cannot apply the same substitutivity tricks directly to $\Rightarrow S_1$ and $\Rightarrow S_2$. As a result, the argumentative strategies mobilized above provide no reasons for denying that (our rational reconstructions of) Shapiro's rules are logically valid.

Moreover, there is at least some evidence in favor of treating $\Rightarrow S_1$ and $\Rightarrow S_2$ as logically valid. John Burgess (building on work by, e.g. (Halldén 1963)) argues that *S5* is the right modal logic for logical validity, understanding "$\Box\Phi$" as "Φ is a logical truth" (Burgess 1999). If this is right, then we can easily construct a binary validity operator obeying $\Rightarrow S_1$ and $\Rightarrow S_2$:

$$\Phi \Rightarrow \Psi =_{df} \Box(\Phi \rightarrow \Psi)$$

Note that, if the language only contains the resources of the modal logic *S5*, then both $\Rightarrow S_1$ and $\Rightarrow S_2$ are themselves logically valid on this interpretation.[26]

Furthermore, although Shapiro does not proceed this way, given the logical validity connective and the truth predicate, it is straightforward to construct a validity predicate:

$$Val(x, y) =_{df} T(x) \Rightarrow T(y)$$

With this predicate in hand, the Beall-Murzi version of the paradox can be reconstructed.

If $\Rightarrow S_1$ and $\Rightarrow S_2$ are, in fact, logically valid rules (or, at the very least, *could be* for all we have seen so far), then where does the Shapiro version of the derivation go wrong? The fallacy lies in the transition from line 1 to line 2. This inference does not rely solely on diagonalization (or whatever non-logical resources were used to obtain a sentence K equivalent to "$T(<K>) \Rightarrow P$". In addition, the inference in question relies on the T-schema (or some equivalent rule for the truth predicate) that allow us to move from:

$$K \leftrightarrow (T(<K>) \Rightarrow P)$$

To:

$$T(<K>) \leftrightarrow (T(<K>) \Rightarrow P)$$

The T-schema, however, is no more logically valid than are VS_1 and VS_2, since it is susceptible to logical substitutivity arguments along lines similar to those given in the last section.[27] As a result, the move from 1 to 2, although truth preserving, is not logically valid. Since $\Rightarrow S_1$ should only apply to sub-proofs that are logically valid, the derivation given in (Shapiro 2012) is

[26]This point also highlights the fact that the logical substitutivity requirement, while providing a necessary condition for the logical validity of a rule of inference, does not provide a sufficient condition. While *S5* satisfies the substitutivity requirement, and the modal operators might, as sketched above, be plausibly thought of as logical operators (and the rules for the modal operators might be plausible thought to be logically valid) when the modal operators are interpreted as codifying the notion of logical validity itself, this clearly does not hold for other modal logics that satisfy the substitutivity requirement. In particular, the points made earlier in this paper make it clear that the rules for the modal operators in the Gödel-Löb provability logic *GL* are not logically valid when "$\Box$" is interpreted as arithmetic provability, yet propositional *GL* satisfies the substitutivity requirement – see, e.g. (Boolos 1993).

[27]Let t_1 be the name of a logical truth Φ_1 and t_2 be the name of a logical falshood Φ_2. Then $\varnothing \vdash \Phi_1$. So, by the T-schema, we have $\varnothing \vdash T(t_1)$. Hence, by substitutivity, we obtain $\varnothing \vdash T(t_2)$. But then, again by the T-schema, we obtain $\varnothing \vdash \Phi_2$. For further details, see (Cook 2012).

fallacious in exactly the same manner as the Beall-Murzi construction (as is the similar construction given in (Whittle 2004)).

Nevertheless, although neither the validity predicate nor the validity connective is susceptible to paradox, they are different in an important and striking way. The natural rules for the validity predicate – VS_1 and VS_2 – are not logically valid, while the corresponding rules for the logical validity connective ($\Rightarrow S_1$ and $\Rightarrow S_2$) are valid (or, at the very least, there seem to be no reasons at present for denying them this status). As a result, in a certain sense it is not the *content* of the logical validity predicate that prevents the rules governing its use – VS_1 and VS_2 – from being logically valid, since presumably, in some loose sense, at least, the content of the logical validity predicate and the content of the logical validity connective are the same. Rather, it is *how* the logical validity predicate codifies that content – in particular, that it is a predicate, and not a connective, and thus that it applies not directly to sentences but to *names of* sentences – that prevents VS_1 and VS_2 from being logically valid.

To sum up: There is no paradox of logical validity since the construction of the paradox requires a context into which we can diagonalize. Such a context requires, in turn, a logical validity predicate (either primitive or defined in terms of a logical validity connective and the truth predicate). Such a predicate, however, requires Gödel coding or some other naming device. The presence of such coding functions, however, brings with it violations of the substitutivity requirement, thus preventing VS_1 and VS_2 from being logically valid. Hence, there is no paradox of logical validity.

6 What is the Status of Logical Validity?

Returning to our primitive validity predicate "$Val(x, y)$" a final question remains to be answered: If the 'correct' rules for the validity predicate – VS_1^L and VS_2 – are not logically valid, then what is their status? Part of the answer is simple, and already familiar: There are no reasons to think that VS_1^L and VS_2 are not truth-preserving, and every reason to think that they are. So, assuming proper care is taken to only use logically valid rules in sub-proofs terminating with applications of VS_1^L, reasoning with the validity predicate in accordance with VS_1^L and VS_2 cannot lead us astray (in the sense of leading us from truths to falsehoods – of course, the point of much of the above is that application of these rules can lead us from logical validities to logical invalidities).

Further, there seems to be no good reason for thinking that the consequences of correct applications of VS_1^L and VS_2 are not analytically true or necessarily true (and again, every reason to think that they are). This observation is critically important. If facts about logical validity – that is, those facts codified by our logical validity predicate – were merely contingent facts that held as a matter of luck or happenstance, then it would be difficult to explain the central role played by logic and logical validity in philosophical and mathematical theorizing. Fortunately, there is nothing contingent or lucky about logical validity, and the central theses governing validity codified by VS_1^L and VS_2 (as well as, perhaps, other principles, such as VS_3 touched on in §1 above) are analytic, necessary, etc. They just are not logically valid.

I will close the paper by pointing out that there are other notions of validity that *are* susceptible to paradoxes of the general shape outlined by (Beall & Murzi 2012). For example, if we were to introduce a predicate "$Val_M(x, y)$" such that:

$$Val_M(<\Phi>, <\Psi>) \text{ iff it is metaphysically necessary that, if } \Phi, \text{ then } \Psi.$$

or a predicate "$Val_A(x, y)$" such that:

$$Val_M(<\Phi>, <\Psi>) \text{ iff it is an analytic truth that, if } \Phi, \text{ then } \Psi.$$

then we would find ourselves faced with genuine paradoxes. The reason is simple – just as any rules that can be applied in a sub-proof terminating in an application of VS_1 must be logically valid, any rule applied in a sub-proof terminating in an application of the"$Val_M(x, y)$" analogue of VS_1 must preserve metaphysical necessity (and similarly, any rule applicable in a sub-proof terminating in an application of the "$Val_A(x, y)$" analogue of VS_1 must preserve analytic truth). But, unlike the case with VS_1 itself, there seems every reason (other than the paradoxes that ultimately arise!) to think that the rules for "$Val_M(x, y)$" do preserve metaphysical necessity, and that the rules for "$Val_A(x, y)$" do preserve analytic necessity.[28]

Thus, there are paradoxes that *can* be formulated in terms of important understandings of validity. But there are no paradoxes that plague the notion of logical validity. As a result, the far-reaching conclusions drawn by (Beall & Murzi 2012), (Field 2008), and (Shapiro 2012) based on the supposed existence of a paradox of logical validity need to be reassessed.

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