SUSZKO'S REDUCTION IN A TOPOS

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I study here Suszko's reduction in toposes. The originality of this paper comes not from the observation that the internal logic of a topos is bivalent because it is in the scope of Suszko's reduction; that is easy. Rather, the originality comes from
(1) recognizing the import of applying Suszko's reduction in a topos-theoretical setting, because the beautiful picture of logic in a topos rests on the ideas that (a) Ω is (or at least can be seen as) a truth-values object, (b) that the internal logic of a topos is in general many-valued and (c) that the internal logic of a topos is in general (with a few provisos) intuitionistic[1];
(2) the construction used to give categorial content to the reduction, and
(3) the extrapolation of the debate about Suszko's thesis to the topos-theoretical framework, which gives us some insight about the scope of another theorem, namely that stating that the internal logic of a topos in general is intuitionistic.

The plan of the paper is as follows. In the first section I expound Suszko's reduction, following closely the presentations in [6], [22] and [24]. In section 2 I show that the internal logic of a topos can be described as algebraically many-valued but logically two-valued. I introduce there the notions of a *Suszkian object* and of *Suszkian bivaluation*, as opposed to a truth values object and a subobject classifier. I prove their existence and uniqueness (up to isomorphism) in a given topos. Even though the main result of the paper is the internalization of Suszkian bivalence in a topos, it is relatively straightforward once the hard part, the characterization of a Suszkian object and a bivaluation, has been done. Finally, in sections 3 to 6 I suggest how logical many-valuedness could be recovered, but at the price of letting the internal logic of a topos become variable. Knowledge of basic category theory (as in chapters of 1, 2 and 4 of [16]) is presupposed and knowledge of basic topos theory would be useful (good sources are [9] and again [16]).

1. Suszko's reduction

Let $\mathbb{F}$ be a non-empty collection of, say, formulas, and let A be a non-empty collection of truth values. Let us assume $A = D^+ \cup D^-$ for suitable disjoint collections $D^+ = \{d_1, d_2, \ldots\}$ and $D^- = \{a_1, a_2, \ldots\}$ of *designated* and *antidesignated* values, respectively. A *semantics* over $\mathbb{F}$ and A is said to be any collection $\mathtt{sem}$ of mappings $\sigma : \mathbb{F} \longrightarrow A$, called *n-valuations* (where 'n' corresponds to a suffix, uni-/bi-/tri-, and so, according to $|A|$, the cardinality of A). Given some n-valued semantics $\mathtt{sem}$ and some formula $\varphi \in \mathbb{F}$, it is said that one has a *model* for φ when there is some $\sigma \in \mathtt{sem}$ such that $\sigma(\varphi) \in D^+$; when this is true for every $\sigma \in \mathtt{sem}$ it

[1] All this is common categorial wisdom. On (a), see for example [9], [11], [12], [13]; on (b) see [2], [3], [4], [9], [11], [12], [16]; on (c) just to name but two important texts where this is asserted see [9] and [13]. These claims are so widely endorsed by topos theorists as accurate readings of some definitions and theorems that it is hardly worth documenting, but I have done it just to show that they appear in several major texts by leading category-theorists.

is said that φ is *validated*. A notion of *entailment* given by a *consequence relation* $\models_{\texttt{sem}} \subseteq \wp(\mathbb{F}) X \mathbb{F}$ associated to the semantics $\texttt{sem}$ is then defined by saying that a formula $\varphi \in \mathbb{F}$ *follows from* a set of formulas $\Gamma \subseteq \mathbb{F}$ whenever all models of all formulas of Γ are also models of φ, that is:

$$\Gamma \models_{\texttt{sem}} \varphi \;\text{ iff }\; \text{for all } \sigma \in \texttt{sem},\ \sigma(\varphi) \in D^{+} \;\text{ whenever }\; \sigma(\Gamma) \subseteq D^{+}$$

I will omit the semantics in the index whenever it is clear from the context. Clauses like $\Gamma,\ \varphi,\ \Delta\ \models\ \psi$ are written to denote $\langle\ \Gamma\ \cup\ \varphi\ \cup\ \Delta,\ \psi\ \rangle \in \models$; such clauses can be called *inferences*.

It was early remarked by Tarski that the above notion of consequence may be abstractly axiomatized as follows. For every Γ, φ, Δ, ψ:
(T1) $\Gamma, \varphi, \Delta \Vdash \varphi$ (reflexivity)
(T2) If $\Delta \Vdash \varphi$ then $\Gamma, \Delta \Vdash \varphi$ (monotonicity)
(T3) If $\Gamma, \varphi \Vdash \psi$ and $\Delta \Vdash \varphi$ then $\Gamma, \Delta \Vdash \psi$ (transitivity)

A logic $\mathcal{L}$ can be defined simply as a structure (typically a set of propositions, formulas, sentences or something similar) together with a consequence relation defined over it. Logics respecting axioms (T1)-(T3) are called *Tarskian*. Notice, in particular, that when $\texttt{sem}$ is a singleton, one also defines a Tarskian logic. A theory will be any subset of a logic. A logic given by some convenient structure and a consequence relation $\Vdash$ is said to be *sound* with respect to some given semantics $\texttt{sem}$ whenever $\Gamma \Vdash \varphi$ implies $\Gamma \models_{\texttt{sem}} \varphi$ (that is, $\Vdash\ \subseteq\ \models_{\texttt{sem}}$); and is said to have a *complete* semantics when $\Gamma \models_{\texttt{sem}} \varphi$ implies $\Gamma \Vdash \varphi$ (that is, $\models_{\texttt{sem}}\ \subseteq\ \Vdash$). A semantics which is both sound and complete for a given logic is often called *adequate*. In case a logic $\mathcal{L}$ is characterized by some n-valued semantics, it will be dubbed a *(general) n-valued logic*, where $n = |A|$; $\mathcal{L}$ will be called *finite-valued* if $n < \aleph_0$, otherwise it will be called *infinite-valued*. Note that, as a logic may have different semantical presentations, the 'n' in n-valued is not necessarily unique for $\mathcal{L}$. Given a family of logics $\{\mathcal{L}_i\}_{i \in I}$, where each $\mathcal{L}_i$ is n_i-valued, it is also said that the logic given by $\mathcal{L}\ =\ \cap_{i \in I} \mathcal{L}_i$ is $\mathrm{Max}_{i \in I}(n_i)$-valued (this means that an arbitrary intersection of Tarskian logics is still a Tarskian logic).

Suszko distinguished between the logical values on the one hand, and algebraic values on the other hand. According to Suszko, many-valued logics resulted from a purely *referential* phenomenon, i.e. from the fact that when one defines homomorphisms between a logic $\mathcal{L} = \langle \mathbb{F},\ \models \rangle$ and one of its models, one can associate with an element of $F\ =\ |\mathbb{F}|$ any number of algebraic values, for these homomorphisms are in fact merely admissible reference assignments for F. Said otherwise, they are *algebraic valuations* of $\mathcal{L}$ over the model. Logical values would play a different role, since they are used to define logical consequence. One of them, *TRUE*, is used to define consequence as follows: If every premise is $TRUE$, then so is (at least one of) the conclusion(s). By contraposition, the other logical value also can be used to explain valid semantic consequence: If the (every) conclusion is *NOT TRUE*, then so is at least one of the premises.

Theorem 1.1 (Suszko's Reduction)**.** *Every Tarskian logic is logically two-valued.*

Proof. For any n-valuation σ of a given semantics $\texttt{sem}$ (n), and every consequence relation based on A_n and D_n^{+}, define $A_2\ =\ \{TRUE, NOT\ TRUE\}$ and $D_2^{+}\ =\ \{TRUE\}$ and set the characteristic total function $b_\sigma : \mathbb{F} \longrightarrow A_2$ to be such that $b_\sigma(\varphi) = TRUE$ if and only if $\sigma(\varphi) \in D_2^{+}$. Now, collect all such bivaluations b_σ's into a new semantics $\texttt{sem}(2)$, and notice that $\Gamma \models_{\texttt{sem}(2)} \varphi$ if and only if $\Gamma \models_{\texttt{sem}(n)} \varphi$. $\square$

Reductive results similar in spirit to Suszko's were presented independently by other logicians, for example Newton da Costa (see e.g. [10]), Dana Scott (cf. [20], [21]) and Richard Routley and Robert K. Meyer [19]. Moreover, there is a family of akin results of different strengths under the label "Suszko's reduction". *Suszko's reduction* in rigor, required from the logic not only to be reflexive, transitive and monotonic, but also *structural*. *Suszko-da Costa's reduction*, which is closer to what I expound here, dropped the structurality requisite. *Suszko-Béziau's reduction* only requires reflexivity from the logic (cf. [22]).

Suszko declared that many-valuedness is "a magnificent conceptual deceit" and he claimed that "(...) there are but two logical values, true and false (...)". This claim is now called *Suszko's thesis* and can be stated more dramatically as "All logics are bivalent" or "Many-valued logics do not exist at all". Reductive results, especially the strongest form (Suszko-Béziau's reduction) seem to be overwhelming evidence in favor of Suszko's thesis because virtually all logics regarded as such are in the scope of this theorem.

A possible way to resist Suszo's thesis is by extending the scope of logics to cover non-Tarskian logics, especially to non-reflexive ones to avoid Suszko-Béziau's reduction, and this reply is what I will discuss more extensively after presenting Suszko's reduction in a categorial setting.

2. Suszkoing toposes...

A *(standard) topos* is a category ${}_S\mathcal{E}$ with equalizers, (binary) products, coequalizers, coproducts, exponentials, and a morphism ${}_S true : \mathbf{1} \longrightarrow {}_S\Omega$, called *subobject classifier*, has the following property:

Comprehension axiom. For each ${}_S\varphi : O \longrightarrow {}_S\Omega$ there is an equalizer of ${}_S\varphi$ and ${}_S true_O$, and each monic $m : M \rightarrowtail O$ is such an equalizer for a unique ${}_S\varphi$. In diagrams, ${}_S true$ is such that for every ${}_S\varphi$ and every object T and morphism $o : T \longrightarrow O$, if $m \circ_S \varphi = m \circ_S true_O$ and $x \circ_S \varphi = x \circ_S true_O$, then there is a unique $h : X \longrightarrow M$ that makes the diagram below commutative:

$$\begin{tikzcd} M \arrow[r, tail, "m"] & O \arrow[r, shift left, "{}_S\varphi"] \arrow[r, shift right, "{}_S true_O"'] & {}_S\Omega \\ & X \arrow[ul, "h"] \arrow[u, "x"'] & \end{tikzcd}$$

Example 2.1. *Let **Set** be the (standard) category of (abstract constant) sets as objects and functions as morphisms. ${}_S\Omega_{\boldsymbol{Set}}$ has only two elements with the order ${}_S false_{\boldsymbol{Set}} < {}_S true_{\boldsymbol{Set}}$. Hence, in this category ${}_S\Omega_{\boldsymbol{Set}} = \boldsymbol{2}_{S\boldsymbol{Set}}$. Thus, for every element t of O, $t : \boldsymbol{1} \longrightarrow O$, $t \in O$ if and only if ${}_S\varphi \circ t = {}_S true_{\boldsymbol{Set}}$, and $t \notin O$ if and only if ${}_S\varphi_m \circ t = {}_S false_{\boldsymbol{Set}}$, since ${}_S false_{\boldsymbol{Set}}$ is the only morphism distinct from ${}_S true_{\boldsymbol{Set}}$.*

Example 2.2. *${}_S S^{\downarrow\downarrow}$ is the category of (standard irreflexive directed multi-)* graphs *and graph structure preserving maps.*[2] *An object of ${}_S S^{\downarrow\downarrow}$ is any pair of sets equipped with a parallel pair of maps $A \underset{t}{\overset{s}{\rightrightarrows}} V$ where A is called the set* arrows *and V is the*

[2]Nice introductions to this category can be found in [23] and [12].

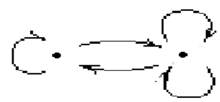

Figure 1

set of dots *(or* nodes *or* vertices*). If a is an element of A (an arrow), then $s(a)$ is called the* source *of a, and $t(a)$ is called the* target *of a.*

Morphisms of ${}_S\boldsymbol{S}^{\downarrow\downarrow}$ are also defined so as to respect the graph structure. That is, a morphism $f:(A \overset{s}{\underset{t}{\rightrightarrows}} V) \longrightarrow (E \overset{s'}{\underset{t'}{\rightrightarrows}} P)$ in ${}_S\boldsymbol{S}^{\downarrow\downarrow}$ is defined to be any pair *of morphisms of* **Set** *$f_a : A \longrightarrow V$, $f_v : E \longrightarrow P$ for which both equations*

$$f_v \circ s = s' \circ f_a$$

$$f_v \circ t = t' \circ f_a$$

*are valid in ${}_S$**Set**. It is said that f preserves the structure of the graphs if it preserves the source and target relations.*

A terminal object in this category, $\boldsymbol{1}_{{}_S S^{\downarrow\downarrow}}$, is any arrow such that its source and target coincide.

This topos provides a simple yet good example of a truth values object with more than two elements. ${}_S\Omega_{S^{\downarrow\downarrow}}$ has the form of a graph like that in Figure 1 above. There are exactly three morphisms $\boldsymbol{1}_{{}_S S^{\downarrow\downarrow}} \longrightarrow {}_S\Omega_{S^{\downarrow\downarrow}}$ in this category, which means that ${}_S\Omega_{S^{\downarrow\downarrow}}$ has three truth values with the order ${}_Sfalse_{S^{\downarrow\downarrow}} < {}_S\binom{s}{t}_{S^{\downarrow\downarrow}} < {}_Strue_{S^{\downarrow\downarrow}}$.

The **Comprehension axiom** enables us to define the more usual standard connectives as operations on ${}_S\Omega$ in a way that they come with certain truth conditions.[3] For the purposes of this paper only the following ones are needed:

Negation. $\neg : {}_S\Omega \longrightarrow {}_S\Omega$

$\neg p = {}_Strue$ if and only if $p = {}_Sfalse$, otherwise $\neg p = {}_Sfalse$

Disjunction. $\vee : {}_S\Omega \times {}_S\Omega \longrightarrow {}_S\Omega$

$(p \vee q) = \sup(p, q)$

Implication. $\Rightarrow : {}_S\Omega \times {}_S\Omega \longrightarrow {}_S\Omega$

$(p \Rightarrow q) = {}_Strue$ if and only if $p \leq q$, otherwise $(p \Rightarrow q) = q$[4]

The *internal logic* of a standard topos will be the algebra of operations of ${}_S\Omega$, that is, the algebra of operations of its object of truth values.[5] In general, ${}_S\Omega$ has more than two elements, and that is why it is said that the internal logic of a topos is in general many-valued. There is a theorem establishing necessary and sufficient conditions for a proposition ${}_Sp$ being the same morphism as ${}_Strue$ in a given standard topos ${}_S\mathcal{E}$. Let '$\models_I$' indicate that logical consequence gives the results as in intuitionistic logic. Then the following theorem holds:

[3] For details see for example [9, § 6.6].

[4] sup and $\leq$ here are relative to the partial order formed by the elements of ${}_S\Omega$ (morphisms with codomain ${}_S\Omega$).

[5] This logic is called *internal* because (i) it is formulated exclusively in terms of the objects and morphisms of the topos in question and (ii) it it is the right to reason about the topos in question, since it is determined by the definition of its objects and morphisms. It is not a canon imposed "externally" to reason about the topos: Using a different logic for that purpose would alter the definitory properties of those objects and morphisms and thus it would not be a logic for the intended objects and morphisms.

Theorem 2.3. *For every proposition* ${}_Sp$, $\models_{{}_S\mathcal{E}} {}_Sp$ *for every topos* ${}_S\mathcal{E}$ *if and only if* $\models_I {}_Sp$.

i.e. ${}_S\Omega$ is a Heyting algebra.[6] Hence the commonplace that the internal logic of a topos is, in general, intuitionistic.

Example 2.4. *The internal logic of* ${}_S\boldsymbol{Set}$ *is classical. For example, in* ${}_S\boldsymbol{Set}$, *every proposition* p *is the same as one and only one of* ${}_Strue_{\boldsymbol{Set}}$ *and* ${}_Sfalse_{\boldsymbol{Set}}$. $\neg \circ_S true_{\boldsymbol{Set}} =_S false_{\boldsymbol{Set}}$ *and* $\neg \circ_S false_{\boldsymbol{Set}} =_S true_{\boldsymbol{Set}}$. *Hence, for any* p, $\neg\neg p = p$. *Also, for any* p $(p \vee \neg p) = \vee \circ \langle p, \neg p\rangle = \sup(p, \neg p) =_S true_{\boldsymbol{Set}}$.

Example 2.5. *In* ${}_SS^{\downarrow\downarrow}$, *negation gives the following identities of morphisms:*

$$\neg\ {}_Strue_{S^{\downarrow\downarrow}} =\ {}_Sfalse_{S^{\downarrow\downarrow}},\quad \neg\ {}_S\tbinom{s}{t}_{S^{\downarrow\downarrow}} = false_{S^{\downarrow\downarrow}},\quad \neg\ {}_Sfalse_{S^{\downarrow\downarrow}} =\ {}_Strue_{S^{\downarrow\downarrow}}$$

Since $(p \Rightarrow q) = {}_Strue$ *if and only if* $(p \wedge q) = p$, *in general* $(\neg\neg p \Rightarrow p) \neq {}_Strue$ *in* $S^{\downarrow\downarrow}$ *because even though* $(\neg\neg p \Rightarrow p) = {}_Strue_{S^{\downarrow\downarrow}}$ *either when* $p = {}_Strue_{S^{\downarrow\downarrow}}$ *or when* $p = {}_Sfalse_{S^{\downarrow\downarrow}}$, $(\neg\neg p \wedge p) \neq \neg\neg p$ *when* $p = {}_S\tbinom{s}{t}_{S^{\downarrow\downarrow}}$. *Given that* $(\neg\neg p \Rightarrow p) \neq {}_Strue_{S^{\downarrow\downarrow}}$ *but there is no formula* Φ *such that* $\Phi = true$ *in classical logic and* $\Phi = false$ *in intuitionistic logic,* $(\neg\neg p \Rightarrow p) = {}_S\tbinom{s}{t}_{S^{\downarrow\downarrow}}$ *when* $p = {}_S\tbinom{s}{t}_{S^{\downarrow\downarrow}}$. *Moreover,* $p \vee \neg p$ *fails to be the same morphism as* ${}_Strue_{S^{\downarrow\downarrow}}$ *since* $(p \vee q) = {}_Strue$ *if and only if either* $p = {}_Strue$ *or* $q = {}_Strue$. *If* $p = {}_S\tbinom{s}{t}_{S^{\downarrow\downarrow}}$, $\neg p = {}_Sfalse_{S^{\downarrow\downarrow}}$, *so neither* $p = {}_Strue_{S^{\downarrow\downarrow}}$ *nor* $\neg p = {}_Strue_{S^{\downarrow\downarrow}}$ *and hence* $(p \vee \neg p) \neq {}_Strue_{S^{\downarrow\downarrow}}$.

The underlying idea behind Suszko's reduction can be expressed in terms of morphisms and compositions as follows. The internal logic of a standard topos is said to be *algebraically* n*-valued* if there are n distinct morphisms $\mathbf{1} \longrightarrow {}_S\Omega$ in the given standard topos. As reductive results have shown, an algebraically n-valued Tarskian logic in general is not *logically* n-valued. Accordingly, the internal logic of a topos is said to be *logically* m*-valued* if its notion of consequence implies that there are m distinct values and this can be internalized in the topos. I give here a definition for the case $m = 2$ and suggest a more general definition of logical many-valuedness for the internal logic of a topos in section 6.

Logical consequence in a topos is assumed to be traditional, Tarskian consequence. q is a consequence of premises Γ if *true* is preserved from premises to the conclusion and is not a consequence if the premises are the same morphism as *true* but the conclusion is not. A *theorem* is a consequence of an empty set of premises, i.e. if it is a morphism which is the same morphism as *true*. A non-theorem is a morphism which is different from *true*. But the *two* values *true* and *not true* (or *untrue*, etc.) are the only values required to define (Tarskian) consequence. Let us give categorial content to this Suszkian distinction.

Definition 2.6. *In a non-degenerate category* $\boldsymbol{C}$ *with subobject classifier, a* Suszkian logical truth values object, *or* Suszkian object *for short, is an object* $\mathscr{S}$ *such that there are exactly two morphisms* $\boldsymbol{1} \underset{\delta^-}{\overset{\delta^+}{\rightrightarrows}} \mathscr{S}$ *and a morphism* $sep : {}_S\Omega \longrightarrow \mathscr{S}$ *such that sep is the unique morphism which satisfies the following properties:*
(S1) $sep \circ p = \delta^+$ *if* $p =_S true$, *and*

[6] In rigor, ${}_Sp$ is a morphism which corresponds to a formula $({}_Sp)^*$ in a possibly different language, but there is no harm if one identifies them, hence the abuse of notation. A proof can be found in [9, see §8.3 for the soundness part and §10.6 for the completeness part].

(S2) $sep \circ p = \delta^-$ *if* $p \neq_S true$

The morphisms δ^+ *and* δ^- *can be collectively denoted by* biv *and are called a Suszkian bivaluation. Thus, the diagram below commutes according to the above definition of* biv *and the conditions (S1) and (S2):*

$$\begin{tikzcd} \mathbf{1} \arrow[r, "p"] \arrow[dr, "biv"'] & {}_S\Omega \arrow[d, "sep"] \\ & \mathscr{S} \end{tikzcd}$$

Theorem 2.7. *A Suszkian object is unique up to isomorphism.*

Proof. Suppose that $\mathscr{S}$ is a Suszkian object and that there is an isomorphism $f : \mathscr{S}' \longrightarrow \mathscr{S}$. Given any two morphisms $x, y : \mathbf{1} \longrightarrow \mathscr{S}'$, collectively denoted biv', one has $f \circ biv' : \mathbf{1} \longrightarrow \mathscr{S}$. By exactness, $biv = f \circ biv'$, and so $biv' = f^{-1} \circ biv$. Thus, $f^{-1} \circ biv$ are the only two arrows from $\mathbf{1}$ to $\mathscr{S}'$. By a similar reasoning, $sep' = f^{-1} \circ sep$. The exactness and uniqueness conditions imply that $\mathscr{S}'$, biv' and sep' satisfy (S1) and (S2), so $\mathscr{S}'$ is also a Suszkian object.

Now suppose that both $\mathscr{S}$ and $\mathscr{S}'$ are Suszkian objects. Suppose now that there is a morphism $f : \mathscr{S}' \longrightarrow \mathscr{S}$ such that $f \circ biv' = f \circ (sep' \circ \varphi) = sep \circ \varphi = biv$. Such an f is unique. Take now a morphism $g : \mathscr{S} \longrightarrow \mathscr{S}'$. Then $f \circ (g \circ biv) = (f \circ g) \circ biv = biv$. But $1_{\mathscr{S}}$ is the unique morphism from $\mathscr{S}$ to $\mathscr{S}$ such that $1_{\mathscr{S}} \circ biv = biv$. Hence $f \circ g = 1_{\mathscr{S}}$. By an analogous reasoning $g \circ f = 1_{\mathscr{S}'}$. Then, f is an isomorphism. $\square$

Remark 2.8. *As a consequence of the definition, there is no morphism* $\psi : \mathbf{1} \longrightarrow \mathscr{S}$ *such that* $\psi \in \delta^+$ *and* $\psi \in \delta^-$*. However, this does not mean that* $\cap \circ (\delta^+, \ \delta^-) = \varnothing$*. This implies that, in general,* $\mathscr{S}$ *is not isomorphic to* $\mathbf{2}$ *in spite of having exactly two morphisms* $\mathbf{1} \longrightarrow \mathscr{S}$ *(nonetheless, it is easily proved that a Suszkian object and* ${}_S\Omega$ *are isomorphic for example in* ***Set****).*

From the very definition of a Suszkian object, for every proposition φ, either $sep \circ \varphi = \delta^+$ or $sep \circ \varphi = \delta^-$. Now, for every theorem Φ, $sep \circ \Phi = \delta^+$, and for every non-theorem Ψ, $sep \circ \Psi = \delta^-$. This justifies the definition of a Suszkian object, but the difficult part is proving the following

Theorem 2.9. *Every (non-degenerate) topos has a Suszkian object.*

which guarantees that the internal logic of a topos is completely characterized by a Suszkian object. First we need to prove the following

Lemma 2.10. *In every standard non-degenerate topos there is an object* $\mathscr{S}$ *such that there are exactly two morphisms from a terminal object* $\mathbf{1}$ *to* $\mathscr{S}$*.*

Proof. Suppose there is no object such that it is the codomain of exactly two morphisms with domain a terminal object $\mathbf{1}$. One has thus three options:
(a) No object X is the codomain of a morphism with domain $\mathbf{1}$.
(b) Every object X is the codomain of only one morphism with domain $\mathbf{1}$.
(c) Every object X is the codomain of three or more distinct morphisms with domain $\mathbf{1}$.

There must be at least the identity morphism for $\mathbf{1}$, hence there is a morphism with domain $\mathbf{1}$, so (a) is impossible. According to (b), ${}_S\Omega$ would have only one morphism $\mathbf{1} \longrightarrow {}_S\Omega$. But then ${}_S true = {}_S false$ and hence the topos would be degenerate, contrary to the hypothesis. Regarding (c), the (domain of the) subobject induced by two of those morphisms would provide the required object. $\square$

Now it is possible to prove THEOREM 2.9:

Proof. Let $\mathscr{S}$ be an object of ${}_S\mathcal{E}$ together with exactly two different morphisms $\mathbf{1} \underset{\delta^-}{\overset{\delta^+}{\rightrightarrows}} \mathscr{S}$. Then there is exactly one morphism $sep\colon {}_S\Omega \longrightarrow \mathscr{S}$ such that makes the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, tail, "m"] & O \arrow[r, "{}_S\varphi"] & {}_S\Omega \arrow[d, "sep"] \\
& \mathbf{1} \arrow[u, "a"] \arrow[ur, "{}_S true"'] \arrow[r, "\delta^+"'] & \mathscr{S}
\end{tikzcd}$$

This means that a standard non-degenerate topos has a Suszkian object. $\square$

Remark 2.11. *Unlike a subobject classifier, a Suszkian object does not necessarily classify subobjects and it does not necessarily count them, either, for it collapses every other proposition different from ${}_S true$ into δ^-. A Suszkian object provides a bivaluation $biv = sep \circ p$ for ${}_S\Omega$, i.e. it says whether a proposition is logically true or not, full stop.*

Example 2.12. *Consider the three truth values in ${}_S S^{\downarrow\downarrow}$. Then $sep \circ_S true_{S^{\downarrow\downarrow}} = \delta^+$ and $sep \circ_S false_{S^{\downarrow\downarrow}} = sep \circ_S \binom{s}{t}_{S^{\downarrow\downarrow}} = \delta^-$. Hence, for example, $sep \circ (\neg\neg p \Rightarrow p) = \delta^-$, for $(\neg\neg p \Rightarrow p) =_S \binom{s}{t}_{S^{\downarrow\downarrow}}$ when $p =_S \binom{s}{t}_{S^{\downarrow\downarrow}}$. Something similar happens with $(p \vee \neg p)$.*

Remark 2.13. *The internal logic of complement-toposes (cf.* [17], [18]*) is Tarskian, too. Even though the subobject classifier and the connectives are described in a different, dual way, the notion of consequence in the internal logic of complement-toposes is the same as that of (ordinary or standard) toposes. Therefore, the Suszkian object can be defined in the same way as it was done here for ordinary toposes. Moreover, the proof of* THEOREM 2.9 *can be copied almost verbatim for complement-toposes (but of course the morphisms equal to "true" are different in ordinary and complement-toposes; their logics have been differently, dually described after all!). This allows us to speak of toposes simpliciter, ignoring for the rest of the paper whether they are standard or not unless otherwise indicated.*

3. ... AND DE-SUSZKOING THEM?

There are is least one more notion of logical consequence which in general is not characterizable by a bivalent semantics and that could introduce interesting complications in the theory of the internal logic of toposes. Consider Malinowski's Q-consequence ("Q" for "Quasi"; cf. [14], [15]):
Q-*consequence.* q is a logical Q-consequence from premises Γ, in symbols $\Gamma \models^Q q$, if and only if any case in which each premise in Γ is not antidesignated is also a case in which q is designated. Or equivalently, there is no case in which each premise in Γ is antidesignated, but in which q fails to be designated.

Thus logical many-valuedness in a topos could be obtained at a different level, by taking it into account from the very characterization of logical consequence. However, this would result in a change in the description of the internal logic, for it would be no longer intuitionistic. The Tarskian properties are indissolubly tied to the canonical characterizations of consequence, but Q-consequence is non-Tarskian: it is not reflexive. Let me exemplify how radical the change would be if Q-consequence is adopted instead of the Tarskian one.

Theorems are those propositions which are consequences of an empty set of premises, so theorems are propositions that are always designated. This is just the usual notion of theoremhood, but whether Q-consequence affects the collection of theorems depends on what are the designated values, because one has to choose by hand, as it were, what are the designated, antidesignated and neither designated nor antidesignated values. If ${}_{\mathcal{E}}true$ is the only designated value as usual, the theorems of the internal logic are the same whether Tarskian or Q-consequence is assumed.

Nonetheless, Q-consequence does affect the validity of inferences even if ${}_{\mathcal{E}}true$ is the only designated value. Unlike Tarskian consequence, Q-consequence is not reflexive. For example, let us assume that ${}_{S}true_{S^{\downarrow\downarrow}}$ is the only designated value in ${}_{S}\mathbf{S}^{\downarrow\downarrow}$ and that ${}_{S}false_{S^{\downarrow\downarrow}}$ is the only antidesignated value. Suppose that $p =_{S} \binom{s}{t}_{S^{\downarrow\downarrow}}$. Then $p \not\models^{Q}_{{}_{S}S^{\downarrow\downarrow}} p$, because Q-consequence requires that if premises are not antidesignated, conclusions must be designated, which is not the case in this example.

4. Other variations in the internal logic

Consider now Frankowski's P-consequence ("P" for "Plausible"; cf. [7], [8]):

P-*consequence*. q is a logical P-consequence from premises Γ, in symbols $\Gamma \models^{P} q$, if and only if any case in which each premise in Γ is designated is also a case in which φ is not antidesignated. Or equivalently, there is no case in which each premise in Γ is designated, but in which q fails to be antidesignated.

In general, P-consequence does affect the collection of theorems. Since theorems are those propositions which are consequences of an empty set of premises, theorems according to P-consequence are those propositions that are not antidesignated. So theorems of the internal logic are not the same as those when Tarskian or Q-consequence are assumed even if ${}_{\mathcal{E}}true$ is taken as the only designated value. For example, let us assume as above that ${}_{S}true_{S^{\downarrow\downarrow}}$ is the only designated value in ${}_{S}\mathbf{S}^{\downarrow\downarrow}$ and that ${}_{S}false_{S^{\downarrow\downarrow}}$ is the only antidesignated value. $p \vee \neg p$ would be a theorem because there is no case in which it is antidesignated.

P-consequence affects also the validity of inferences. Remember that unlike Tarskian consequence, P-consequence is not transitive. Suppose that $p = {}_{S}true_{S^{\downarrow\downarrow}}$, $q = {}_{S}\binom{s}{t}_{S^{\downarrow\downarrow}}$ and $r = {}_{S}false_{S^{\downarrow\downarrow}}$. Thus $p \models^{P}_{{}_{S}S^{\downarrow\downarrow}} q$ and $q \models^{P}_{{}_{S}S^{\downarrow\downarrow}} r$, but $p \not\models^{P}_{{}_{S}S^{\downarrow\downarrow}} r$, because P-consequence requires that if premises are designated, conclusions must be not antidesignated, which is not the case in this example.[7]

5. Is this logical consequence?

An obvious worry at this point is whether these strange notions of consequence are notions of logical consequence at all. I sketch five arguments supporting the idea that they do not lead us so far off the usual business of logic.

[7]Similar arguments to the above can be given to show how Q- and P-consequence modify the internal logic of a complement-topos.

First, one of the non-Tarskian notions of consequence is well-known, non-monotonic logic, but non-reflexive and non-transitive consequence relations are not so popular. However, there is no prima facie reason as to why it is possible to do without monotonicity but not without reflexivity or transitivity: As properties of a relation of logical consequence they seem to be *pari passu*.

Beall and Restall [1] say that they are not on equal footing "because preservation of designated values (from premises to conclusions) is a reflexive and transitive relation". This begs the question, for (and this is a second argument for non-Tarskian logics) they are precisely asking for ways of logically connecting premises and conclusions others than preservation of designated values.

Third, designatedness is a case of non-antidesignatedness, so Tarskian consequence is a case of a more general notion which also encompasses Q- and P-consequence: *Preservation of non-antidesignatedness.* For example, Q-consequence deals with preservation of non-antidesignated values but in such a strong way that it rather forces passing from non-antidesignated values to designated values. Similarly, P-consequence is preservation of non-antidesignated values but in such a weak way that it allows passing from designated values to some non-designated values (but never from designated to antidesignated values!). Let me call this notion of consequence *TMF-consequence* (for Tarski, Malinowski and Frankowski) and define it as follows: q is a logical TMF-consequence from premises Γ, in symbols $\Gamma \models^{TMF} q$, if and only if any case in which each premise in Γ is not antidesignated is also a case in which q is not antidesignated. Or equivalently, there is no case in which each premise in Γ is not antidesignated, but in which q fails to be antidesignated. Again equivalently, if there is a case in which q is antidesignated, then at least one premise in Γ is also antidesignated.

Fourth, a good signal that we are not very far of logic is that under minimal classical constraints on the structure of truth values, these notions of consequence are indistinguishable from the traditional one. If there are only two truth values, *true* and $false$ with their usual order, the collections of designated and antidesignated values exhaust all the possible values, designated = not antidesignated and not designated = antidesignated. But if this were a feature merely of classical logic, surely Q- and P-consequence would have arisen before they did. However, these notions of consequence collapse if the collections of designated and antidesignated values are supposed to be mutually exclusive and collectively exhaustive with respect to the total collection of values given, as is assumed in most popular logics.

Finally, another good signal that we are still in the business of logic is that if one uses cognitive states like acceptance and rejection to define validity, these different notions of consequence arise almost naturally, for example:

- An argument is TMF-valid if and only if, if the premises are not rejected, then conclusions are also not rejected. Equivalently, if the conclusions are rejected, premises are rejected too. The relevant dialogical properties are those *not rejected*, so there is more than one property that can be forwards-preserved. The preservation of these properties is required, but the direction is not important since *not rejected* determines another property, whose name depends of what the components of the first property are taken to be (for example, it can be *accepted* or *either accepted or undecided*), which makes the collections of properties mutually exclusive (but probably not collectively exhaustive).

- An argument is Tarskian-valid if and only if, if the premises are accepted, then the conclusions are also accepted. Equivalently, if the conclusions are non-accepted, the premises are non-accepted either. The relevant dialogical property is *accepted*, so there is only one property forwards-preserved. That property determines a property, *not accepted*, which makes the collections of properties mutually exclusive and collectively exhaustive.

Similar variations can be obtained using also probabilistic, conceivability, epistemic or whatnot specifications of consequence. Thus, the easy answer to the question of this section is that, at least technically, non-Tarskian notions of consequence such as Q- and P-consequence are as legitimate as non-classical logics are. More elaborate answers could be given along the lines of non-monotonic logics. For example, Q-consequence would serve to "tie" to conclusions more certain than the premises, whereas P-consequence would allow to "jump" to conclusions less certain than the premises. Thus, the route of exploring notions of logical consequence other than the Tarskian one provides a way to give Suszko's thesis up.[8]

6. Logical m-valuedness

A problem at this point is whether those other notions of consequence can be internalized in a topos. Without trying to settle that question here, I will probe an idea at least for consequences based on some form of forwards-preservation.

A topos is said to be *logically m-valued* if
(1) the assumed notion of consequence, $\Vdash$, implies that there are m logical values;
(2) there is an object $\mathfrak{V}$ such that it is the codomain of exactly m morphisms with domain $\mathbf{1}$ such that to each logical value implied by $\Vdash$ corresponds one and only one morphism from $\mathbf{1}$ to $\mathfrak{V}$; and
(3) there is a unique morphism $\mathfrak{sep}: \Omega \longrightarrow \mathfrak{V}$ such that $\mathfrak{sep}$ satisfies the following properties:

(3.1) for every $\delta_i : \mathbf{1} \longrightarrow \mathfrak{V}$ there is a $p : \mathbf{1} \longrightarrow \Omega$ such that $\mathfrak{sep} \circ p = \delta_i$

(3.2) If $p \Vdash q$ implies that p and q have certain $\Vdash$-logical values v_i and v_j, respectively, then if $\mathfrak{sep} \circ p = \delta_i$, $\mathfrak{sep} \circ q = \delta_j$ (where δ_i corresponds to v_i and δ_j corresponds to v_j).

The morphisms $\delta_1, \ldots \delta_m$ can be collectively denoted by m-val and are called *a logical m-valuation (based on $\Vdash$)*. Thus, the diagram below commutes according to the definition of m-val just given and the conditions (1)-(3):

$$\begin{tikzcd} \mathbf{1} \arrow[r, "p"] \arrow[dr, "m-val"'] & \Omega \arrow[d, "\mathfrak{sep}"] \\ & \mathfrak{V} \end{tikzcd}$$

The morphism δ_i such that $\mathfrak{sep} \circ_{\mathcal{E}} true = \delta_i$ will be called "morphism of designated values" and will be denoted "δ^+".

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[8][24] is a good source of inspiration for other notions logical consequence. Abstraction on the notions of logical consequence could go further up to a definition of a logical structure analogous to that of an algebraic structure given in Universal Algebra such that other notions of consequence and particular logics appear as specifications of that structure: That is the project of Universal Logic, see [5] for an introduction. However, I will stop generalization here because of limitations of space and because it has been enough to show that the many-valuedness of topos logic is not an easy issue.

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