Forcing and the Omitting Type Theorem, institutionally

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Abstract. In the context of proliferation of many logical systems in the area of mathematical logic and computer science, we present a generalization of forcing in institution-independent model theory which is used to prove an abstract Omitting Types Theorem (OTT). We instantiate this general result to many first-order logics, which are, roughly speaking, logics whose sentences can be constructed from atomic formulae by means of Boolean connectives and classical first-order quantifiers. These include first-order logic (**FOL**), logic of order-sorted algebra (**OSA**), preorder algebra (**POA**), partial algebras (**PA**), as well as their infinitary variants $\mathbf{FOL}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$, $\mathbf{POA}_{\omega_1,\omega}$, $\mathbf{PA}_{\omega_1,\omega}$. In addition to the first technique for proving the OTT, we develop another one, in the spirit of institution-independent model theory, which consists of borrowing the Omitting Types Property (OTP) from a simpler institution across an institution comorphism. As a result we export the OTP from **FOL** to first-order partial algebras (**FOPA**) and higher-order logic with Henkin semantics (**HNK**), and from the institution of $\mathbf{FOL}_{\omega_1,\omega}$ to $\mathbf{FOPA}_{\omega_1,\omega}$ and $\mathbf{HNK}_{\omega_1,\omega}$. The second technique successfully extends the domain of application of OTT to (non classical) logical systems for which the standard methods may fail.

Introduction

A type is a set of formulas in finite number of variables. A type Δ with free variables $x_1 \ldots x_n$ is principal for a theory T if there exists a finite set of formulas p with free variables $x_1 \ldots x_n$ such that $T \cup p \models \Delta$. The classical OTT states that if T is a complete theory and $(\Delta_n : n \in \mathbb{N})$ is a sequence of non-principal types of T, then there is a model of T omitting all the types Δ_n. The OTT was proved by Henkin [31] and Orey [38] using the method of diagrams and it has many applications in classical model theory. Forcing is a technique invented by Paul Cohen, for proving the independence of the continuum hypothesis from the other axioms of Zermelo-Fraenkel set theory [12,13]. A. Robinson [42] developed an analogous theory of forcing in model theory, and Barwise [6] extended Robinson's theory to infinitary logic and used it to give a new proof of the OTT. An early contribution on forcing and the omitting types for infinitary logic $\mathcal{L}_{\omega_1,\omega}$ is [32]. For recent developments of the result see [2,3,4].

The framework adopted here is the theory of institutions [23] which is a category-based formalization of the intuitive notion of logical system. Institutions constitute a meta-theory on logical systems similar to how universal algebra constitute a meta-theory for groups and rings. The institution theory arose within computing science, by abstracting away from the realities of conventional logics, with the ambition of doing as much as possible at the level of abstraction, independent of the details of any particular logical system. In addition to the large use in algebraic specification theory where institutions became the most fundamental mathematical structure underlying the formal specification languages, there have been substantial developments towards an abstract institutional model theory [44,46,15,18,16,28,40,29,39,11,27]. See [19] for a monography dedicated to this topic.

The present paper studies the abstract notions of OTP and forcing in the framework of institutions, and points out many particular instances to concrete logics. In institutional model theory the forcing technique was introduced in [27] and it was used to prove an abstract first-order completeness theorem. The result was obtained as a consequence of the research on a syntactic forcing property. In the present paper we investigate a semantic forcing property which was studied in classical model theory by Robinson, Barwise and Keisler. As an outcome of this investigation we obtain institution-independent versions of some well-known results in classical model theory:

1. Downward Löwenheim-Skolem Theorem (”any consistent theory has a countable model”), and
2. Omitting Types Theorem (“any non-principal type has a model which omits it”).

The categorical assumptions used here are easy to check in concrete logics such that the abstract theorems can be instantiated to many institutions, some of them described explicitly here, and others just mentioned. Another advantage of the present research is the applicability to both finitary and infinitary cases which is due to the use of forcing technique.

However, there are examples of more refined institutions which cannot be cast in this abstract framework and for which we believe that the standard methods of proving OTT cannot be replicated. Therefore, in addition to the first technique for establishing OTP, we develop another one, in the spirit of institutional model theory. Instead of developing directly the result within a given institution, one may “borrow” it from a simpler institution via an adequate encoding, expressed as institution *comorphisms* [26]. More concretely, here we prove a generic theorem for OTP along an institution comorphisms $I \to I'$ such that if I' has the OTP and the institution comorphism is *conservative*, then I can be established to have the OTP. We illustrate the applicability of our borrowing result with examples: we “export” the OTP from first-order logic to higher order logic with Henkin semantics and from the first-order logic to first-order partial algebras.

The paper is organized as follows. The first technical section introduces the institution theoretic preliminaries and recalls necessary fundamental concepts of institution-independent model theory such as internal logic, basic sets of sentences, reachable models. The next section recalls the forcing technique of institutional model theory. In Section 3 we develop an institution-independent version of the OTT applicable uniformly to both finitary and inifinitary cases. Section 4 studies the translation of the OTP along the institution comorphisms and illustrates its applicability power with examples which cannot be captured in the previous abstract setting. Section 5 concludes the paper and discusses the future work.

We assume that the reader is familiar with the basic notions of category theory. See [33] for the standard definitions of category, functor, pushout, etc., which are omitted here.

1 Institutions

The concept of institution formalizes the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [23].

Definition 1. *An* institution $I = (\mathbb{S}ig^I, \mathbb{S}en^I, \mathbb{M}od^I, \models^I)$ *consists of*

1. *a category* $\mathbb{S}ig^I$*, whose objects are called* signatures*,*
2. *a functor* $\mathbb{S}en^I : \mathbb{S}ig^I \to \mathbb{S}et$*, providing for each signature a set whose elements are called* (Σ-)sentences*,*
3. *a functor* $\mathbb{M}od^I : (\mathbb{S}ig^I)^{op} \to \mathbb{CAT}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)morphisms*,*
4. *a relation* $\models^I_\Sigma \subseteq |\mathbb{M}od^I(\Sigma)| \times \mathbb{S}en^I(\Sigma)$ *for each signature* $\Sigma \in |\mathbb{S}ig^I|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \to \Sigma'$ *in* $\mathbb{S}ig^I$*, the following* satisfaction condition *holds:*

$$M' \models^I_{\Sigma'} \mathbb{S}en^I(\varphi)(e) \text{ iff } \mathbb{M}od^I(\varphi)(M') \models^I_\Sigma e$$

for all $M' \in |\mathbb{M}od^I(\Sigma')|$ *and* $e \in \mathbb{S}en^I(\Sigma)$*.*

We denote the *reduct* functor $\mathbb{M}od^I(\varphi)$ by $\_\restriction_\varphi$ and the sentence translation $\mathbb{S}en^I(\varphi)$ by $\varphi(\_)$. When $M = M'\restriction_\varphi$ we say that M is a φ-reduct of M' and M' is a φ-expansion of M. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^I$ may be simply denoted by $\mathbb{S}ig$.

*Example 1 (First Order Logic (***FOL***) [23]).* Signatures are first-order many-sorted signatures (with sort names, operation names and predicate names); sentences are the usual closed formulae of first-order logic built over atomic formulae given either as equalities or atomic predicate formulae; models are the usual first-order structures; satisfaction of a formula in a structure is defined in the standard way.

*Example 2 (Preorder Algebra (***POA***) [21,22]).* The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

*Example 3 (Order Sorted Algebra (**OSA**) [25]).* An order-sorted signature $(S,\leq,F)$ consist of an algebraic signature (S,F) with a partial ordering $(S,\leq)$ such that the following *monotonicity* condition is satisfied: $\sigma \in F_{w_1\to s_1} \cap F_{w_2\to s_2}$ and $w_1 \leq w_2$ imply $s_1 \leq s_2$. A morphism of **OSA**-signatures $\varphi : (S,\leq,F) \to (S',\leq',F')$ is just a morphism of algebraic signatures $(S,F) \to (S',F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$. Given an order-sorted signature $(S,\leq,F)$, an order-sorted $(S,\leq,F)$-algebra is a (S,F)-algebra M such that $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and $\sigma \in F_{w_1\to s_1} \cup F_{w_2\to s_2}$ and $w_1 \leq w_2$ imply $M_\sigma^{w_1,s_1} = M_\sigma^{w_2,s_2}|_{M_{w_1}}$. Given two order-sorted $(S,\leq,F)$-algebras M and N, an order-sorted $(S,\leq,F)$-morphism $h : M \to N$ is a (S,F)-morphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}|_{M_{s_1}}$.

An **OSA** signature $(S,\leq,F)$ is *regular* iff for each $\sigma \in F_{w_1\to s_1}$ and each $w_0 \leq w_1$ there is a unique least element in the set $\{(w,s) \mid \sigma \in F_{w\to s} \text{ and } w_0 \leq w\}$. For regular signatures $(S,\leq,F)$, any F-term t has a least sort $LS(t)$ and the initial $(S,\leq,F)$-algebra can be defined as a term algebra, cf. [25]. Let $(S,\leq,F)$ be an order-sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S iff $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. A partial ordering $(S,\leq)$ is *filtered* iff for all $s_1,s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* iff every connected component of it is filtered. An order-sorted signature $(S,\leq,F)$ is *locally filtered* iff $(S,\leq)$ is locally filtered, and it is *coherent* iff it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent.

The atoms of the signature $(S,\leq,F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are closed formulas built by application of Boolean connectives and quantification to the equational atoms. Order sorted algebras were extensively studied in [25,24,35].

*Example 4 (Partial Algebra (**PA**) [41,7]).* A partial algebraic signature (S,F) consists of a set S of sorts and a set F of partial operations. We assume that there is a distinguished constant on each sort $\perp_s : s$. Signature morphisms map the sorts and operations in a compatible way, preserving $\perp_s$. A partial algebra is just like an ordinary algebra but interpreting the operations of F as partial rather than total functions; $\perp_s$ is always interpreted as undefined. A *partial algebra homomorphism* $h : A \to B$ is a family of (total) functions $\{h_s : A_s \to B_s\}_{s\in S}$ indexed by the set of sorts S of the signature such that $h_s(A_\sigma(a)) = B_\sigma(h_w(a))$ for each operation $\sigma : w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

We consider one kind of "base" sentences: *existence equality* $t \stackrel{e}{=} t'$. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal. The definedness predicate and strong equality can be introduced as notations: $def(t)$ stands for $t \stackrel{e}{=} t$ and $t \stackrel{s}{=} t'$ stands for $(t \stackrel{e}{=} t') \vee (\neg def(t) \wedge \neg def(t'))$. The sentences are formed from these "base" sentences by Boolean connectives and quantification over variables.

*Example 5 (First Order Partial Algebra (**FOPA**)).* Here we consider the institution **FOPA** as employed by the specification language CASL [5].

Its signatures consist of tuples (S,TF,PF), where TF is a family of sets of total function symbols and PF is a family of sets of partial function symbols such that $TF_{w\to s} \cap PF_{w\to s} = \emptyset$ for each arity w and any sort s. Models consist of algebras interpreting each total symbol in TF as a total function and each partial symbol in PF as a partial function. The sentences are constructed from existence equalities $t \stackrel{e}{=} t'$ by means of Boolean connectives and first-order quantification over total constant symbols.

*Example 6 (Higher order logic with Henkin semantics (**HNK**)).* Higher order logic with Henkin semantics has been introduced and studied in [9] and [30]. In the present paper we consider a simplified version close to the "higher order algebra" of [43] which does not consider λ-abstractions. This is not loss of power, since $\lambda x.t$ can always be replaced by a new constant symbol f together with the axiom $(\forall x) f x = t$.

For any set S of sorts, let $\overrightarrow{S}$ be the set of S-types defined as the least set such that $S \subseteq \overrightarrow{S}$ and $s_1 \to s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$. A **HNK**-signature is a tuple (S, F), where S is a set of sorts and F is a family of sets of constants $F = (F_s)_{s \in \overrightarrow{S}}$. A signature morphism $\varphi : (S, F) \to (S', F')$ consists of a function $\varphi^{st} : S \to S'$ and a family of functions between operation symbols $(\varphi^{op}_s : F_s \to F'_{\varphi^{type}(s)})_{s \in \overrightarrow{S}}$ where $\varphi^{type} : \overrightarrow{S} \to \overrightarrow{S}$ is the natural extension of φ^{st} to $\overrightarrow{S}$. For every signature (S, F), a (S, F)-model interprets each

1. sort $s \in S$ as a set, and
2. function symbol $\sigma \in F_s$ as a an element of M_s, where for each types $s_1, s_2 \in \overrightarrow{S}$, $M_{s_1 \to s_2} \subseteq [M_{s_1} \to M_{s_2}] = \{f \text{ function } \mid f : M_{s_1} \to M_{s_2}\}$.

An (S, F)-model morphism $h : M \to N$ interprets each type $s \in \overrightarrow{S}$ as a function $h_s : M_s \to N_s$ such that $h(M_\sigma) = N_\sigma$, for all function symbols $\sigma \in F$, and the following diagram commutes

$$\begin{array}{ccc} M_{s_1} & \xrightarrow{f} & M_{s_2} \\ {\scriptstyle h_{s_1}}\downarrow & & \downarrow{\scriptstyle h_{s_2}} \\ N_{s_1} & \xrightarrow[h_{s_1 \to s_2}(f)]{} & N_{s_2} \end{array}$$

for all types $s_1, s_2 \in \overrightarrow{S}$ and functions $f \in M_{s_1 \to s_2}$.

An (S, F)-equation is of the form $t_1 = t_2$ where t_1 and t_2 are terms of the same type. Sentences are constructed from equations by iteration of Boolean connectives and of higher order quantification.

Assumption 1 *For all institutions above we assume that:*

- *the signatures consist of countable number of symbols,*
- *the models have non-empty carriers, and*
- *the signature morphisms allow mapping constants to terms.*

The first condition implies that the sets of sentences of any signature of the institutions above are countable. The second condition assures that every injective signature morphism $\varphi : \Sigma \to \Sigma'$ is *conservative*, i.e. every Σ-model has a φ-expansion. The last condition allows us to treat substitutions in the comma category of signature morphisms (see subsection 1.3 for details).

Example 7 (Infinitary logic $\mathbf{FOL}_{\omega_1,\omega}$*).* This is the infinitary version of first-order logic allowing disjunctions of countable sets of sentences. Similarly we may define $\mathbf{POA}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$, $\mathbf{PA}_{\omega_1,\omega}$, $\mathbf{FOPA}_{\omega_1,\omega}$, and $\mathbf{HNK}_{\omega_1,\omega}$. Note that the set of sentences over a signature is uncountable even if the signatures consist of a countable number of symbols.

Example 8 (Institution of presentations). In any institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, a presentation is a pair (Σ, E) consisting of a signature $\Sigma \in |\mathbb{S}ig|$ and a set of Σ-sentences E. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism such that $E' \models \varphi(E)$. Note that presentation morphisms are closed under the composition. The institution of presentations of I, denoted by $I^{pres} = (\mathbb{S}ig^{pres}, \mathbb{S}en^{pres}, \mathbb{M}od^{pres}, \models^{pres})$ is defined as follows:

- $\mathbb{S}ig^{pres}$ is the category of presentations of I,
- $\mathbb{S}en^{pres}(\Sigma, E) = \mathbb{S}en(\Sigma)$,
- $\mathbb{M}od^{pres}(\Sigma, E)$ is the full subcategory of $\mathbb{M}od(\Sigma)$ of models satisfying E, and
- $M \models^{pres}_{(\Sigma,E)} e$ iff $M \models_\Sigma e$, for each (Σ, E)-model M and Σ-sentence e.

Let $\mathbb{S}ig^{cpres}$ be the full subcategory of $\mathbb{S}ig^{pres}$ consisting of countable presentations, i.e. presentations (Σ, E) for which the set of Σ-sentences E is countable. One can easily define the institution I^{cpres} of countable presentations over an arbitrary institution I.

Definition 2 (Compactness). *An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is compact whenever $E \models_\Sigma e$ implies the existence of a finite subset $E_f \subseteq E$ such that $E_f \models_\Sigma e$.*

Definition 3 (Finitary sentences). *In any institution a Σ-sentence ρ is* finitary *iff it can be written as $\varphi(\rho_f)$ where $\varphi : \Sigma_f \to \Sigma$ is a signature morphism such that Σ_f is a finitely presented signature* [1] *and ρ_f is a Σ_f sentence. An institution* has finitary sentences *when all its sentences are finitary.*

This condition usually means that the sentences contain only a finite number of symbols. Only the infinitary logics, such as $\mathbf{FOL}_{\omega_1,\omega}$, do not fulfill this condition.

Definition 4 (Finitary signature morphisms). *We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ is finitary if it is finitely presented in the category $\Sigma/\mathbb{S}ig$.*

In typical institutions the extensions of signatures with finite number of symbols are finitary.

1.1 Basic sets of sentences

A set of sentences $E \subseteq Sen(\Sigma)$ is called *basic* [15] if there exists a Σ-model M_E such that

$$M \models E \text{ iff there exists a morphism } M_E \to M$$

for all Σ-models M. If in addition the morphism $M_E \to M$ is unique then the set E is called *epi basic*.

Lemma 1. *Any set of atomic sentences in* **FOL***,* **POA***,* **OSA** *and* **PA** *is basic.*

[1] An object A in a category $\mathcal{C}$ is called *finitely presented* ([1]) if

- for each directed diagram $D : (J, \leq) \to \mathcal{C}$ with co-limit $\{Di \xrightarrow{\mu_i} B\}_{i \in J}$, and for each morphism $A \xrightarrow{g} B$, there exists $i \in J$ and $A \xrightarrow{g_i} Di$ such that $g_i;\mu_j = g$,

- for any two arrows g_i and g_j as above, there exists $i \leq k, j \leq k \in J$ such that $g_i;D(i \leq k) = g_j;D(j \leq k) = g$.

Proof. In **FOL** the basic model M_E for a set E of atomic (S,F,P)-sentences is constructed a follows: on the quotient $(T_F)/_{\equiv_E}$ of the term model T_F by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{\equiv_E}, \ldots, t_n/_{\equiv_E}) \mid \pi(t_1, \ldots, t_n) \in E\}$. A similar construction as the preceding holds for **OSA** provided that the order-sorted signatures are coherent. By defining an appropriate notion of congruence for **POA**-models compatible with the preorder (see [20] or [11]) one may obtain the same result for **POA**. In **PA** for a set of existence equations E we define T_E as the set of the sub-terms appearing in E. Note that T_E can be organized as a partial algebra by defining each partial operation symbol σ by $(T_E)_\sigma(t) = \begin{cases} \sigma(t) & \text{if } \sigma(t) \in T_E \\ \text{undefined,} & \text{otherwise} \end{cases}$, for all appropriate strings of terms t. The basic model M_E will be the quotient of T_E by the partial congruence induced by the existence equations in E.

In **FOPA**, any set of ground existence equations is basic (a proof of this fact can be found in [11]) . In **HNK**, not every set of ground equations is basic [10].

1.2 Internal logic

The following institutional notions dealing with the logical connectives and quantifiers were defined in [45]. Let Σ be a signature of an institution,

- a Σ-sentence $\neg e$ is a *(semantic) negation* of the Σ-sentence e when for every Σ-model M we have $M \models_\Sigma \neg e$ iff $M \not\models_\Sigma e$,
- a Σ-sentence $e_1 \vee e_2$ is a *(semantic) disjunction* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \vee e_2$ iff $M \models_\Sigma e_1$ or $M \models_\Sigma e_2$, and
- a Σ-sentence $(\exists\chi)e'$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathbb{S}ig$ and $e' \in \mathbb{S}en(\Sigma')$, is a *(semantic) existential χ-quantification* of e' when for every Σ-model M we have $M \models_\Sigma (\exists\chi)e'$ iff $M' \models_{\Sigma'} e'$ for some χ-expansion M' of M.

Distinguished negation $\neg\_$, disjunction $\bigvee\_$ [2], and existential quantification $(\exists\chi)\_$ are called *first-order constructors* and they have the semantical meaning defined above. Throughout this paper we assume the following commutativity of the first-order constructors with the signature morphisms: for each $\varphi : \Sigma \to \Sigma_1$ and any Σ-sentence

- $\neg e$ we have $\varphi(\neg e) = \neg\varphi(e)$;
- $\bigvee E$ we have $\varphi(\bigvee E) = \bigvee \varphi(E)$;
- $(\exists\chi)e'$, where $\chi : \Sigma \to \Sigma'$, there exists $\begin{tikzcd} \Sigma' \arrow[r, dotted, "\varphi'"] & \Sigma'_1 \\ \Sigma \arrow[u, "\chi"] \arrow[r, "\varphi"'] \arrow[ur, phantom, "pushout"] & \Sigma_1 \arrow[u, dotted, "\chi'"'] \end{tikzcd}$ s.t. $\varphi((\exists\chi)e') = (\exists\chi')\varphi'(e')$.

Since in concrete examples of institutions the quantified sentences are identified modulo renaming of variables we assume another rather mild condition: for every signature morphism $\varphi : \Sigma \to \Sigma_1$, each Σ-sentence $(\exists\chi)e'$ and any Σ_1-sentence $(\exists\chi')e'_1$

[2] We will use the symbol $\bigvee$ to represent the most general kind of disjunction even if it is finitary.

– if $\varphi((\exists\chi)e') = (\exists\chi')e'_1$, where $\chi : \Sigma \to \Sigma'$ and $\chi' : \Sigma' \to \Sigma'_1$, then there exists $\varphi' : \Sigma' \to \Sigma'_1$ such that
$$\begin{tikzcd} \Sigma' \arrow[r, dotted, "\varphi'"] & \Sigma'_1 \\ \Sigma \arrow[u, "\chi"] \arrow[r, "\varphi"'] & \Sigma_1 \arrow[u, "\chi'"'] \end{tikzcd}$$
is a pushout and $\varphi'(e') = e'_1$.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. Based on these constructors for sentences we can also define $\bigwedge$, *false*, $(\forall\chi)\_$ using the classical definitions.

1.3 Reachable models

Definition 5. *Consider two signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$. A signature morphisms $\theta : \Sigma_1 \to \Sigma_2$ such that $\chi_1;\theta = \chi_2$ is called a substitution morphism between χ_1 and χ_2.*

A more general treatment of substitutions may be found in [17]. However, in order to keep the presentation as simple as possible, we treat substitutions in the comma category of signature morphisms.

Definition 6. *Let $\mathcal{D}$ be a subcategory of signature morphism. We say that a Σ-model M is $\mathcal{D}$-reachable if for each signature morphism $\chi : \Sigma \to \Sigma'$ in $\mathcal{D}$, each χ-expansion M' of M determines a substitution $\theta : \chi \to 1_\Sigma$ such that $M \upharpoonright_\theta = M'$.*

In concrete examples of institutions $\mathcal{D}$ consists of signature morphisms used for quantifications, i.e. extensions of signatures with finite number of constants. A model M is reachable if the elements of M are exactly the interpretations of the terms.

Proposition 1. *[27] In **FOL**, **OSA**, **POA** and **PA**, a model is $\mathcal{D}$-reachable iff its elements consist only of interpretations of terms, where $\mathcal{D}$ is the class of signature morphisms used for quantification, i.e. signature extensions with finite number of constants.*

Remark 1. For any set E of atomic sentences in **FOL**, **POA**, **OSA** and **PA**, there exists a model M_E, defining E as basic set of sentences, which is $\mathcal{D}$-reachable, where $\mathcal{D}$ consists of signature extensions with finite number of constants.

In **FOPA**, the basic models defining the sets of ground existence equations as basic sets of sentences, are not $\mathcal{D}$-reachable in the sense of Definition 6 (see [11]).

2 Forcing and Generic Models

All the results in this section can be found in [27].

Definition 7 (First order fragments). *Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\Sigma \in |\mathbb{S}ig|$ a signature. A $\mathcal{D}$-*first-order fragment ($\mathcal{D}$-fragment, for short) over Γ*, where Γ is a set of Σ-sentences, is an extension $\mathcal{L}$ of Γ such that*

1. *every sentence of $\mathcal{L}$ is constructed from the sentences of Γ by means of negation, disjunction (possible infinitary) and existential quantification over the signature morphisms in $\mathcal{D}$, and*
2. *$\mathcal{L}$ is closed to*
   (a) *negation, i.e. if $e \in \mathcal{L}$ then $\neg e \in \mathcal{L}$.*
   (b) *"sub-sentence" relation, i.e.*
      – *if $\neg e \in \mathcal{L}$ then $e \in \mathcal{L}$,*
      – *if $\bigvee E \in \mathcal{L}$ then $e \in \mathcal{L}$ for all $e \in E$, and*
      – *if $(\exists\chi)e' \in \mathcal{L}$, where $\chi \in \mathcal{D}$ and $\theta : \chi \to 1_\Sigma$, then $\theta(e') \in \mathcal{L}$.*

Framework 1 *Throughout this section we consider*

– *an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$,*
– *a sub-functor $\mathbb{S}en_0$ of $\mathbb{S}en$,*
– *a broad subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, and*

we work within a $\mathcal{D}$-fragment $\mathcal{L}$ over $\mathbb{S}en_0(\Sigma)$.

An instance of this abstract framework is **FOL**, where

– $\mathbb{S}en_0$ is the sentence functor which associates to each signature the set of all atomic sentences, and
– $\mathcal{D}$ is the subcategory of signature morphisms which consists of signature extensions with a finite number of constant symbols.

Definition 8. *A forcing property for $\mathbb{S}en_0(\Sigma)$ is a tuple $\mathbb{P} = \langle P, \leq, f\rangle$ such that:*

1. *$\langle P, \leq\rangle$ is a partially ordered set with a least element 0,*
2. *f is a function which associates with each $p \in P$ a set $f(p)$ of sentences in $\mathbb{S}en_0(\Sigma)$,*
3. *$f(p) \subseteq f(q)$ whenever $p \leq q$, and*
4. *for each set of sentences $E \subseteq \mathbb{S}en_0(\Sigma)$ and any sentence $e \in \mathbb{S}en_0(\Sigma)$ if $E \subseteq f(p)$ and $E \models e$ then there is $q \geq p$ such that $e \in f(q)$.*

The elements of P are called *conditions*. We will define the forcing relation $\Vdash \subseteq P \times \mathcal{L}$ associated to a forcing property $\mathbb{P} = (P, f, \leq)$.

Definition 9. *Let $\mathbb{P} = \langle P, f, \leq\rangle$ be a forcing property for $\mathbb{S}en_0(\Sigma)$. The relation $p \Vdash e$, read p forces e, is defined by induction on e, for $p \in P$ and $e \in \mathcal{L}$, as follows:*

– For $e \in \mathbb{S}en_0(\Sigma)$. *$p \Vdash e$ if $e \in f(p)$.*
– For $\neg e \in \mathcal{L}$. *$p \Vdash \neg e$ if there is no $q \geq p$ such that $q \Vdash e$.*
– For $\bigvee E \in \mathcal{L}$. *$p \Vdash \bigvee E$ if $p \Vdash e$ for some $e \in E$.*
– For $(\exists\chi)e \in \mathcal{L}$. *$p \Vdash (\exists\chi)e$ if $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$.*

We say that *p weakly forces e*, in symbols $p \Vdash^w e$, iff $p \Vdash \neg\neg e$. The above definition is a generalization of the forcing studied in [42], [6] and [32].

Lemma 2. *Let $\mathbb{P} = (P, f, \leq)$ be a forcing property for $\mathbb{S}en_0(\Sigma)$, and $e \in \mathcal{L}$.*

1. *$p \Vdash^w e$ iff for each $q \geq p$ there is a condition $r \geq q$ such that $r \Vdash e$.*

2. *If $p \leq q$ and $p \Vdash e$ then $q \Vdash e$.*
3. *If $p \Vdash e$ then $p \Vdash^w e$.*
4. *We can not have both $p \Vdash e$ and $p \Vdash \neg e$.*

Definition 10. *Let $\mathbb{P} = (P, f, \leq)$ be a forcing property for $\mathbb{S}en_0(\Sigma)$. A subset $G \subseteq P$ is said to be a generic iff*

1. *$p \in G$ and $q \leq p$ implies $q \in G$.*
2. *$p, q \in G$ implies that there exists $r \in G$ such that $p \leq r$ and $q \leq r$.*
3. *for each sentence $e \in \mathcal{L}$ there exists a condition $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$.*

Lemma 3. *If $\mathcal{L}$ is countable then every p belongs to a generic set.*

Definition 11. *Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property for $\mathbb{S}en_0(\Sigma)$.*

1. *M is a model for $G \subseteq P$ if for every sentence $e \in \mathcal{L}$*

$$M \models e \text{ iff } G \Vdash e$$

2. *M is a generic model for $p \in P$ if there is a generic set $G \subseteq P$ such that $p \in G$ and M is a model for G.*

Proposition 2. *Assume that*

1. *I_0 is compact,*
2. *every set of sentences in I_0 is basic, and*
3. *for each $E \subseteq \mathbb{S}en_0(\Sigma)$ there exists a basic model M_E that is $\mathcal{D}$-reachable.*

Then there is a $\mathcal{D}$-reachable model for every generic set G.

Theorem 1. *(Generic model theorem) Under the conditions of Proposition 2, if $\mathcal{L}$ is countable then there is a generic $\mathcal{D}$-reachable model for each condition $p \in P$.*

The following is a corollary of the generic model theorem.

Corollary 1. *Under the condition Theorem 1 for every condition $p \in P$ and any sentence $e \in \mathcal{L}$ we have that $p \Vdash^w e$ iff $M \models e$ for each generic model M for p which is $\mathcal{D}$-reachable.*

3 Omitting Types

In this section we investigate a forcing property studied by Robinson [42] and Barwise [6] in classical model theory and we use it to prove a very general form of OTT.

3.1 Preliminaries

Given an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, for each signature morphism $\chi : \Sigma \to \Sigma'$, a Σ-model M *χ-realizes* a set Δ of Σ'-sentences, if there exists a χ-expansion M' of M which satisfies Δ. We say that M *χ-omits* Δ if does not χ-realize Δ. The key theorem of this section gives sufficient institution-independent conditions for a theory $\Gamma \subseteq \mathbb{S}en(\Sigma)$ to have a model which omits Δ. As in classical model theory, the central idea is the notion of a theory locally omitting a set of sentences. For each signature morphism $\chi : \Sigma \to \Sigma'$, a set Γ of Σ-sentences *locally χ-realizes* a set Δ of Σ'-sentences iff there is a finite set p of Σ'-sentences such that:

1. $\chi(\Gamma) \cup p$ is consistent, and
2. $\chi(\Gamma) \cup p \models'_{\Sigma} \Delta$

Γ *locally χ-omits* Δ if does not locally χ-realize Δ.

Definition 12. *An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ has the $\mathcal{D}$-Omitting Types Property ($\mathcal{D}$-*OTP*), where $\mathcal{D} \subseteq \mathbb{S}ig$ is a broad subcategory of signature morphisms, when for any*

- *countable set of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$,*
- *sequence of signature morphisms $(\Sigma \xrightarrow{\chi_n} \Sigma_n \in \mathcal{D} : n \in \mathbb{N})$, and*
- *countable sets of sentences $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n) : n \in \mathbb{N})$*

if Γ locally χ_n-omits Δ_n, for all $n \in \mathbb{N}$, then there is a Σ-model M of Γ which χ_n-omits Δ_n, for all $n \in \mathbb{N}$.

In classical model theory, the models of interest are constructed in an extension $\mathcal{L}_C$ of the initial language $\mathcal{L}$ with an infinite but countable set of constants C (see [32,38,31,8]). The following definition is from [27] and it gives the categorical properties of the extension $\mathcal{L} \hookrightarrow \mathcal{L}_C$ that we need to obtain our results.

Definition 13. *Let $\mathcal{D} \subseteq \mathbb{S}ig$ be a subcategory of signature morphisms. We say that $\Sigma \xrightarrow{v} \Sigma'$ is a $\mathcal{D}$-extension of Σ if it is the vertex of a directed co-limit $(u_i \xrightarrow{v_i} v)_{i \in J}$ of a directed diagram $(u_i \xrightarrow{u_{i,j}} u_j)_{(i \leq j) \in (J, \leq)}$ in $\Sigma/\mathcal{D}$ [3] such that*

1. *for all $i \in J$, v_i is conservative, and*
2. *for all signature morphisms $\Sigma_i \xrightarrow{\chi_i} \Sigma'_i \in \mathcal{D}$ there exists a conservative substitution $\chi_i \xrightarrow{\chi_{i,j}} u_{i,j} \in (\Sigma_i/\mathbb{S}ig)$.*

Take for example **FOL** and assume that $\mathcal{D}$ is the class of signature extensions with finite number of constants. For every signature $\Sigma = (S, F, P)$ consider a set C of new constant symbols (C does not contain any symbol from Σ) such that

- C_s is an infinite set for all sorts $s \in S$, and
- $C_s \cap C_{s'} = \emptyset$ for all sorts $s, s' \in S$.

[3] $\Sigma \xrightarrow{u_i} \Sigma_i \in \mathcal{D}$ for all $i \in J$ and $\Sigma_i \xrightarrow{u_{i,j}} \Sigma_j \in \mathcal{D}$ for all $(i, j) \in (J, \leq)$.

The inclusion $\Sigma \overset{u}{\hookrightarrow} \Sigma(C)$, where the signature $\Sigma(C) = (S, F \cup C, P)$, is the vertex of the directed co-limit $((\Sigma \overset{u_i}{\hookrightarrow} \Sigma(C_i)) \overset{v_i}{\hookrightarrow} (\Sigma \overset{v}{\hookrightarrow} \Sigma(C)))_{C_i \subseteq C finite}$ of the directed diagram $((\Sigma \overset{u_i}{\hookrightarrow} \Sigma(C_i)) \overset{u_{i,j}}{\hookrightarrow} (\Sigma \overset{u_j}{\hookrightarrow} \Sigma(C_j))_{C_i \subseteq C_j \subseteq C finite}$. Since C is infinite, for every signature extension $\chi_i : \Sigma(C_i) \hookrightarrow \Sigma(C_i \cup X)$, where X is a finite set of new constants, there exists an injective mapping $\chi_{i,j} : C_i \cup X \to C_j$ for some $j \in J$ such that the restriction $\chi_{i,j} \mid_{C_i} : C_i \to C_j$ is the inclusion. Hence, the following diagram commutes.

$$
\begin{tikzcd}
\Sigma(C_i \cup X) \arrow[rr, dotted, "\chi_{i,j}"] & & \Sigma(C_j) \\
& \Sigma(C_i) \arrow[ul, hook, "\chi_i"] \arrow[ur, hook', "u_{i,j}"'] &
\end{tikzcd}
$$

Since $\chi_{i,j} : \Sigma(C_i \cup X) \to \Sigma(C_j)$ is injective and the models have non empty carriers, $\chi_{i,j} : \Sigma(C_i \cup X) \to \Sigma(C_j)$ is conservative.

Definition 14. *Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}en_0$ a sub-functor of $\mathbb{S}en$. We say that I is a $\mathcal{D}$-first-order institution over $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, Mod, \models)$ if $\mathbb{S}en(\Sigma)$ is a $\mathcal{D}$-fragment over $\mathbb{S}en_0(\Sigma)$, for all signatures $\Sigma \in |\mathbb{S}ig|$.*

For example **FOL** is a $\mathcal{D}$-first-order institution over $\mathbf{FOL}_0$, where

- $\mathbf{FOL}_0$ is the restriction of **FOL** to atomic sentences, and
- $\mathcal{D}$ is the subcategory of signature morphisms which consists of signature extensions with a finite number of constant symbols.

3.2 A semantic forcing property

Framework 2 *In this section we work within a $\mathcal{D}$-first-order institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over I_0, with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$, such that*

1. *the sentences in I_0 are finitary,*
2. *the sentences in I_0 are not formed by applying the first-order constructors,*
3. *every signature Σ has a $\mathcal{D}$-extension, and*
4. *every signature morphism in $\mathcal{D}$ is conservative and finitary.*

We have the following consequence of the finiteness of the "atomic" sentences.

Lemma 4. *For any $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ as in Definition 13 we have $Sen_0(\Sigma') = \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$.*

Similar results as the above Lemma can be found also in [39] or [27].

If $v : \Sigma \to \Sigma'$ is a $\mathcal{D}$-extension as in Definition 13 then we denote by $\mathcal{L}_v$ the set of sentences $\bigcup_{i \in J} v_i(\mathbb{S}en(\Sigma_i))$. We have the following consequence of Lemma 4 and the finiteness of signature morphisms in $\mathcal{D}$.

Proposition 3. *[27] $\mathcal{L}_v$ is a $\mathcal{D}$-fragment over $\mathbb{S}en_0(\Sigma)$, where $v:\Sigma\to\Sigma'$ is a $\mathcal{D}$-extension as in Definition 13.*

Definition 15. *Let $v:\Sigma\to\Sigma'$ be a signature morphism, $\mathcal{M}\subseteq\mathbb{M}od(\Sigma)$ a class of models, and $\mathcal{L}\subseteq\mathbb{S}en(\Sigma')$ a $\mathcal{D}$-fragment. We define the forcing property $\mathbb{P}(v,\mathcal{M},\mathcal{L})=\langle P,\leq,f\rangle$ as follows*

1. *$P=\{p\subseteq\mathcal{L}$ finite $\mid$ there is $M'\in\mathbb{M}od(\Sigma')$ s.t. $M'\upharpoonright_v\in\mathcal{M}$ and $M'\models p\}$,*
2. *$f(p)=p\cap\mathbb{S}en_0(\Sigma')$ for all $p\in P$, and*
3. *$\leq$ is the inclusion between sets of sentences.*

In [32] the conditions of $\mathcal{P}$ are called *finite pieces of $\mathcal{M}$*.

Proposition 4. *$\mathbb{P}(v,\mathcal{M},\mathcal{L})=\langle P,\leq,f\rangle$ defined above is a forcing property.*

Proof. Assume a condition $p\in P$ and a set of sentences $E\subseteq f(p)$ such that $E\models e$ where $e\in\mathbb{S}en_0(\Sigma)$. There is $M'\in\mathbb{M}od(\Sigma')$ such that $M'\upharpoonright_v\in\mathcal{M}$ and $M'\models p$. We have $M'\models E$ which implies $M'\models e$ meaning that $p\cup\{e\}\in P$ and $p\cup\{e\}\Vdash e$.

Lemma 5. *Let $v:\Sigma\to\Sigma'$ be a $\mathcal{D}$-extension as in Definition 13, $\mathcal{M}$ a class of Σ-models, and $\mathcal{L}\subseteq\mathcal{L}_v$ a $\mathcal{D}$-fragment. $\mathbb{P}(v,\mathcal{M},\mathcal{L})$ has the following properties:*

1. *if $p\in P$ and $\bigvee E\in p$ then $p\cup\{e\}\in P$ for some $e\in E$.*
2. *if $p\in P$ and $(\exists\chi)e\in p$ (where $\chi:\Sigma'\to\Sigma'_1$) there exists a substitution $\theta:\chi\to 1_{\Sigma'}$ such that $p\cup\{\theta(e)\}\in P$.*

Proof. We proceed as follows:

1. Assume $\bigvee E\in p$, where p is a condition in P. There is a Σ'-model M' such that $M'\upharpoonright_v\in\mathcal{M}$ and $M'\models p$. We have $M\models\bigvee E$ which implies $M'\models e$ for some $e\in E$ and we get $p\cup\{e\}\in P$.
2. Assume $(\exists\chi)e\in p$, where $p\in P$. There is a Σ'-model M' such that $M'\upharpoonright_v\in\mathcal{M}$ and $M'\models p$. Since $p\subseteq\bigcup_{i\in J}v_i(\mathbb{S}en(\Sigma_i))$ is finite there is $p_i\subseteq\mathbb{S}en(\Sigma_i)$, where $i\in J$, such that $v_i(p_i)=p$. We have $v_i((\exists\chi_i)e_i)=(\exists\chi)e$ for some $(\exists\chi_i)e_i\in\mathbb{S}en(\Sigma_i)$ (where $\chi_i:\Sigma_i\to\Sigma'_i$). By our assumptions there exists $v'_i:\Sigma'_i\to\Sigma'_1$ such that
$$\begin{tikzcd} \Sigma'_i \arrow[r, dotted, "v'_i"] & \Sigma'_1 \\ \Sigma_i \arrow[u, "\chi_i"] \arrow[r, "v_i"'] & \Sigma' \arrow[u, "\chi"'] \end{tikzcd}$$
is a pushout and $e=v'_i(e_i)$. By Definition 13 there exists $(i\leq j)\in(J,\leq)$ and a substitution $\chi_{i,j}:\chi_i\to u_{i,j}$ with $\chi_{i,j}$ conservative as a signature morphism.

$$\begin{tikzcd} & \Sigma'_i \arrow[rr, dotted, "v'_i"] \arrow[d, dotted, "\chi_{i,j}"] & & \Sigma'_1 \\ \Sigma_i \arrow[ur, dotted, "\chi_i"] \arrow[r, "u_{i,j}"'] & \Sigma_j \arrow[r, "v_j"'] & \Sigma' \arrow[ur, "\chi"] & \end{tikzcd}$$

Because
$$\begin{tikzcd} \Sigma'_i \arrow[r, dotted, "v'_i"] & \Sigma'_1 \\ \Sigma_i \arrow[u, "\chi_i"] \arrow[r, "v_i"'] & \Sigma' \arrow[u, "\chi"'] \end{tikzcd}$$
is a pushout and $\chi_i;(\chi_{i,j};v_j) = v_i;1_{\Sigma'}$ there exists $\theta : \Sigma'_1 \to \Sigma'$ such that $v'_i;\theta = (\chi_{i,j};v_j)$ and $\chi;\theta = 1_{\Sigma'}$.

$$\begin{tikzcd} & \Sigma'_i \arrow[rr, dotted, "v'_i"] \arrow[d, dotted, "\chi_{i,j}"] & & \Sigma'_1 \arrow[d, dotted, "\theta"] \\ \Sigma_i \arrow[ur, dotted, "\chi_i"] \arrow[r, "u_{i,j}"'] & \Sigma_j \arrow[r, "v_j"'] & \Sigma' \arrow[ur, "\chi"] \arrow[r, "1'_\Sigma"'] & \Sigma' \end{tikzcd}$$

By the satisfaction condition $M' \upharpoonright_{v_i} \models_{\Sigma_i} p_i$ and we have $M' \upharpoonright_{v_i} \models_{\Sigma_i} (\exists \chi_i) e_i$. There exists a χ_i-expansion M'_i of $M' \upharpoonright_{v_i}$ such that $M'_i \models_{\Sigma'_i} e_i$. Because $\chi_{i,j}$ is conservative there exists a $\chi_{i,j}$-expansion M_j of M'_i and by the satisfaction condition $M_j \models_{\Sigma_j} \chi_{i,j}(\chi_i(p_i) \cup \{e_i\})$. Since v_j is conservative there exists a v_j-expansion M'' of M_j and by satisfaction condition $M'' \models_{\Sigma'} (\chi_{i,j};v_j)(\chi_i(p_i) \cup \{e_i\}) = p \cup \{\theta(e)\}$. Note that $M'' \upharpoonright_{v_i} = M' \upharpoonright_{v_i}$ which implies $M'' \upharpoonright_v \in \mathcal{M}$ and because $M'' \models_{\Sigma'} p \cup \{\theta(e)\}$ we obtain $p \cup \{\theta(e)\} \in P$.

Proposition 5. *Let $v : \Sigma \to \Sigma'$ be a $\mathcal{D}$-extension as in Definition 13, $\mathcal{M}$ a class of Σ-models, and $\mathcal{L} \subseteq \mathcal{L}_v$ a $\mathcal{D}$-fragment. $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ has the following property:*

$$\textit{there exists } q \geq p \textit{ such that } q \Vdash e \textit{ iff } p \cup \{e\} \in P$$

for every sentence $e \in \mathcal{L}$ and each condition $p \in P$.

Proof. We proceed by induction on the structure of the sentence e.

For $e \in \mathbb{S}en_0(\Sigma')$. If there is $q \geq p$ such that $q \Vdash e$ then $e \in q$. We have $p \cup \{e\} \subseteq q$ and by the definition of $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ we obtain $p \cup \{e\} \in P$. For the converse implication take $q = p \cup \{e\}$.

For $\neg e$. By the induction hypothesis applied to e, for all $q \in P$ we have

$$\text{for every } r \geq q, r \nVdash e \iff q \cup \{e\} \notin P$$

which implies that for all $q \in P$ we have

$$q \Vdash \neg e \iff q \cup \{e\} \notin P$$

We need to prove

$$\text{there exists } q \geq p \text{ such that } q \cup \{e\} \notin P \iff p \cup \{\neg e\} \in P$$

Assume that there is $q \geq p$ such that $q \cup \{e\} \notin P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M \models_{\Sigma'} q$ and since $q \cup \{e\} \notin P$ we have $M' \models \neg e$. We obtain $M' \models_{\Sigma'} q \cup \{\neg e\}$ and in particular $M' \models_{\Sigma'} p \cup \{\neg e\}$ meaning that $p \cup \{\neg e\} \in P$. For the converse implication, take $q = p \cup \{\neg e\}$.

For $\bigvee E$. If there is $q \geq p$ such that $q \Vdash \bigvee E$, then there is $e \in E$ such that $q \Vdash e$. By the induction hypothesis, $p \cup \{e\} \in P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models_{\Sigma'} p \cup \{e\}$. We obtain $M' \models_{\Sigma'} p \cup \{\bigvee E\}$ meaning that $p \cup \{\bigvee E\} \in P$.

For the converse implication assume that $p \cup \{\bigvee E\} \in P$. By Lemma 5 (1) there is $e \in E$ such that $p \cup \{\bigvee E, e\} \in P$. By induction hypothesis applied to e we have $q \Vdash e$ for some $q \geq p \cup \{\bigvee E\}$. Hence there exists $q \geq p$ such that $q \Vdash \bigvee E$.

For $(\exists \chi)e$. Assume that there is $q \geq p$ such that $q \Vdash (\exists \chi)e$. By the definition of forcing relation there exists a substitution $\chi' : \chi \to 1_{\Sigma'}$ such that $q \Vdash \chi'(e)$. By induction $p \cup \{\chi'(e)\} \in P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models_{\Sigma'} p \cup \{\chi'(e)\}$. We obtain $M' \models_{\Sigma'} p \cup \{(\exists \chi)e\}$ meaning that $p \cup \{(\exists \chi)e\} \in P$.

For the converse implication assume that $p \cup \{(\exists \chi)e\} \in P$ where $\chi : \Sigma \to \Sigma'$. By Lemma 5 (2) there exists a substitution $\chi' : \chi \to 1_{\Sigma}$ such that $p \cup \{(\exists \chi)e,\ \chi'(e)\} \in P$. By the induction hypothesis applied to $\chi'(e)$, there exists $q \geq p \cup \{(\exists \chi)e\}$ such that $q \Vdash \chi'(e)$. Therefore, by the definition of forcing relation $q \Vdash (\exists \chi)e$.

Corollary 2. *For each each forcing property $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$, where v, $\mathcal{M}$ and $\mathcal{L}$ are as in Proposition 5, we have:*

1. *For each condition $p \in P$, any generic model M for p satisfies p, and*
2. *If $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$ and $v(\Gamma) \subseteq \mathcal{L}$ then every generic model satisfies Γ.*

Proof. We proceed as follows:

1. Let $G \subseteq P$ be the generic set such that $p \in G$ and M a model for G. We prove that $M \models e$ for all $e \in p$.
   Let e be an arbitrary sentence in p. Since $G \subseteq P$ is a generic set there exists $q \in G$ such that either $q \Vdash e$ or $q \Vdash \neg e$. Suppose that $q \Vdash \neg e$ then there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 2 (2) $r \Vdash \neg e$. By Proposition 5 since $e \in r$ there exists $r' \geq r$ such that $r' \Vdash e$. Using Lemma 2 (2) again we get $r' \Vdash \neg e$ which is a contradiction. Therefore $q \Vdash e$ and since M is a model for G we have that $M \models e$.
2. Let M be a a generic model for G and $e \in \Gamma$. Since G is generic there is $p \in G$ such that $p \Vdash e$ or $p \Vdash \neg e$. Assuming that $p \Vdash \neg e$ since $p \cup \{e\} \in P$, by Proposition 5 there exists $q \geq p$ such that $q \Vdash e$. By Lemma 2 (2) $q \Vdash \neg e$ which is a contradiction. Therefore $p \Vdash e$ which implies $M \models_{\Sigma'} e$. Since e was arbitrary we get $M \models_{\Sigma'} \Gamma$.

3.3 Main results

Theorem 2 (Downward Löwenheim-Skolem Theorem). *Let $I = (\mathbb{S}ig, Sen, \mathbb{M}od, \models)$ be a $\mathcal{D}$-first-order institution with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$ such that all conditions in Framework 2 are fulfilled:*

1. *the sentences in I_0 are finitary,*
2. *the sentences in I_0 are not formed by applying the first-order constructors,*
3. *every signature has a $\mathcal{D}$-extension, and*
4. *every signature morphism in $\mathcal{D}$ is conservative and finitary.*

In addition we assume that

5. *for all signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, the substitutions $\theta : \chi \to 1_{\Sigma'}$ form a countable set.*

For every countable and consistent set Γ of Σ-sentences and each $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ there exists a $\mathcal{D}$-reachable Σ'-model M' such that $M' \upharpoonright_v \models_\Sigma \Gamma$.

Proof. Consider an extension $v : \Sigma \to \Sigma'$ of Σ as in Definition 13 and let $\mathcal{L} \subseteq \mathbb{S}en(\Sigma')$ be the least fragment which contains $v(\Gamma)$.

Using the condition "for all $\Sigma' \xrightarrow{\chi} \Sigma'_1 \in \mathcal{D}$ the substitutions $\theta : \chi \to 1_{\Sigma'}$ form a countable set", one can easily prove by induction that $\mathcal{L}$ is countable. Let $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$ and consider the forcing property $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ defined in the previous subsection. By Theorem 1 there is a generic $\mathcal{D}$-reachable Σ'-model M' for $\emptyset$, by Corollary 2, $M' \models v(\Gamma)$, and by the satisfaction condition $M' \upharpoonright_v \models \Gamma$.

Corollary 3. *In* **FOL***, any consistent set of sentences has a countable model.*

Proof. Let $\mathcal{D}$ be the subcategory of signature morphisms which consists of signature extensions with a finite number of constants, and Γ a set of Σ-sentences. Let $\mathcal{C}$ be a set of new constant symbols, such that C_s is infinite and countable for all sorts s, and $C_s \cap C_{s'} = \emptyset$ for all sorts s and s' such that $s \neq s'$. The inclusion $\Sigma \hookrightarrow \Sigma(C)$ is a $\mathcal{D}$-extension and since Σ consists of a countable set of symbols

1. Γ is countable, and
2. the substitutions $\theta : C \to \Sigma(C)$ form a countable set.

By Theorem 2 there is a $\mathcal{D}$-reachable $\Sigma(C)$-model M which satisfies Γ. By Proposition 1, the elements of M consist only of interpretations of terms, and since $\Sigma(C)$ is countable, we have that M is countable.

Similar corollaries as above hold also for **POA**, **OSA** and **PA**. As for their infinitary variants, we obtain that any consistent and countable set of sentences has a countable model.

Theorem 3 (Omitting Types). *Let $I = (\mathbb{S}ig, Sen, \mathbb{M}od, \models)$ be a $\mathcal{D}$-first-order institution with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$ such that all conditions in Framework 2 are fulfilled:*

1. *the sentences in I_0 are finitary,*
2. *the sentences in I_0 are not formed by applying the first-order constructors,*
3. *every signature has a $\mathcal{D}$-extension, and*
4. *every signature morphism in $\mathcal{D}$ is conservative and finitary.*

In addition we assume that

5. *for all signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, the substitutions $\theta : \chi \to 1_{\Sigma'}$ form a countable set,*
6. *for all Σ-sentences e_1 and e_2 there is a Σ-sentence ρ semantic equivalent with $e_1 \wedge e_2$, and*

7. *for every substitution* $\theta : \chi_1 \to \chi_2$*, where* $\Sigma \overset{\chi_1}{\to} \Sigma_1, \Sigma \overset{\chi_2}{\to} \Sigma_2 \in \mathcal{D}$*, and each* Σ_2*-sentence* e_2 *there exists a* Σ_1*-sentence* e_1 *semantic equivalent with* $(\exists\theta)e_2$*.*

The institution I *has the* $\mathcal{D}$*-OTP.*

Proof. Let $v : \Sigma \to \Sigma'$ be an extension of Σ as in Definition 13. Let $\Gamma \subseteq \mathbb{S}en(\Sigma)$, $\Sigma \overset{\chi_n}{\to} \Sigma_n \in \mathcal{D}$ and $\Delta_n \subseteq \mathbb{S}en(\Sigma_n)$ be as in Definition 12. For all $n \in \mathbb{N}$ consider the following pushout of signature morphisms

$$\begin{tikzcd} \Sigma_n \arrow[r, dotted, "v_n"] & \Sigma'_n \\ \Sigma \arrow[u, "\chi_n"] \arrow[r, "v"'] & \Sigma' \arrow[u, dotted, "\chi'_n"'] \end{tikzcd}$$

such that $v((\exists\chi_n)\rho) = (\exists\chi'_n)v_n(\rho)$ for all sentences $(\exists\chi_n)\rho \in \mathbb{S}en(\Sigma)$. For the sake of simplicity we will make the following notations $\Gamma' = v(\Gamma)$, and $\Delta'_n = v_n(\Delta_n)$, for all $n \in \mathbb{N}$.

Let $\mathcal{L} \subseteq \mathbb{S}en(\Sigma')$ be the least fragment which includes

1. Γ', and
2. $\theta(\Delta'_n)$, for all $n \in \mathbb{N}$ and any $\theta : \chi'_n \to 1_{\Sigma'}$.

Since the substitutions $\theta : \chi_n \to 1_{\Sigma'}$ form a countable set, $\mathcal{L}$ is countable. Consider the forcing property $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ with $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$.

Firstly we prove that for all natural numbers $n \in \mathbb{N}$, conditions $p \in P$, and substitutions $\theta : \chi'_n \to 1_{\chi'}$ there exists $\delta \in \Delta'_n$ such that $p \cup \{\neg\theta(\delta)\} \in P$. Let $n \in \mathbb{N}$, $p \in P$, and $\chi'_n \overset{\theta}{\to} 1_{\Sigma'} \in \Sigma'/\mathbb{S}ig$. Since χ_n is finitary, there exists a substitution $\theta_i : \chi_n \to u_i$, where $i \in J$, such that $\theta_i; v_i = v_n; \theta$. By the definition of $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ there exists $p_j \subseteq \mathbb{S}en(\Sigma_j)$, where $j \in J$, such that $p = v_j(p_j)$. Let $k \in J$ such that $k \geq i$ and $k \geq j$ and we have $\theta_k; v_k = v_n; \theta$ and $p = v_k(p_k)$, where $\theta_k = \theta_i; u_{i,k}$ and $p_k = u_{j,k}(p_j)$.

$$\begin{tikzcd} & \Sigma_n \arrow[rr, dotted, "v_n"] \arrow[d, dotted, "\theta_k"] & & \Sigma'_n \arrow[d, "\theta"] \\ \Sigma \arrow[ur, dotted, "\chi_n"] \arrow[r, "u_k"'] & \Sigma_k \arrow[r, "v_k"'] & \Sigma' \arrow[ur, "\chi'_n"] \arrow[r, dotted, "1_{\Sigma'}"'] & \Sigma' \end{tikzcd}$$

By our hypothesis, there is a Σ_n-sentence ρ_n semantically equivalent with $(\exists\theta_k)\bigwedge p_k$. $u_k(\Gamma) \cup p_k$ consistent implies $\chi_n(\Gamma) \cup \{\rho_n\}$ consistent. Since Γ locally χ_n-omits Δ_n there exists $\delta \in \Delta_n$ such that $\chi_n(\Gamma) \cup \{\rho_n, \neg\delta\}$ is consistent which implies $u_k(\Gamma) \cup p_k \cup \{\neg\theta_k(\delta)\}$ is consistent. Since v_k is conservative, $\Gamma' = v_k(u_k(\Gamma))$, $v_k(p_k) = p$ and $v_k(\neg\theta_k(\delta)) = \neg\theta(v_n(\delta))$, we have $\delta' = v_n(\delta) \in \Delta'_n$ and $\Gamma' \cup p \cup \{\neg\theta(\delta')\}$ is consistent.

Secondly, we construct a generic set G such that all generic $\mathcal{D}$-reachable models for G will χ'_n-omit Δ'_n, for all $n \in \mathbb{N}$. Note that the substitutions $\theta : \chi'_n \to 1_{\Sigma'}$, where $n \in \mathbb{N}$, form a countable set. Let $(\theta_m : m \in \mathbb{N})$ be an enumeration of all such substitutions. Since the fragment $\mathcal{L}$ is also countable let $(e_m : m \in \mathbb{N})$ be an enumeration of $\mathcal{L}$. We form an increasing chain of conditions $p_0 \leq p_1 \ldots \leq p_m \ldots$ such that for all $m \in \mathbb{N}$

1. $p_{m+1} \Vdash e_m$ or $p_{m+1} \Vdash \neg e_m$, and
2. there exists $\delta \in \Delta'_n$ such that $\neg\theta_m(\delta) \in p_{m+1}$, where $\theta_m : \chi'_n \to 1_{\Sigma'}$.

Let $p_0 = \emptyset$ and assuming that we already have the condition p_m we construct p_{m+1} as follows: if $p_m \Vdash \neg e_m$ then take $q = p_m$ else take $q \geq p_m$ such that $q \Vdash e_m$; assuming that $\theta_m : \chi'_n \to 1_{\Sigma'}$, by the first part of the proof there exists $\delta \in \Delta'_n$ such that $q \cup \{\neg\theta_m(\delta)\} \in P$; take $p_{m+1} = q \cup \{\neg\theta_m(\delta)\}$. The set $G = \{q \mid q \leq p_m \text{ for some } m \in \mathbb{N}\}$ is generic. Let M be a $\mathcal{D}$-reachable model for G. We show that M χ'_n-omits Δ'_n for all $n \in \mathbb{N}$. Let M_n be an χ'_n-expansion of M. Since M is $\mathcal{D}$-reachable there exists a substitution $\theta_m : \chi'_n \to 1_{\Sigma'}$ such that $M \upharpoonright_{\theta_m} = M_n$.

$$\begin{tikzcd}
\Sigma'_n \arrow[rr, dotted, "\theta_m"] & & \Sigma' \\
& \Sigma' \arrow[ul, "\chi'_n"] \arrow[ur, "1_{\Sigma'}"'] &
\end{tikzcd}$$

By the definition of G there exists $\delta \in \Delta'_n$ such that $\neg\theta_m(\delta) \in p_{m+1}$. Since M is a generic model for p_{m+1} by Corollary 2 $M \models_{\Sigma'} p_{m+1}$ and we have $M \models_{\Sigma'} \neg\theta_m(\delta)$. By the satisfaction condition $M_n \models_{\Sigma'_n} \neg\delta$.

Finally, by Proposition 2 there is a generic $\mathcal{D}$-reachable model M for G and by the satisfaction condition $(M \upharpoonright_v)$ χ_n-omits Δ_n for all $n \in \mathbb{N}$.

In the following we discuss the applicability of Theorem 3 by making an analysis of its underlying conditions in the typical example of **FOL**.

Condition 5. In **FOL** holds because we assumed that the symbols of any signature form a countable set.

Condition 6. **FOL** has finite conjunctions.

Condition 7. Given the signature morphisms $\Sigma \hookrightarrow \Sigma(X)$ and $\Sigma \hookrightarrow \Sigma(Y)$, where $\Sigma = (S, F, P)$, and X, Y are finite sets of constant symbols, for each substitution $\theta : X \to T_F(Y)$ and any $\Sigma(Y)$-sentence e_2, $(\exists\theta)e_2$ is semantically equivalent to $(\exists Y)e_2 \wedge (\bigwedge_{x \in X} x = \theta(x))$.

Corollary 4. **FOL***,* **POA***,* **OSA***,* **PA** *and their infinitary variants* $\mathbf{FOL}_{\omega_1,\omega}$*,* $\mathbf{POA}_{\omega_1,\omega}$*,* $\mathbf{OSA}_{\omega_1,\omega}$*,* $\mathbf{PA}_{\omega_1,\omega}$ *have the $\mathcal{D}$-omitting types property, where $\mathcal{D}$ consists of signature extensions with finite number of constants.*

In **FOPA**, the models defining the sets of ground existential atoms as basic sets of sentences are not $\mathcal{D}$-reachable in the sense of Definition 6. In **HNK**, not all sets of atoms are basic. Hence, **FOPA** and **HNK** do not fall into the framework of Theorem 3.

4 Borrowing Omitting Types

In this section we borrow the OTP along institution mappings for more expressive logical systems which are encoded into the institutions of presentations of less refined institutions. Therefore we also need to lift the OTP from a base institution to the institution of (countable) presentations. The institution mappings used here for borrowing results is that of institution comorphisms [26] previously known as *plain maps* [34] or *representations* [47].

Definition 16. *Let $I' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ and $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be two institutions. An* institution comorphism $(\phi, \alpha, \beta) : I \to I'$ *consists of*

- *a functor $\phi : \mathbb{S}ig \to \mathbb{S}ig'$, and*
- *two natural transformations $\alpha : \mathbb{S}en \Rightarrow \phi; \mathbb{S}en'$ and $\beta : \phi^{op}; \mathbb{M}od' \Rightarrow \mathbb{M}od$ such that the following satisfaction condition for institution morphisms holds:*

$$M' \models'_{\phi(\Sigma)} \alpha_\Sigma(e) \text{ iff } \beta_\Sigma(M') \models_\Sigma e$$

for every signature $\Sigma \in \mathbb{S}ig$, each $\phi(\Sigma)$-model M', and any Σ-sentence e.

4.1 Main results

Theorem 4. *Let $(\phi, \alpha, \beta) : I \to I'$ be an institution comorphism as in Definition 16 such that*

- *I and I' have negations and $\alpha_\Sigma(\neg e) = \neg \alpha_\Sigma(e)$ for all Σ-sentence e, and*
- *for all signatures $\Sigma \in |\mathbb{S}ig|$*
  - *α_Σ is surjective modulo $\models\!\mid$ [4], i.e. for all $\phi(\Sigma)$-sentences ρ' there exists a Σ-sentence ρ such that $\alpha_\Sigma(\rho) \models\!\mid \rho'$, and*
  - *β_Σ is conservative [5].*

Then I has the $\mathcal{D}$-OTP if I' has the $\mathcal{D}'$-omitting type property for some broad subcategory $\mathcal{D}' \subseteq \mathbb{S}ig'$ of signature morphisms such that $\phi(\mathcal{D}) \subseteq \mathcal{D}'$.

Proof. Assume $\Gamma \subseteq \mathbb{S}en(\Sigma)$ locally χ_n-omits Δ_n for all $n \in \mathbb{N}$, where $\Sigma \in |\mathbb{S}ig|$, $(\Sigma \xrightarrow{\chi_n} \Sigma_n \in \mathcal{D} : n \in \mathbb{N})$ and $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n) : n \in \mathbb{N})$.

We show that Γ' locally χ'_n-omits Δ'_n, where $\Gamma' = \alpha_\Sigma(\Gamma)$, $\chi'_n = \phi(\chi_n)$ and $\Delta'_n = \alpha_{\Sigma_n}(\Delta_n)$. Let $p' \subseteq \mathbb{S}en'(\phi(\Sigma_n))$ finite. Since α_{Σ_n} is surjective modulo $\models\!\mid$ there is $p \subseteq \mathbb{S}en(\Sigma_n)$ finite such that $\alpha_{\Sigma_n}(p) \models\!\mid p'$. There is $\delta \in \Delta_n$ such that $\chi_n(\Gamma) \cup p \cup \{\neg\delta\}$ is consistent. Since β_{Σ_n} is conservative, $\chi'_n(\Gamma') \cup \alpha_{\Sigma_n}(p) \cup \{\neg \alpha_{\Sigma_n}(\delta)\}$ is consistent which implies $\chi'_n(\Gamma') \cup p' \cup \{\neg\delta'\}$ is consistent, where $\delta' = \alpha_{\Sigma_n}(\delta) \in \Delta'$.

$\phi(D) \subseteq \mathcal{D}'$ implies $\chi'_n \in \mathcal{D}'$. Since I' has $\mathcal{D}'$-OTP, there is a $\phi(\Sigma)$-model M' of Γ' which χ'_n-omits Δ'_n, for all $n \in \mathbb{N}$. This implies that $\beta_\Sigma(M')$ satisfies Γ and χ_n-omits Δ_n, for all $n \in \mathbb{N}$.

Proposition 6 (Lifting OTP to Presentations). *For any institution I which has $\mathcal{D}$-OTP, the institution I^{cpres} of countable presentations of I has $\mathcal{D}^{cpres}$-OTP, where $\mathcal{D}^{cpres}$ consists of signature morphisms of the form $\chi : (\Sigma, E) \to (\Sigma', E')$ such that $\chi \in \mathcal{D}$ and $E' \models\!\mid \chi(E)$.*

Proof. For every signature morphism $(\Sigma, E) \xrightarrow{\chi} (\Sigma', E') \in \mathcal{D}^{cpres}$, each countable set of Σ-sentences Γ, and any countable set Δ of Σ'-sentences, the following are equivalent:

[4] In any institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, $E \models\!\mid E'$ when $E \models E'$ and $E' \models E$, for all sets of Σ-sentences E and E', where $\Sigma \in |\Sigma|$.

[5] For all Σ-models M there exists a $\phi(\Sigma)$-model M' such that $\beta_\Sigma(M') = M$.

1. In I^{cpres}, Γ locally χ-omits Δ, where $\chi : (\Sigma, E) \to (\Sigma, E')$.
2. In I, $\Gamma \cup E$ locally χ-omits Δ, where $\chi : \Sigma \to \Sigma$.

Let $\Gamma \subseteq \mathbb{S}en(\Sigma)$ countable, $((\Sigma, E) \xrightarrow{\chi_n} (\Sigma_n, E_n) \in \mathcal{D}^{cpres} : n \in \mathbb{N})$ and $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n) : n \in \mathbb{N})$ such that Δ_n is countable. Assume Γ χ_n-omits Δ_n in I^{cpres}. We have $\Gamma \cup E$ χ_n-omits Δ_n in I. Since I has $\mathcal{D}$-OTP, there exists a Σ-model M of $\Gamma \cup E$ such that M χ_n-omits Δ_n, where $\chi_n : \Sigma \to \Sigma_n$ is regarded as a signature morphism in $\mathcal{D}$. Then in I^{cpres}, M is a (Σ, E)-model of Γ which χ_n-omits Δ_n, where $\chi_n : (\Sigma, E) \to (\Sigma_n, E_n)$ is regarded as a signature morphism in $\mathcal{D}^{cpres}$.

4.2 Omitting Types in FOPA

In order to establish that **FOPA** has the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite number of total constant symbols, we need to set the parameters of Theorem 4. We recall the definition of a comorphism $(\phi, \alpha, \beta) : \mathbf{FOPA} \to \mathbf{FOL}^{cpres}$ which can be found, for example, in [40] or [37].

- Each **FOPA**-signature (S, TF, PF) is mapped to the **FOL** presentation $((S, TF, \overline{PF}), E_{(S,TF,PF)})$ where $\overline{PF}_{ws} = PF_{w \to s}$ for all $w \in S^*$ and $s \in S$, and $E_{(S,TF,PF)} = \{(\forall X \uplus \{y, z\})\sigma(X, y) \wedge \sigma(X, z) \Rightarrow (y = z) \mid \sigma \in PF\}$.
- Each $((S, TF, \overline{PF}), E_{(S,TF,PF)})$-model M is mapped to the algebra $\beta_{(S,TF,PF)}(M)$ such that
  - $\beta_{(S,TF,PF)}(M)_x = M_x$ for each $x \in S$ or $x \in TF$,
  - for each $\sigma \in PF$, $\beta_{(S,TF,PF)}(M)_\sigma(m) = n$ when $(m, n) \in M_\sigma$.
- For all **FOPA**-signatures (S, TF, PF)
  - $\alpha(t \stackrel{e}{=} t') = (\exists X \uplus \{x_0\}) bind(t, x_0) \wedge bind(t', x_0)$, where for each (S, TF, PF)-term t and variable x, $bind(t, x)$ is a finite conjunction of atoms defined by
  $$bind(\sigma(t_1, \dots, t_n), x) = \bigwedge_{1 \le i \le n} bind(t_i, x_i) \wedge \begin{cases} \sigma(x_1, \dots, x_n) = x & \text{if } \sigma \in TF \\ \sigma(x_1, \dots, x_n, x) & \text{if } \sigma \in PF \end{cases}$$
  where X is the set of new constant symbols introduced by $bind(t, x_0)$ and $bind(t', x_0)$.
  - commutes with the first-order constructors on sentences.

Corollary 5. **FOPA** *has the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite number of total constant symbols.*

Proof. By Proposition 6, we lift the OTP from **FOL** to $\mathbf{FOL}^{cpres}$. Then we apply Theorem 4 to the above comorphism. Note that for all **FOPA**-signatures we have

- $\beta_{(S,TF,PF)}$ is conservative because it is an isomorphism, and
- $\alpha_{(S,TF,PF)}$ is surjective modulo $\models\mid$ because it is surjective modulo $\models\mid$ on atoms and it commutes with the first-order constructors on sentences.

Therefore the conditions of Theorem 4 are fulfilled and we infer that **FOPA** has the OTP.

Similarly one can define a comorphism $\mathbf{FOPA}_{\omega_1,\omega} \to \mathbf{FOL}^{cpres}_{\omega_1,\omega}$ and establish in a similar manner as above that $\mathbf{FOPA}_{\omega_1,\omega}$ has the OTP.

4.3 Omitting Types in HNK

We borrow the OTP for **HNK** along a comorphism $(\phi,\alpha,\beta) : \mathbf{HNK} \to \mathbf{FOEQL}^{cpres}$ which is defined using the ideas from [36].

- Each **HNK**-signature (S,F) is mapped to the presentation $((\overrightarrow{S},\overrightarrow{F}), E_{(S,F)})$ where
  - $\overrightarrow{S}$ is the set of all types over S,
  - $\overrightarrow{F}_s = F_s$ for each $s \in \overrightarrow{S}$, $\overrightarrow{F}_{[(s\to s')s]\to s'} = \{app_{s,s'}\}$ for all $s,s' \in \overrightarrow{S}$ and $F_{w\to s} = \emptyset$ otherwise.
  - $E_{(S,F)} = \{(\forall\{f,g,x\})app_{s,s'}(f,x) = app_{s,s'}(g,x) \Rightarrow (f = g) \mid s,s' \in \overrightarrow{S}\}$
- $\beta_{(S,F)}(M) = \overline{M}$, where $\overline{M}_s = \{\overline{m} \mid m \in M_s\}$ for all types $s \in \overrightarrow{S}$, such that
  - for each $s \in S$ and $m \in M_s$, we have $\overline{m} = m$, and
  - for each $s \to s' \in \overrightarrow{S}$ and $m \in M_{s\to s'}$, the function $\overline{m} : \overline{M}_s \to \overline{M}_{s'}$ is defined as follows: $\overline{m}(\overline{x}) = \overline{apply_{s,s'}(m,x)}$, for all $x \in M_s$.
- α is defined as the canonical extension of the mapping on the terms α^{term} defined by $\alpha^{term}(tt') = apply(\alpha^{term}(t), \alpha^{term}(t'))$.

The reader may complete the details of this definition (such as the definitions of ϕ on the signature morphisms and of the $\beta_{(S,F)}$ on the model morphisms) by herself/himself or may look into [10].

Now we have established all the necessary conditions for the applications of Theorem 4 for the comorphism $\mathbf{HNK} \to \mathbf{FOEQL}^{cpres}$.

Corollary 6. **HNK** *has the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite numbers of types.*

5 Conclusions and Future Work

We have lifted the OTP from the conventional model theory to the institution independent framework and we have developed two ways of obtaining the OTP

1. within an arbitrary institution by using the forcing technique introduced in [27]; as instances of our abstract results we have obtained the OTP for first-order logic, preorder algebra, order-sorted algebra, partial algebra and also their infinitary variants.
2. by transporting it (backwards) along the institution comorphisms; we have illustrated the applicability power of our method by deriving the OTP to first-order partial algebra and higher-order logic with Henkin semantics; as in the previous case, the abstract results can be applied to infinitary variants of the institutions we have just mentioned.

Our work is justified by the the institution-independent status of the results, and the multitude of instances of the abstract theorems. Due to the use of forcing, our work covers uniformly both finitary and ininitary case. We obtained also an abstract version of the famous Downward Löwenheim-Skolem Theorem. The interested reader may complete the details for borrowing this result along instition comorphisms to first-order partial algebra and higher-order logic with Henkin semantics.

In the future we are planning to apply our results to other logics such as institutions with predefined types [14]. We expect our methods to be applicable to most of the multitude of combinations between the logics discussed here, such as order-sorted algebra with transitions. An interesting topic for the future work would be forcing and the OTP for modal logics.

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6 Exiled proofs

Section 2:

Proof (Lemma 2).

1. $p \Vdash^w e$ iff $p \Vdash \neg\neg e$ iff for each $q \geq p$, $q \nVdash \neg e$ iff for each $q \geq p$, there exists $r \geq q$ such that $r \Vdash e$.
2. By induction on e.
   For $e \in \mathbb{S}en_0(\Sigma)$. The conclusion follows from $f(p) \subseteq f(q)$.
   For $\neg e \in \mathcal{L}$. We have $p \Vdash \neg e$. Suppose towards a contradiction $q \nVdash \neg e$, then by definition of forcing there is $q' \geq q$ such that $q' \Vdash e$. Therefore there is $q' \geq p$ such that $q' \Vdash e$, thus $p \nVdash \neg e$, which is a contradiction.
   For $\bigvee E \in \mathcal{L}$. $p \Vdash e$ for some $e \in E$. By induction $q \Vdash e$ which implies $q \Vdash \bigvee E$.
   For $(\exists\chi)e \in \mathcal{L}$. Since $p \Vdash (\exists\chi)e$ then $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $q \Vdash \theta(e)$, and by the definition of forcing relation $q \Vdash (\exists\chi)e$.
3. It follows easily from 1 and 2.
4. Obvious.

Proof (Lemma 3). The proof of this lemma is similar to the one in [32]. Since $\mathcal{L}$ is countable let $\{e_n \mid n < \omega\}$ be an enumeration of $\mathcal{L}$. We form a chain of conditions $p_0 \leq p_1 \leq \ldots$ in P as follows. Let $p_0 = p$. If $p_n \Vdash \neg e_n$, let $p_{n+1} = p_n$, otherwise choose $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash e_n$. The set $G = \{q \in P \mid q \leq p_n \text{ for some } n < \omega\}$ is generic and contains p.

Proof (Proposition 2). Let T be the set of all sentences of $\mathcal{L}$ which are forced by G. Let $B = \mathbb{S}en_0(\Sigma) \cap T$. We prove that for each $e \in \mathcal{L}$ $M_B \models e$ iff $e \in T$ by induction on e.

For $e \in \mathbb{S}en_0(\Sigma)$. Suppose $M_B \models e$ then we have $B \models e$ and by the hypothesis there is $B' \subseteq B$ finite such that $B' \models e$. Since G is generic there exists $p \in G$ such that $B' \subseteq f(p)$. Suppose towards a contradiction that $e \notin T$ which because G is generic leads to $\neg e \in T$. Then there is $q \in G$ such that $q \Vdash \neg e$. Since G is generic there is $r \in G$ such that $r \geq p$ and $r \geq q$. We have $B' \subseteq f(r)$ and using Lemma 2(2) we obtain $r \Vdash \neg e$. By the definition of forcing property $r' \Vdash e$ for some $r' \geq r$ and and by Lemma 2(2) $r' \Vdash \neg e$ which is a contradiction. If $e \in T$ then $e \in B$ and $M_B \models e$.

For $\neg e \in \mathcal{L}$. Exactly one of e, $\neg e$ is in T. Since G is generic there is $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$. Therefore $e \in T$ or $\neg e \in T$. Suppose towards a contradiction that e ,$\neg e \in T$, then there exists $p, q \in G$ such that $p \Vdash e$ and $q \Vdash \neg e$. By the definition of generic sets there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 1(2) $r \Vdash e$ and $r \Vdash \neg e$ which is a contradiction.

Let $\neg e \in T$. Suppose that $M_B \models e$, then by induction we have $e \in T$, which is a contradiction. Therefore $M_B \models \neg e$. Now if $M_B \models \neg e$, then e is not in T, therefore $\neg e \in T$.

For $\bigvee E \in \mathcal{L}$. If $M_B \models \bigvee E$ then $M_B \models e$ for some $e \in E$. By induction $e \in T$. We have $p \Vdash e$ for some $p \in G$ and we obtain $p \Vdash \bigvee E$. Thus, $\bigvee E \in T$. Now if $\bigvee E \in T$ then $e \in T$, for some $e \in E$. Therefore, by induction, $M_B \models e$ and thus $M_B \models \bigvee E$.

For $(\exists\chi)e \in \mathcal{L}$. Assume that $M_B \models (\exists\chi)e$ where $\chi : \Sigma \to \Sigma'$. There exists a χ-expansion N of M_B such that $N \models e$. Because M_B is $\mathcal{D}$-reachable there exists a substitution $\theta : \chi \to 1_\Sigma$ such that $M_B \restriction_\theta = N$. By the satisfaction condition $M_B = N \restriction_\chi \models \theta(e)$.

By induction $\theta(e) \in T$ which implies $(\exists\chi)e \in T$. For the converse implication assume that $p \Vdash (\exists\chi)e$ for some $p \in G$. We have that $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $M_B \models \theta(e)$ which implies $M_B \upharpoonright_\theta \models e$. Since $(M_B \upharpoonright_\theta) \upharpoonright_\chi = M_B$ we obtain $M_B \models (\exists\chi)e$.

Proof (Theorem 2). By Lemma 3 there is a generic set $G \subseteq P$ such that $p \in G$ and by Proposition 2 there is a $\mathcal{D}$-reachable model M for G.

Proof (Corollary 1). Suppose $p \Vdash^w e$ and M is a generic model for p which is also $\mathcal{D}$-reachable. We have $p \Vdash \neg\neg e$ which implies $M \models \neg\neg e$ and $M \models e$. Now for the converse implication suppose that $p \nVdash^w e$. There is $q \Vdash \neg e$ for some $q \geq p$. By Proposition 2 there is a generic model M for q which is also $\mathcal{D}$-reachable; this implies $M \models \neg e$. But M is also a generic model for p.

Section 3:

Proof (Lemma 4). We show $\mathbb{S}en_0(\Sigma') \subseteq \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$. Let $e \in \mathbb{S}en_0(\Sigma')$. Since e is finitary it can be written as $\varphi(e_f)$ where $\varphi : \Sigma_f \to \Sigma'$ is a signature morphism such that Σ_f is finitely presented in the category $\mathbb{S}ig$. By finiteness of Σ_f there exists a signature morphism $\varphi_i : \Sigma_f \to \Sigma_i$ such that $\varphi_i; v_i = \varphi$. We have that $e = v_i(\varphi_i(e_f))$. Therefore $\mathbb{S}en_0(\Sigma') = \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$.

Proof (Proposition 3). By Lemma 4 we have that $\mathbb{S}en_0(\Sigma) \subseteq \mathcal{L}_v$.

The most difficult closer property to prove is the closure of $\mathcal{L}_v$ to substitutions. The remaining cases are straightforward. Let $(\exists\chi)e \in \mathcal{L}_v$, where $\chi : \Sigma' \to \Sigma'_1$, and a substitution $\theta : \chi \to 1_{\Sigma'}$. By the definition of $\mathcal{L}_v$, $(\exists\chi)e = v_k((\exists\chi_k)e_k)$ for some $(\exists\chi_k)e_k \in \mathbb{S}en(\Sigma_k)$, where $\chi_k : \Sigma_k \to \Sigma'_k$. By our assumptions, there exists $v'_k : \Sigma'_k \to \Sigma'_1$ such that

$$\begin{tikzcd} & \Sigma'_k \arrow[rr, dotted, "v'_k"] & & \Sigma'_1 \\ \Sigma_k \arrow[ur, "\chi_k"] \arrow[rr, "v_k"'] & & \Sigma' \arrow[ur, "\chi"'] & \end{tikzcd}$$

is a pushout. Since χ_k is finitary and $(u_{k,i} \xrightarrow{v_i} v_k)_{(k \leq i) \in (J, \leq)}$ is a directed co-limit in the category $\Sigma_k / \mathbb{S}ig$, there exists $\theta_k : \chi_k \to u_{k,j}$, where $k \leq j$ such that $\theta_k; v_j = v'_k; \theta$.

$$\begin{tikzcd} & \Sigma'_k \arrow[rr, dotted, "v'_k"] \arrow[d, dotted, "\theta_k"] & & \Sigma'_1 \arrow[d, "\theta"] \\ \Sigma_k \arrow[ur, "\chi_k"] \arrow[r, "u_{k,j}"'] & \Sigma_j \arrow[r, "v_j"'] & \Sigma' \arrow[ur, "\chi"] \arrow[r, "1_\Sigma"'] & \Sigma' \end{tikzcd}$$

Therefore $\theta(e) = \theta(v'_k(e_k)) = v_j(\theta_k(e_k)) \in \mathcal{L}_v$.