Saturated Logic: its syntax

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Abstract

In this paper we investigate the expressive power of Gentzen sequent calculi and the scope of Cut-elimination theorem. For this purpose, we introduce (propositional, non-modal, with Associative and Exchange) Saturated Logic, prove Cut-elimination and study the relationship between connectives and Weakening or Contraction rules. Compared to the most common logics, Saturated Logic maximizes, within the 'standard' sequent calculi frame, the number of connectives extending the language and expanding the stock of inference rules: hence its name, as it fills, *saturates* the space of the inference forms.

Saturated Logic can be seen as a refinement of one or more known logics, plausibly able to promote the development of new logics; moreover, it can be seen not only as a 'logic-to-use', but above all as an attempt to develop a 'toolbox', useful to reformulate the presentation of logical calculi and make comparisons among them, as well as able to shed new light on given logics or on the organization of the logic space. See in this perspective, Saturated Logic might help to further understand the Cut-elimination theorem, as well as the relationship between Structural Rules and meta-language on the one side and Operational Rules and object-language on the other side.

Contents

1 Introduction: looking for unity in multiplicity

In the words of Béziau, ‘as universal algebra is a general theory of algebraic structures, so the universal logic is a general theory of logical structures’. According to this perspective, the universal logic is not a new logic, but a way to unify the multiplicity of logics [2]. This research field embodies what might be called *weak assumption*: the universal logic is a single *theoretical framework* within which to deal with the multiplicity of logics.

On the other hand, the substructural logics [3, 8, 11, 12, 6, 9] are now one of the most prolific areas of research and thus providing interesting consequences to the field of logic as a whole, and not only. In this area one can find a ‘propensity’ to the generalization that goes in the direction of the themes and issues of universal logic. Overall, it can be concluded that the sequent calculi (and their variants) have been a unifying element for the wide and diverse range of so-called substructural logics, confirming their position as an important and very fruitful tool in logic. This research field invites us to make what might be called *strong assumption*: Universal Logic is a unique *logical system* that subsumes the multiplicity of logics. With different sensitivities, some systems that embody this ‘tension’ are: Unified Logic of Girard [5], Display Logic of Belnap [7] and Basic Logic [13] of Sambin. The present research lies in that trend.

1.1 Motivations

In a Gentzen sequent calculus[1] for a logic [4, 15, 10], binary Operational Rules are usually divided into:

- ‘context-sensitive rules’, where the binary component of the rule treats the contexts by identification and the unary component takes only one auxiliary formula, as for example in the following case for the Conjunction (cf. ‘with’ & in Linear Logic)

$$\wedge_1 \frac{\Gamma, \chi \in \{\varphi, \psi\} \vdash \Delta}{\Gamma, \varphi \wedge_1 \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge_1 \psi, \Delta} \wedge_1$$

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[1] Note: the sequents considered here are ‘finite sequences’.

- ‘context-free rules’,[2] where the binary component treats the contexts by juxtaposition and the unary component takes two auxiliary formulas, as in the following case for the Conjunction (cf ‘times’ $\otimes$ in Linear Logic)

$$\wedge_2\ \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge_2 \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma' \vdash \psi, \Delta'}{\Gamma, \Gamma' \vdash \varphi \wedge_2 \psi, \Delta, \Delta'}\ \wedge_2$$

With the advent of Linear Logic the use, respectively, of the expressions ‘additive rule’ and ‘multiplicative rule’[3] has become well established: we use the pair multiplicative/additive in a way compatible with its use in Linear Logic and, although it is a little more specialized, immediately obvious. It is immediate to observe that in literature, in order to define the rules of inference, some left-right pairings between multiplicative and additive form of the rules were simply not considered or were not considered in a systematic way within a single framework (in particular, all the rules that are here classified as *heterogeneous*). Let us see respectively two examples:

- among the Operational Rules, the pair $(\wedge_3 L, \wedge_3 R)$

$$\wedge_3\ \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge_3 \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma' \vdash \psi, \Delta}{\Gamma, \Gamma' \vdash \varphi \wedge_3 \psi, \Delta}\ \wedge_3$$

  defines the behaviour of the binary connective $\wedge_3$, a sort of Conjunction where the left component of the rule $\wedge_3 L$ is *multiplicative*, while the right component of the rule $\wedge_3 R$ is *multiplicative-to-left* (of the proof symbol or ‘turnstile’), and *additive-to-right* (of the proof symbol), which classifies it as a heterogeneous rule;

- among the Structural Rules, the pair $(\vdash_4 L, \vdash_4 R)$

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[2] The choice of the oppositional couple ‘context-sensitive/context-free’ for the nomenclature is motivated by the fact that the first type of rules applies if a certain condition is met on the contexts, while the second type is free of conditions and it is borrowed from the theory of Chomsky grammars. Similar oppositional expressions are ‘context-sharing/context-private’, where the reference is clear in a paradigm that interprets formulas as resources that can be shared or not (cf. Bunched Logic). Conversely, in the case of ‘extensional/intensional’ the reference is tied to historical reasons more than anything else: starting from Boole, connectives of Conjunction and Disjunction expressed using the first form were interpreted in the operations of sets intersection and sets union called ‘extensional’, then the modal logics have proved capable of handling even the so-called intensional expressions as opposed to the extensional and, by similarity, the connectives expressible in the second form are called ‘intensional’ as opposed to extensional connectives (see Relevant Logic).

[3] The choice of the couple ‘multiplicative/additive’ is motivated by the fact that in the so-called Phase semantics, the first semantics for Linear Logic proposed by Girard, each context-free connective can be expressed by a ‘multiplication’ (i.e. a product of phases) while each context-sensitive connective can be expressed by a ‘sum’ (i.e. a union of phases also called direct sum).

$$\vdash_4 \frac{}{\varphi \vdash_4 \varphi} \qquad \frac{\Gamma \vdash_4 \varphi, \Delta \qquad \Gamma', \varphi \vdash_4 \Delta}{\Gamma, \Gamma' \vdash_4 \Delta} \vdash_4$$

defines the behaviour of the metalinguistic turnstile $\vdash_4$ (together with the metalinguistic comma), a sort of relation where the left component of the rule $\vdash_4 L$ (a sort of Identity) is *multiplicative-to-left* and *multiplicative-to-right*, while the right component of the rule $\vdash_4 R$ (a sort of Cut) is *multiplicative-to-left* and *additive-to-right*, which classifies it as a heterogeneous rule.

Therefore, a systematic treatment of possible inference rule forms is required: this is the main conceptual motivation for introducing Saturated Logic. Here we shall only consider *standard sequent calculi* which require few and 'reasonable' restrictions on the form of inference rules: Saturated Logic is designed to exhaust the combinatorial space of inferential behaviours in such frame.

1.2 Symmetries

In Saturated Logic each pair of Operational Rules corresponds to a *symmetric* pair of rules,[4] therefore: each Conjunction form corresponds to a Disjunction form and vice versa, so as each Implication form corresponds to a *Disimplication* form and vice versa. Here are two examples:

- both $(\wedge_5 L, \vee_6 R)$ and $(\vee_6 L, \wedge_5 R)$ are symmetric pairs

$$\wedge_5 \frac{\Gamma, \chi \in \{\varphi, \psi\} \vdash \Delta}{\Gamma, \varphi \wedge_5 \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \chi \in \{\varphi, \psi\}, \Delta}{\Gamma \vdash \varphi \vee_6 \psi, \Delta} \vee_6$$

$$\vee_6 \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \vee_6 \psi \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma' \vdash \psi, \Delta'}{\Gamma, \Gamma' \vdash \varphi \wedge_5 \psi, \Delta, \Delta'} \wedge_5$$

  in fact the component $\wedge_5 L$ (resp. $\wedge_5 R$) corresponds to the symmetric component $\vee_6 R$ (resp. $\vee_6 L$) and vice versa, and therefore the connective $\wedge_5$ is the symmetric of $\vee_6$ and vice versa;

- both $(\rightarrow_7 L, \leftarrow_8 R)$ and $(\leftarrow_8 L, \rightarrow_7 R)$ are symmetric pairs

$$\rightarrow_7 \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \psi \vdash \Delta}{\Gamma, \Gamma', \varphi \rightarrow_7 \psi \vdash \Delta} \qquad \frac{\Gamma, \psi \vdash \Delta \qquad \Gamma \vdash \varphi, \Delta'}{\Gamma \vdash \varphi \leftarrow_8 \psi, \Delta, \Delta'} \leftarrow_8$$

$$\leftarrow_8 \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \leftarrow_8 \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \rightarrow_7 \psi, \Delta} \rightarrow_7$$

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[4] See the concept of symmetry as developed in Linear Logic, Basic Logic and Display Logic.

in fact the component $\rightarrow_7 L$ (resp. $\rightarrow_7 R$) corresponds to the symmetric component $\leftarrow_8 R$ (resp. $\leftarrow_8 L$) and vice versa, and therefore the connective $\rightarrow_7$ is the symmetric of $\leftarrow_8$ and vice versa.

The *Multiplicative (structure) Saturated Logic* $M.LSat$ results from taking multiplicative-to-left and to-right Identity and multiplicative-to-left and to-right Cut, and by defining all the combinatorial predictable Operational Rules in the sense previously specified.[5] In this article, we consider explicitly only the *Multiplicative Saturated Logic with Exchange and Associativity* $AEM.LSat$, i.e. we assume also associativity and commutativity of the comma. Similar Structural Rules as primitive rules of calculus allow to not consider systematically primitive Operational Rules that discriminate in the way of order or assembly the contexts in the conclusion sequent or the auxiliary formulas in the main formula.

1.3 Advantages

About the Cut-elimination we observe:

- in $AEM.LSat$ some pairs of rules allow to derive Weakening W or Contraction C (Left or Right): it becomes meaningful to identify such pairs of rules (here classified as *structured*, in opposition to the others classified as *unstructured*);
- in this article we sketch the proof of Cut-elimination (specified below as 'Cut-containment') for the system $AEM.LSat$: it becomes meaningful to systematically investigate which subsystems are Cut-free, or what constraints preserve the Cut-elimination;
- always about the syntactic side, we reserve to investigate the relationship between the concept of Cut-containment and some attempts to define general properties that guarantee the syntactic Cut-elimination or its preservation.[6]

About the 'space of logics' organization we could argument:

- Basic Logic confirms to be a fundamental system, since its connectives are 'ambiguous' in the sense that they can 'evolve' (going up in the cube of the extensions of Basic Logic) in both their corresponding *homogeneous* or *heterogeneous* version (in the sense of Saturated Logic);

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[5] Of course, it assumes that one has defined a language with as many symbols of connectives as you need, i.e. one for each pair of rules of introduction.

[6] See the set of properties identified by Belnap in Display Logic [7, 12] or the propagation property identified by Terui [14].

- Classic Logic confirms more clearly to be a ‘superior limit’, since its *saturation* coincides with $AEM.LSat$;[7]
- many logics (all those exhibit a standard sequent calculus?) would be sublogics of Saturated Logic in a clear sense: that is, as appropriate choices of a subset of the rules of $LSat$.

1.4 Structure of the paper

In section 2 crucial notation and terminology is introduced: note that a list of connectives rules is not explicitly given in the paper, but they can be unambiguously reconstructed from conventions on the shape of the symbols given in this section. In section 3 some conceptual links are highlighted between Structural and Operational rules: this discussion helps to assess the meaning of Saturated Logic in the context of the Universal Logic research program, but the reader who is only interested in the technical results can skip it. In section 4 the system $AEM.LSat$ is introduced and the derivability of Weakening and Contraction as second rules (by use of some Operational rules) and the interderivability between Conjunctions connectives are studied. In section 5 the Cut-elimination for $AEM.LSat$ is specified as Cut-tradeoff theorem and it is sketched.

2 Introducing the notation

Part of this work has been the definition of a notation that would allow one to easily and immediately manipulate and compare a large number of symbols in a uniform way. In particular, the notational conventions established here bring into correspondence the (form of) symbols of the formulas language with the (form of) Operational Rules. Therefore, one can use the symbols of language to refer to and, at the same time, instantly recall the rules of calculus. In addition, it becomes possible to formulate or prove theorems by directly referring to the form of symbols.

For the complete list of $AEM.LSat$ connectives, see below Table 1: here the connectives are divided into *family* and *subfamily*, where, for example, the symbol $\wedge$ belongs to the *cancellative* family and *multiplicative* subfamily, the symbol $^{-}\!\!\wedge$ belongs to the *adjunctive* family and *(west-over-)sub-additive* subfamily, while the symbol $^{-}\!\!\wedge\!\!^{-}$ belongs to the *adjunctive* family and *(west-east-over-)sub-additive* subfamily.

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[7] We reserve to investigate the space of the logics in a next job, so here we do not specify over what we mean by ‘saturation’. If one is interested in keeping all the $AEM.LSat$ connectives in a single system and at the same time at preventing their ‘collapse’, it seems likely to proceed in the direction of Linear Logic, namely: introducing modalities, or in the direction of Display Logic, namely: considering further punctuation than the comma, in order to deal with a lot of distinct *families* of connectives.

cancellative family

muliplic.	$\wedge$ $\to$ $\top$	$\vee$ $\leftarrow$ $\bot$
almost-mul.	$\wedge\vert$ $\to\vert$ $\top\vert$	$\vert\vee$ $\vert\leftarrow$ $\vert\bot$
semi-mul.	$\vert\wedge$ $\vert\to$ $\vert\top$	$\vee\vert$ $\leftarrow\vert$ $\bot\vert$
sub-mul.	$\vert\wedge\vert$ $\vert\to\vert$ $\vert\top\vert$	$\vert\vee\vert$ $\vert\leftarrow\vert$ $\vert\bot\vert$

adjunctive family

sub-add.	$\overline{\wedge}$ $\overline{\to}$ $\overline{\leftarrow}$ $\overline{\top}$ $\overline{\bot}$	${}^{-}\wedge$ ${}^{-}\to$ ${}^{-}\leftarrow$	$\wedge^{-}$ $\to^{-}$ $\leftarrow^{-}$	${}^{-}\wedge^{-}$ ${}^{-}\to_{-}$ ${}_{-}\leftarrow^{-}$	$\overline{\underline{\to}}$ $\overline{\underline{\top}}$	$\overline{\underline{\leftarrow}}$ $\overline{\underline{\bot}}$	${}_{-}\vee_{-}$ ${}^{-}\leftarrow_{-}$ ${}_{-}\to^{-}$	${}_{-}\vee$ ${}_{-}\leftarrow$ ${}_{-}\to$	$\vee_{-}$ $\leftarrow_{-}$ $\to_{-}$	$\underline{\vee}$ $\underline{\leftarrow}$ $\underline{\to}$ $\underline{\bot}$ $\underline{\top}$
semi-add.	$\overline{\wedge}\vert$ $\overline{\to}\vert$ $\vert\overline{\leftarrow}$ $\overline{\top}\vert$ $\vert\overline{\bot}$	${}^{-}\wedge\vert$ ${}^{-}\to\vert$ $\vert{}^{-}\leftarrow$	$\wedge^{-}\vert$ $\to^{-}\vert$ $\vert\leftarrow^{-}$	${}^{-}\wedge^{-}\vert$ ${}^{-}\to_{-}\vert$ $\vert{}_{-}\leftarrow^{-}$	$\overline{\underline{\to}}\vert$ $\overline{\underline{\top}}\vert$	$\vert\overline{\underline{\leftarrow}}$ $\vert\overline{\underline{\bot}}$	$\vert{}_{-}\vee_{-}$ $\vert{}^{-}\leftarrow_{-}$ ${}_{-}\to^{-}\vert$	$\vert{}_{-}\vee$ $\vert{}_{-}\leftarrow$ ${}_{-}\to\vert$	$\vert\vee_{-}$ $\vert\leftarrow_{-}$ $\to_{-}\vert$	$\vert\underline{\vee}$ $\vert\underline{\leftarrow}$ $\underline{\to}\vert$ $\vert\underline{\bot}$ $\underline{\top}\vert$
almost-add.	$\vert\overline{\wedge}$ $\vert\overline{\to}$ $\overline{\leftarrow}\vert$ $\vert\overline{\top}$ $\overline{\bot}\vert$	$\vert{}^{-}\wedge$ $\vert{}^{-}\to$ ${}^{-}\leftarrow\vert$	$\vert\wedge^{-}$ $\vert\to^{-}$ $\leftarrow^{-}\vert$	$\vert{}^{-}\wedge^{-}$ $\vert{}^{-}\to_{-}$ ${}_{-}\leftarrow^{-}\vert$	$\vert\overline{\underline{\to}}$ $\vert\overline{\underline{\top}}$	$\overline{\underline{\leftarrow}}\vert$ $\overline{\underline{\bot}}\vert$	${}_{-}\vee_{-}\vert$ ${}^{-}\leftarrow_{-}\vert$ $\vert{}_{-}\to^{-}$	${}_{-}\vee\vert$ ${}_{-}\leftarrow\vert$ $\vert{}_{-}\to$	$\vee_{-}\vert$ $\leftarrow_{-}\vert$ $\vert\to_{-}$	$\underline{\vee}\vert$ $\underline{\leftarrow}\vert$ $\vert\underline{\to}$ $\underline{\bot}\vert$ $\vert\underline{\top}$
additive	$\vert\overline{\wedge}\vert$ $\vert\overline{\to}\vert$ $\vert\overline{\leftarrow}\vert$ $\vert\overline{\top}\vert$ $\vert\overline{\bot}\vert$	$\vert{}^{-}\wedge\vert$ $\vert{}^{-}\to\vert$ $\vert{}^{-}\leftarrow\vert$	$\vert\wedge^{-}\vert$ $\vert\to^{-}\vert$ $\vert\leftarrow^{-}\vert$	$\vert{}^{-}\wedge^{-}\vert$ $\vert{}^{-}\to_{-}\vert$ $\vert{}_{-}\leftarrow^{-}\vert$	$\vert\overline{\underline{\to}}\vert$ $\vert\overline{\underline{\top}}\vert$	$\vert\overline{\underline{\leftarrow}}\vert$ $\vert\overline{\underline{\bot}}\vert$	$\vert{}_{-}\vee_{-}\vert$ $\vert{}^{-}\leftarrow_{-}\vert$ $\vert{}_{-}\to^{-}\vert$	$\vert{}_{-}\vee\vert$ $\vert{}_{-}\leftarrow\vert$ $\vert{}_{-}\to\vert$	$\vert\vee_{-}\vert$ $\vert\leftarrow_{-}\vert$ $\vert\to_{-}\vert$	$\vert\underline{\vee}\vert$ $\vert\underline{\leftarrow}\vert$ $\vert\underline{\to}\vert$ $\vert\underline{\bot}\vert$ $\vert\underline{\top}\vert$
	over-	west-over-	east-over-	west-east-over- east-under-west-over- west-under-east-over-	under-over-	over-under-	east-over-west-under- west-over-east-under- east-west-under-	west-under-	east-under-	under-

Table 1: connectives of *AEM.LSat*

In Table 1 the connectives of each subfamily are listed in the following order:

- first the connectives of the *genus* ‘junction’ and, respectively, of *species* ‘Conjunction’ and ‘Disjunction’
- then the connectives of the *genus* ‘implicature’ and, respectively, of *species* ‘Implication’ and ‘Disimplication’
- finally the connectives of the *genus* ‘truth constants’ and, respectively, of *species* ‘Truth’ and ‘False’

For each row of table, proceeding from the extremes toward the center, the symbols occur in pairs with a double symmetry: along a horizontal axis and along a vertical axis that orthogonally meet each other at the midpoint of each symbol. For example, each symbol in the pair $(\,|\overset{-}{\rightarrow}\,,\,\underset{-}{\leftarrow}|\,)$ is the symmetric of the other, both along the vertical axis and along the horizontal axis that cross them in the midpoint. Similarly for $(\wedge, \vee)$ and so on.

Such observation is meaningful in Saturated Logic since the notation, i.e. the form of symbols, mirrors the form of the rules and therefore ‘symmetry into notation’ corresponds to ‘symmetry into rules’.

Definition 2.1 *The unary (resp. zero-ary) rule for a binary (resp. zero-ary) connective introduces a main formula that consists of two subformulas (resp. contexts*[8]*), which we call*

- visible-formula *(resp.* -context*) if it is already in the premise sequent*
- gosth-formula *(resp.* -context*) if it appears for the first time in the conclusion sequent*

The rules having at least one ghost component (i.e. a formula or context) can be ‘simulated’ in two steps by an application of Weakening that introduces the ghost-components and by the application of the multiplicative rule corresponding to the considered connective. Therefore, we say that such a main component is obtained by *adjunction* (hence the name of ‘adjunctive family’).

Definition 2.2 *The binary (resp. zero-ary) rule for a binary (resp. zero-ary) connective introduces a main formula and a context in each side, which we call*

- as-it-is-context *if it consists of the concatenation of contexts on the same side in the premise sequents (resp. if it consists of the copy of the context on the same side in the premise sequent);*

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[8] To left and to right of the turnstile.

- halved-context *if it consists of one and only one copy of contexts on the same side in the premise sequents (resp. in the premise sequent) and if these are equal to each other.*

The rules having at least a halved-context can be 'simulated' in two steps by an application of the multiplicative rule corresponding to the considered connective and by a suitable number of applications of Contraction (and eventually Exchange). Therefore, we say that such a main context is obtained by *cancellation* (hence the name of 'cancellative family').

For the symbolic notation of a connective - binary or zero-ary - we adopt the following conventions (see below for some examples):

- a horizontal dash, eventually compounded, is named *horizontal bar*. It means that the rule with a smaller number of premises - unary in the case of a binary connective and zero-ary in the case of a zero-ary connective - is additive, in particular:

  the horizontal bar that looks long, centered and superscribed is named *over bar* (resp. *under bar* if subscribed) and it means that the ghost-component can be 'simulated' to the left (resp. to the right) of the turnstile, without discriminating whether it will appear to the left or to the right of the connective - binary or zero-ary;

  the horizontal bar that looks compounded by an over bar and by an under bar is named *over-under bar* or *under-over bar* and it means that the ghost-component can be 'simulated' both to the left and to the right of the turnstile;[9]

  the horizontal bar that looks short and left-aligned is named *west bar* (resp. *east bar* if right-aligned) and it discriminates more than a long horizontal bar indicating also that the ghost-formula appears to the left (resp. to right) of the binary connective;[10]

  the horizontal bar that looks compounded by a west bar and by an east bar is named *west-east bar* or *east-west bar* and it means that the ghost-formula appears both to the left and to the right of the binary connective;[11]

- the vertical bar that looks to left is named *left bar* (resp. *right bar* if to right). It means that the rule with a major number of premises

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[9] i.e. it happens that: the main formula is compounded by two ghost-formulas, a sort of limited Weakening introducing just formulas with the considered connective as main; or it happens that: a ghost-context is introduced to right and another one is introduced to left.

[10] In more details, as the case, here we consider: *west-over* or *west-under bar*, *east-over* or *east-under bar*. We do not regard contexts because the Exchange rule allow to forget similar rules in the case of zero-ary connectives.

[11] In more details, as the case, here we consider: *west-east-over*, *east-under-west-over*, *west-under-east-over*, *east-over-west-under*, *west-over-east-under*, *east-west-under*.

- binary in the case of a binary connective and unary in the case of a zero-ary connective - is multiplicative to left (resp. to right) of the turnstile, i.e. the halved-context can be 'simulated' to left (resp. to right).

Consider the Conjunction almost-additive $|\overline{\wedge}$

- the symbol exhibits an over bar, the correspondent rule with the minor number of premises is $|\overline{\wedge}\,L$, which is unary and it has to be written in additive form to left of the turnstile, but without discriminating if the ghost-formula appears to the left or to the right of the visible-formula

  $$\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A\,|\overline{\wedge}\,B \vdash \Delta} \quad \text{or} \quad \dfrac{\Gamma, B \vdash \Delta}{\Gamma, A\,|\overline{\wedge}\,B \vdash \Delta}$$

  this rule can be simulated in this way

  $$\dfrac{W\,\dfrac{\Gamma, A\ \overline{}\ \vdash \Delta}{\Gamma, A, B \vdash \Delta}}{\Gamma, A \wedge B \vdash \Delta} \quad \text{or} \quad \dfrac{W\,\dfrac{\Gamma, \ \overline{}\ B \vdash \Delta}{\Gamma, A, B \vdash \Delta}}{\Gamma, A \wedge B \vdash \Delta}$$

  where the horizontal bar has a mnemonic value: in some sense it is as if this bar were the marker of an empty seat, which has to be immediately filled with a new occurrence of formula;

- the symbol exhibits a left bar, the correspondent rule with the major number of premises is $|\overline{\wedge}\,R$, this is binary and it has to be written in additive form to the left of the turnstile

  $$\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta'}{\Gamma \vdash A\,|\overline{\wedge}\,B, \Delta, \Delta'}$$

  this rule can be simulated in this way

  $$\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta'}{\Gamma, \Gamma \vdash A \wedge B, \Delta, \Delta'} \atop C, E\ \dfrac{}{|\ \Gamma \vdash A \wedge B, \Delta, \Delta'}$$

  where the vertical bar has a mnemonic value: in some sense it is as if this bar marked the cancellation of a copy of the occurrences in as-it-is-context.

By symmetry, similar considerations apply in respect of over-bar, right bar and the right side of the turnstile.[12]

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[12] If we consider modal expansions of the Saturated Logic, the notation for exponentials of the Linear Logic 'of course $!A$' and 'why not $?A$' can be uniformly replaced respectively by $\overline{A}$ and $\underline{B}$. Note that $\Gamma, \overline{A} \wedge B \vdash \Delta$ is not the same as $\Gamma, A \overline{\wedge} B \vdash \Delta$, as a possible proof that involves the first sequent must first verify that $\Gamma, \overline{A}, B \vdash \Delta$ with the 'formula' A in additive form and freely in the context, while in the latter case the formula A is only a 'subformula' of a formula in additive form.

At this point, the *ratio* of adopted notational conventions should be clear and other few examples follow. In the case of almost-additive Conjunction $|\bar{\wedge}$, the expression 'to right of the auxiliary visible-formula' for $|\bar{\wedge}L$ results in the following rule

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,|\bar{\wedge}\, B \vdash \Delta}$$

In the case of almost-additive Conjunction $|\overline{\wedge}$, the expression 'both to left and to right of the (eventual) auxiliary visible-formula' for $|\overline{\wedge}L$ results in the following rule

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \,|\overline{\wedge}\, B \vdash \Delta}$$

Finally, in the case of almost-additive constant for Truth $|\overline{\overline{\top}}$

- the symbol exhibits an over bar, the correspondent rule with a minor number of premises is $|\overline{\overline{\top}}R$, which is zero-ary and it has to be written in additive form to the left of turnstile

$$\frac{}{\Gamma \vdash |\overline{\overline{\top}}}$$

- the symbol exhibits a left bar, the correspondent rule with a major number of premises is $|\overline{\overline{\top}}R$, which is unary and it has to be written in additive form to the left of turnstile

$$\frac{\Gamma, \Gamma \vdash \Delta}{\Gamma, |\overline{\overline{\top}} \vdash \Delta}$$

3 Structure and operations

In the natural deduction calculi and in the sequent calculi the status of Operational Rules (RO) is clear: in accordance with Gentzen, it can be argued that they specify, in whole or in part, the meaning of the connectives symbols. Conversely, the status of Structural Rules (RS) is not entirely clear: of course it can be said that they specify the allowable manipulations of the structure of a sequent, while one can see that they do not involve a direct manipulation of any symbol for the connectives. Therefore, one may asks what is the connection between Structural Rules and Operational Rules.

3.1 Structural and operational viewpoint

If one refuses a holistic viewpoint (so to say, in the spirit of the calculi à la Hilbert) and embraces instead an inferential viewpoint (so to say, in the spirit of natural deduction and sequent calculi), it becomes possible to distinguish between an *operational meaning* (given by the introduction rules for each connective) and a *global meaning* (given by the provable sequents containing each connective).

The positions expressed in contemporary research differ considerably depending on the weight accorded to the Structural Rules and may vary between the following two extremes that we might call

- *operational viewpoint*: the SR correspond to restrictions on discharge of assumptions and are tied to a particular formalism, therefore: they do not play any role in determining the content of the connectives, the OR specify the operational content of the logical constants;
- *structural viewpoint*: the SR specify different ways to 'compose' or 'assembly' the premises (at a higher-level language, the language of punctuation marks), the OR explicit these possibilities (in a lower-level language, the language of formulas), therefore: they have to be considered as 'translation rules'.

The *Linear Logic*, now one of the most influential research fields, encompasses the structural viewpoint: Girard's statement 'a logic is essentially a set of structural rules' is renown.

A third and alternative viewpoint, respectively, in both its versions, is fundamental and important in two general approaches to the logic: the *Display Logic* and the *Basic Logic*, and probably it is no coincidence if both these logics aim to 'unification' in the field of logic. We can distinguish between those we might call

- *interactionist-from-above viewpoint*: the OR are not sufficiently selective, because they say something about the metalinguistic signs and the contexts on the same side of the main formula. Therefore, it is desirable to find systems that meet the *display property*:

  each part of a sequent is isolable, since one can always switch to an equivalent sequent within which it represents the whole precedent or the whole succedent

  namely that, for each sequent S_1, there exists an equivalent sequent S_2 such that both of the following hold

  for all $\Pi \subseteq \Gamma$, $S_1 = \Gamma \vdash \Delta \Leftrightarrow \Pi \vdash \Sigma = S_2$

  for all $\Sigma \subseteq \Delta$, $S_1 = \Gamma \vdash \Delta \Leftrightarrow \Pi \vdash \Sigma = S_2$
- *interactionist-from-below viewpoint*:[13] the meaning of a connective 'is also determined by the contexts in its rules, which may be carriers of

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[13] About the four 'views' exposed, respectively see the viewpoint i) *nihilistic*, ii) *relativistic*, iii) *of indeterminacy (of the meaning) - first version*, e iv) *- second version* in [11].

hidden information on the behaviour of the connective ’. Therefore, it is desirable that a system satisfies the *visibility property*:

the OR do not have any context on the same side of the main formula and auxiliary formulas

namely that, for each sequent $S = \Gamma \vdash \Delta$ in a proof,[14] at least one of the following holds

$\Gamma = \varphi$ or $\Gamma = \varnothing$

$\Delta = \psi$ or $\Delta = \varnothing$

One can see a certain analogy between the first version and how to proceed ‘from above’ characteristic of coinductive definition and recursion, as well as between the second version and how to proceed ‘from below’ characteristic of inductive definition and recursion. Note that: if definitions and recursion are performed on a inductive set, proceeding by induction or by coinduction makes no difference, otherwise distinguishing among Antifoundation principles and investigating the relationship between not-well-founded and well-founded objects become important.[15] In order to feed this suggestion and for the moment no further, we propose the following diagram

$$\underbrace{\begin{array}{cc}
\text{induction principle} & \text{coinduction principle} \\
\text{compositional principle of meaning} & \text{holistic principle of meaning} \\
\text{operational meaning} & \text{global meaning} \\
\text{Operational Rules} & \text{Structural Rules} \\
\text{operational viewpoint} & \text{structural viewpoint} \\
\text{interactionist-from-below viewpoint} & \text{interactionist-from-above viewpoint}
\end{array}}_{\text{interactionistic viewpoint}}$$

The Structural Rules are the bearers of global meaning and the rules for the connectives or Operational Rules are the bearers of the operational meaning. Only within a calculus it makes sense to question about the meaning of a connective that is generated in the *interaction* between Structural Rules and Operational Rules.

Given a calculus, if a connective is always isolable on the one side of the sequent in which it appears (displaying property) or a connective is always introduced in isolation on the one side (visibility property), then the role of *active contexts* (i.e. contexts on the same side of the main and auxiliary formulas) becomes easier to control because it can be reduced to the role of *passive contexts* (i.e. the contexts on the opposite side of the main and auxiliary formulas).

---

[14] It does not happen that $\Gamma = \Delta = \varnothing$, because the empty sequents are not allowed.

[15] In the scope of set theory see [1] on various Antifoundation axioms.

The Saturated Logic allows that a connective may be heterogeneous, i.e. 'multiplicative-to-left and additive-to-right' or 'additive-to-left and multiplicative-to-right' and, as such, it is both (pre-)determined by passive contexts of its rules and (over-)determined by the active contexts of its rules.

3.2 Saturated calculi and saturation matrix

Definition 3.1 *A* standard sequent calculus *is a sequent calculus where for each primitive left (resp. right) Operational Rule, the auxiliary formulas are generic*[16] *and the main formula introduces to left (resp. to right) one and only one zero-ary or binary connective (the unary connettives, i.e. Negations and Denegations, are not primitive but defined).*

A saturated sequent calculus*, if not otherwise specified, is a standard sequent calculus where Weakening, Contraction, Associativity and Exchange are not allowed as primitive rules but, eventually, as derived rules.*

Definition 3.2 *In a saturated sequent calculus the primitive* Operational Rules *take a finite number of forms, that can be grouped according to two antonymous features:*

- homogeneous*, if both to left and to right the behaviour is the same (multiplicative on the either sides or, exclusively, additive on the either sides)*
- heterogeneous*, if to left and to right the behaviour is different (multiplicative on one side and additive on the other side)*

A finer classification is obtained by considering two other antonymous features:

- unstructured*, if the rule does not allow to derive Weakening, Contraction, Associativity or Exchange (on any sides)*
- structured*, if the rule allows to derive Weakening, Contraction, Associativity or Exchange (at least on one side)*

Definition 3.3 *A* saturation matrix *is a matrix where the features 'homogeneous' and 'heterogeneous' correspond to the columns, and the features 'unstructured' and 'structured' to the rows.*

Therefore: each rule Inf of a saturated sequent calculus is framed into one and only one crossing of the saturation matrix.

---

[16] I.e. they are atomic or compound without any constraint on the main connective and, therefore, they can be representable by a metavariable for formulas.

homogeneous	**heterogeneous**	
Inf_1	Inf_2	**unstructured**
Inf_3	Inf_4	**structured**

Table 2: saturation matrix

4 The system *AEM.LSat*

From a syntactic perspective, the Multiplicative (structure) Saturated Logic with Exchange and Associativity *AEM.LSat* is a saturated calculus where the primitive Structural Rules are: Multiplicative Identity and Multiplicative Cut; Associativity to left and to right; Exchange to left and to right; and the primitive Operational Rules are all those corresponding to the introduction of connectives in Table 1 (73 to left and 73 to right, with a total of 146), that is: 40 junctions, of which 20 Conjunctions and 20 Disjunctions; 80 implicatures, of which 40 Implications and 40 Disimplications; 32 truth constants, of which 16 constants for the Truth and 16 constants for the False.

The notational conventions introduced in the section 2 allow us to uniquely reconstruct the rules pair of each connective out of its symbol shape.

4.1 The form of Operational Rules

In regard to the *static* binary connectives, let us consider the case of Conjunction (the case of Disjunction is totally symmetric). The proposed systematic classification distinguishes 20 connectives divided into two subsets, which we called *cancellative family* and *adjunctive family*: the 4 Conjunctions of the first family do not exhibit any horizontal bar, on the contrary of the 16 Conjunctions of the second family, which is further divisible in *subfamilies*. Each (sub)family respect to the x-axis of Table 1 exhibits a gradation of left bar and right bar ordered according to the diagrams in Table 3.

The bars may be regarded as parameters, where the presence or absence of the over bar determines the membership to either one or the other of two families, while the presence or absence of vertical bars measures the degree of membership to each family (maximum for the vertices with two outgoing

cancellative family

$$\begin{tikzcd}
 & \text{mul.} & \\
 & \wedge \arrow[dl] \arrow[dr] & \\
\substack{\text{semi}\\ \text{-mul.}}\ |\wedge \arrow[dr] & & \wedge|\ \substack{\text{almost}\\ \text{-mul.}} \arrow[dl] \\
 & |\wedge| & \\
 & \text{sub-mul.} &
\end{tikzcd}$$

adjunctive family

subfam. over-

$$\begin{tikzcd}
 & \text{sub-add.} & \\
 & \overline{\wedge} & \\
\substack{\text{almost}\\ \text{-add.}}\ |\overline{\wedge} \arrow[ur] & & \overline{\wedge}|\ \substack{\text{semi}\\ \text{-add.}} \arrow[ul] \\
 & |\overline{\wedge}| \arrow[ul] \arrow[ur] & \\
 & \text{add.} &
\end{tikzcd}$$

subfam. west-over-

$$\begin{tikzcd}
 & \text{sub-add.} & \\
 & \overset{-\,}{\wedge} & \\
\substack{\text{almost}\\ \text{-add.}}\ |\overset{-\,}{\wedge} \arrow[ur] & & \overset{-\,}{\wedge}|\ \substack{\text{semi}\\ \text{-add.}} \arrow[ul] \\
 & |\overset{-\,}{\wedge}| \arrow[ul] \arrow[ur] & \\
 & \text{add.} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{sub-add.} & \\
 & \overset{\,-}{\wedge} & \\
\substack{\text{almost}\\ \text{-add.}}\ |\overset{\,-}{\wedge} \arrow[ur] & & \overset{\,-}{\wedge}|\ \substack{\text{semi}\\ \text{-add.}} \arrow[ul] \\
 & |\overset{\,-}{\wedge}| \arrow[ul] \arrow[ur] & \\
 & \text{add.} &
\end{tikzcd}$$

subfam. east-over-

$$\begin{tikzcd}
 & \text{sub-add.} & \\
 & \overset{--}{\wedge} & \\
\substack{\text{almost}\\ \text{-add.}}\ |\overset{--}{\wedge} \arrow[ur] & & \overset{--}{\wedge}|\ \substack{\text{semi}\\ \text{-add.}} \arrow[ul] \\
 & |\overset{--}{\wedge}| \arrow[ul] \arrow[ur] & \\
 & \text{add.} &
\end{tikzcd}$$

subfam. west-east-over-

Table 3: families of Conjunctions

arcs, intermediate for the vertices with an ingoing arc and an outgoing arc, minimum for the vertices with two ingoing arcs).

Table 4 shows the saturation matrix for the junctions grouped into the subfamilies respect to the y-axis of Table 1), where the correspondent derivable[17] rule of Weakening or Contraction is reported under each structured rule; in particular, below each structured rule 'sub conditione' the derivable rule is labeled by a superscript indicating a specification.

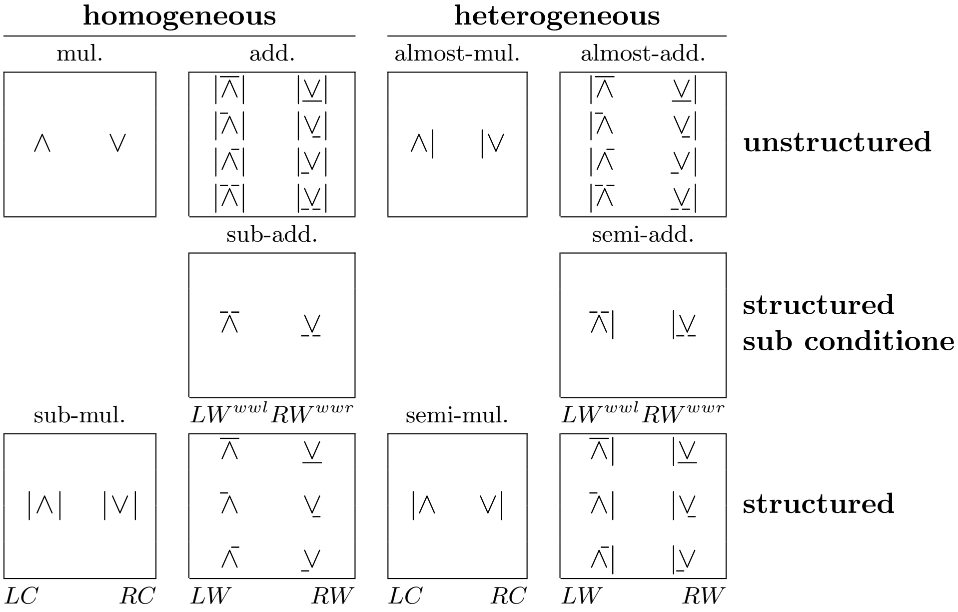

Table 4: saturation matrix for junctions

In regard to *dynamic* binary connectives, let us consider the case of Implication (the case of Disimplication is totally symmetric). The proposed systematic classification distinguishes 40 connectives: the 4 Implications of the *cancellative family* do not exhibit any horizontal bar, on the contrary of the 36 Implications of the *adjunctive family*, which is further divisible in *subfamilies*.

Each (sub)family respect to the x-axis of Table 1 exhibits a gradation of left bar and right bar ordered according to the diagrams in Table 5.[18]

---

[17] Assuming the structural rules of multiplicative Identity and multiplicative Cut.

[18] Even here, the bars can be considered as the parameters according to the same conventions established in the case of the static connectives.

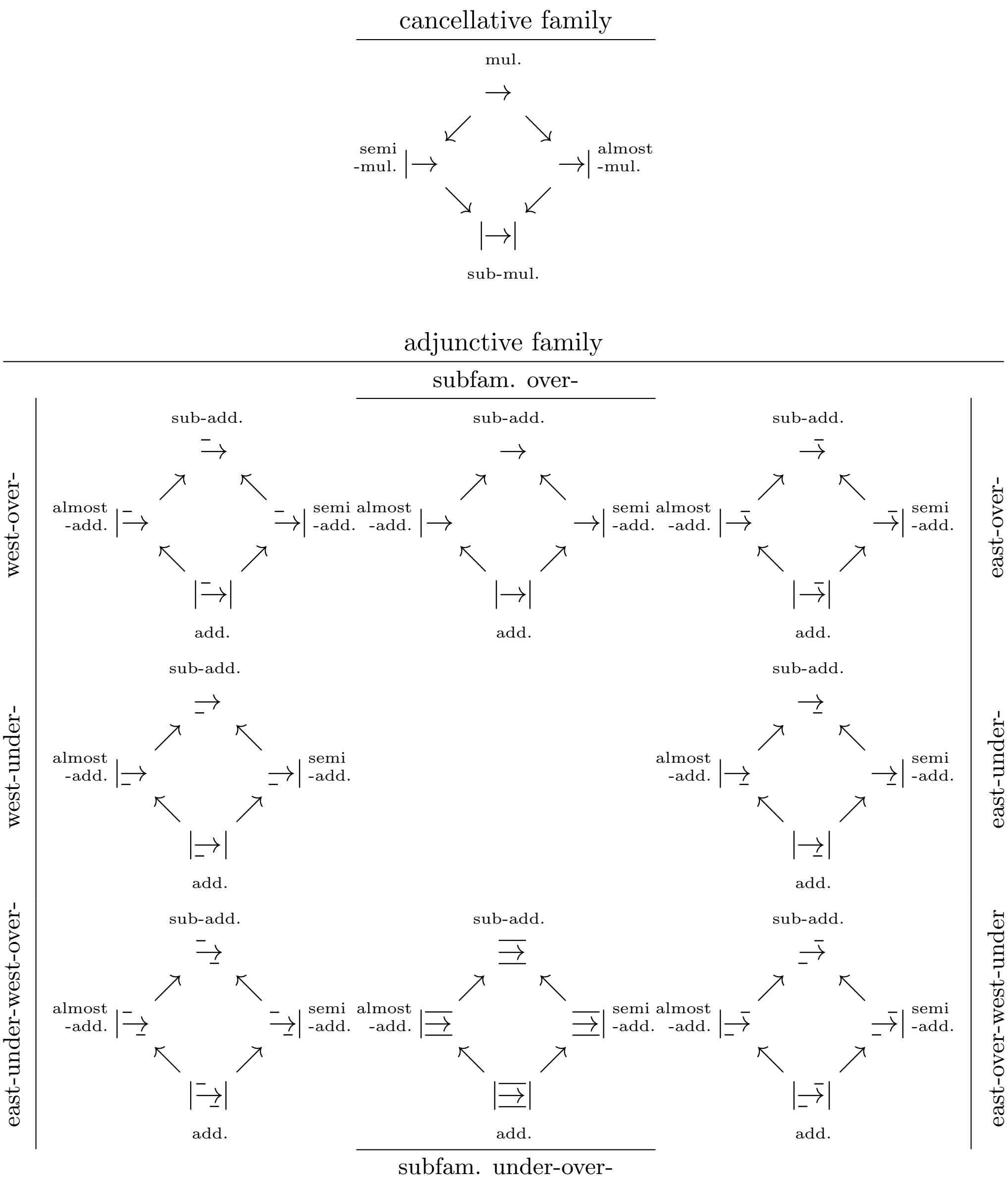

Table 5: families of Implications

Table 6 shows the saturation matrix for the implicatures grouped into the subfamilies respect to the y-axis of Table 1: even here, the correspondent derivable rule of Weakening or Contraction is reported under each structured rule and structured rule ‘sub conditione’; in particular in the last case, the derivable rule is labeled by a superscript indicating a specification or its derivability under a certain condition.

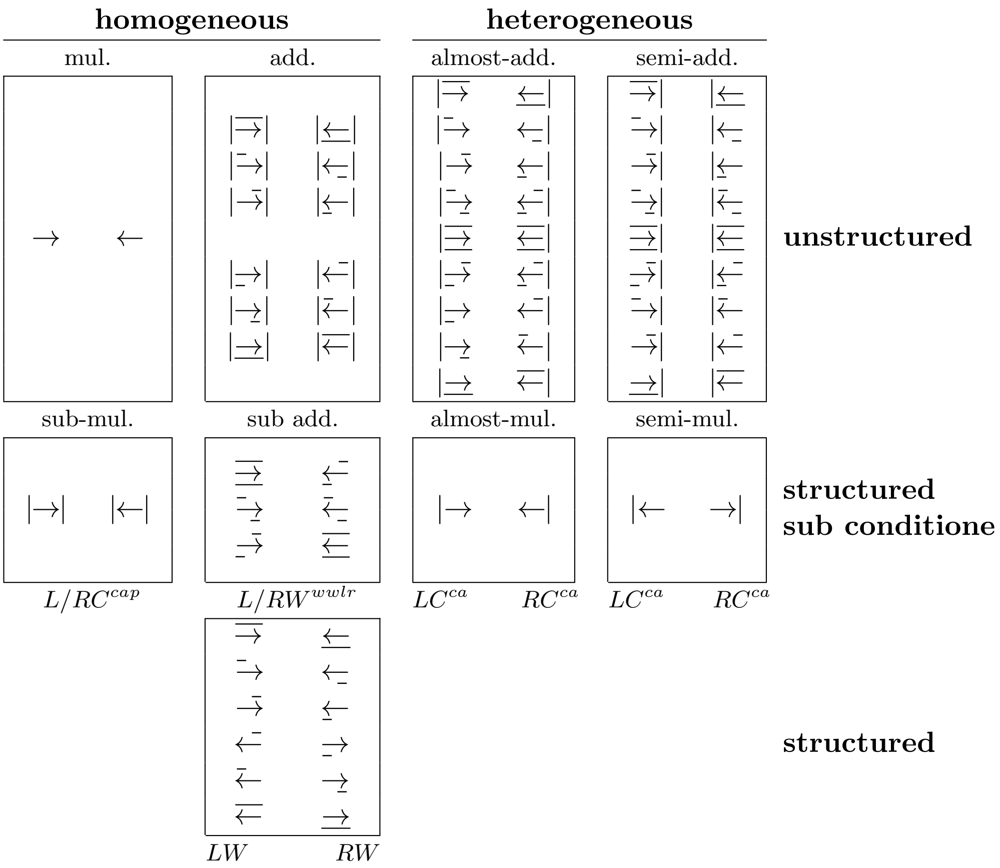

Table 6: saturation matrix for implicatures

In regard to the zero-ary connectives, let us consider the case of the Truth (the case of the False is totally symmetric). The systematic classification proposed distinguishes 16 connectives: the 4 constants for the Truth of the *cancellative family* do not exhibit any horizontal bar, on the contrary of the 12 constants for the Truth of the *adjunctive family*, which is further divisible in *subfamilies*. Each (sub)family respect to the x-axis of Table 1 exhibits a gradation of left bar and right bar ordered according to the diagrams in Table 7.[19]

---

[19] Even here, the bars can be considered as the parameters according to the same conventions established in the case of the binary connectives.

cancellative family

$$\begin{tikzcd}
 & \overset{\text{mul.}}{\top} \arrow[dl] \arrow[dr] & \\
\text{semi-mul.}\,|\top \arrow[dr] & & \top|\,\text{almost-mul.} \arrow[dl] \\
 & \underset{\text{sub-mul.}}{|\top|} &
\end{tikzcd}$$

adjunctive family

subfam. over-

$$\begin{tikzcd}
 & \overset{\text{sub-add.}}{\overline{\top}} & \\
\text{almost-add.}\,|\overline{\top} \arrow[ur] & & \overline{\top}|\,\text{semi-add.} \arrow[ul] \\
 & \underset{\text{add.}}{|\overline{\top}|} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

subfam. under-

$$\begin{tikzcd}
 & \overset{\text{sub-add.}}{\underline{\top}} & \\
\text{almost-add.}\,|\underline{\top} \arrow[ur] & & \underline{\top}|\,\text{semi-add.} \arrow[ul] \\
 & \underset{\text{add.}}{|\underline{\top}|} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overset{\text{sub-add.}}{\overline{\underline{\top}}} & \\
\text{almost-add.}\,|\overline{\underline{\top}} \arrow[ur] & & \overline{\underline{\top}}|\,\text{semi-add.} \arrow[ul] \\
 & \underset{\text{add.}}{|\overline{\underline{\top}}|} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

subfam. under-over-

Table 7: families of constants for Truth

Table 8 shows the saturation matrix for the truth constants grouped into the subfamilies respect to the y-axis of Table 1: even here, the correspondent derivable rule of Weakening or Contraction is reported under each structured rule.

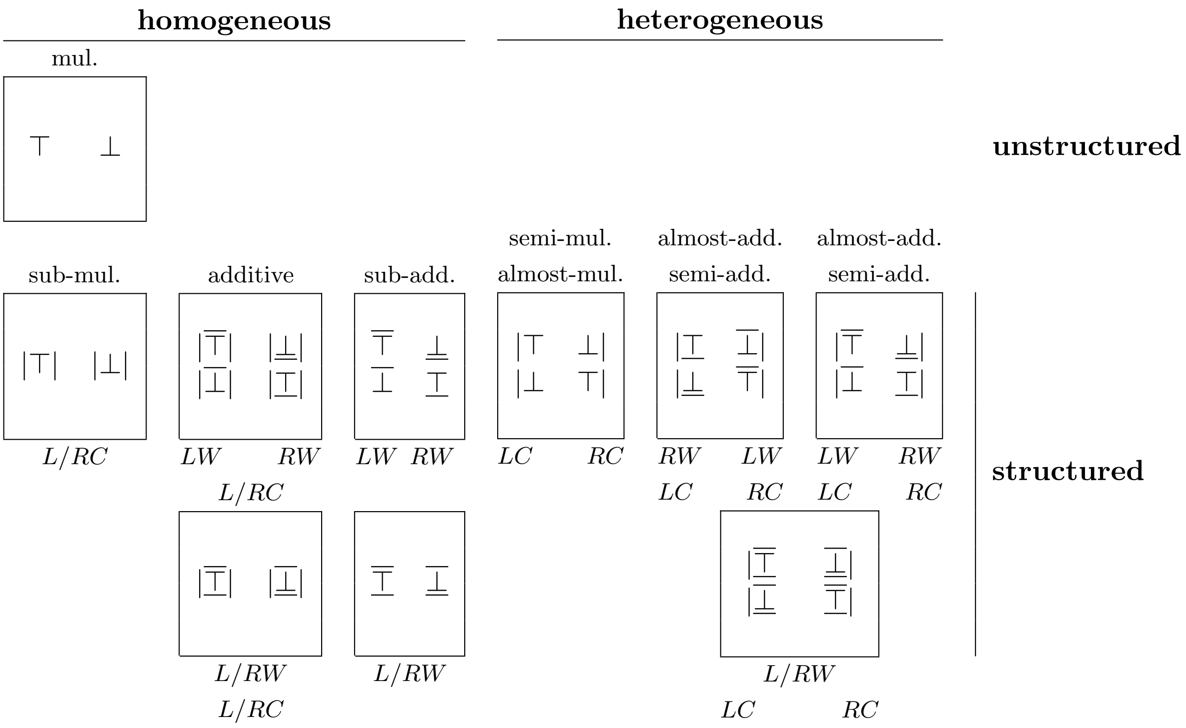

Table 8: saturation matrix for truth constants

4.2 Derivability of Weakening and Contraction

The distinction between structured and unstructured rules is justified by the following two theorems.

Theorem 4.1 *Assume multiplicative Identity and multiplicative Cut. Left Contraction and Right Contraction can be obtained as derived rules in the following cases:*

	left bar	*right bar*	*orizontal-over bar*	*orizontal-under bar*	*derived rules*	*sub-con.*
Cong.	✓	*indifferent*	×	*impossible*	*LC*	
Disg.	*indifferent*	✓	*impossible*	×	*RC*	
implic.	✓	×	×	×	*LC*	*ca*
implic.	×	✓	×	×	*RC*	*ca*
implic.	✓	✓	×	×	*L/RC*	*cap*
truth c.	✓	×	*indifferent*	*indifferent*	*LC*	
truth c.	×	✓	*indifferent*	*indifferent*	*RC*	
truth c.	✓	✓	*indifferent*	*indifferent*	*L/RC*	

*The abbreviations marked by 'c' specify that the Contraction can be obtained as a derived rule only under certain constraints (*sub-conditione*), in particular 'ca': the occurrences which must be contracted do not exhibit any* *active context; 'cap': the occurrences which must be contracted do not exhibit any context (active or passive).*

Below there are some examples of short proofs

Proof

Contraction

Left Right

$$\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A \vdash A \,|\!\wedge A} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, A, A \vdash \Delta\end{matrix}}{\Gamma, A \,|\!\wedge A \vdash \Delta}}{\Gamma, (A) \vdash \Delta} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash A, A, \Delta\end{matrix}}{\Gamma \vdash A \vee\!|\, A, \Delta} \qquad \dfrac{A \vdash A \qquad A \vdash A}{A \vee\!|\, A \vdash A}}{\Gamma \vdash (A), \Delta}$$

$$\dfrac{\dfrac{A \vdash A}{\vdash A \,|\!\rightarrow A} \qquad \dfrac{A \vdash A \qquad \begin{matrix}\vdots\\ A, A \vdash \Delta\end{matrix}}{A, A \,|\!\rightarrow A \vdash \Delta}}{(A) \vdash \Delta} \qquad \dfrac{\dfrac{A \vdash A}{\vdash A \rightarrow\!|\, A} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma \vdash A, A\end{matrix} \qquad A \vdash A}{\Gamma, A \rightarrow\!|\, A \vdash A}}{\Gamma \vdash (A)}$$

$$\dfrac{\vdash |\top \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \Gamma \vdash \Delta\end{matrix}}{\Gamma, |\top \vdash \Delta}}{(\Gamma) \vdash \Delta} \qquad \dfrac{\vdash \top| \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \Delta, \Delta\end{matrix}}{\Gamma, \top| \vdash \Delta}}{\Gamma \vdash (\Delta)}$$

■

Theorem 4.2 *Assume multiplicative Identity and multiplicative Cut. Left Weakening and Right Weakening can be obtained as derived rules in the following cases:*

	left bar	*right bar*	*orizontal-over bar*	*orizontal-under bar*	*derived rules*	*sub-con.*
Cong.	×	*indifferent*	✓	*impossible*	*LW*	*(wwl)**
Disg.	*indifferent*	×	*impossible*	✓	*RW*	*(wwr)**
implic.	×	×	✓	×	*LW*	
implic.	×	×	×	✓	*RW*	
implic.	×	×	✓	✓	*L/RW*	*wwlr*
truth c.	*indifferent*	*indifferent*	✓	×	*LW*	
truth c.	*indifferent*	*indifferent*	×	✓	*RW*	
truth c.	*indifferent*	*indifferent*	✓	✓	*L/RW*	

The abbreviations marked by 'ww' specify that the derived rule is special form of Weakening introducing two occurrences of formulas (also different), in particular 'wwl': either to left, 'wwr': either to right, 'wwlr': one to left and the other to right (of the turnstile).

**Note: the constraints 'wwl' applies iff the horizontal over bar is a west-east-over bar, and the 'wwr' applies iff the orizontal under bar is a east-west-under bar, otherwise there are not constraints for junctions.*

Below there are some examples of the shorts proofs

Proof

Weakening

Left | Right

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \,\overline{\wedge}|\, B} \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma, B \vdash \Delta\end{array}}{\Gamma, A \,\overline{\wedge}|\, B \vdash \Delta}}{\Gamma, (A), B \vdash \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A, \Delta\end{array}}{\Gamma \vdash A \,|\underline{\vee}\, B, \Delta} \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \,|\underline{\vee}\, B \vdash A, B}}{\Gamma \vdash A, (B), \Delta}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B, \Delta\end{array}}{\Gamma \vdash A \overline{\to} B, \Delta} \qquad \dfrac{A \vdash A \qquad B \vdash B}{A, A \overline{\to} B \vdash B}}{\Gamma, (A) \vdash B, \Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array}}{\Gamma \vdash A \underline{\to} B, \Delta} \qquad \dfrac{A \vdash A \qquad B \vdash B}{A, A \underline{\to} B \vdash B}}{\Gamma, A \vdash (B), \Delta}$$

$$\dfrac{\Gamma \vdash \overline{\top}| \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma' \vdash \Delta, \Delta\end{array}}{\Gamma', \overline{\top}| \vdash \Delta}}{(\Gamma), \Gamma' \vdash \Delta} \qquad \dfrac{\vdash \underline{\top}|, \Delta \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta', \Delta'\end{array}}{\Gamma, \underline{\top}| \vdash \Delta'}}{\Gamma \vdash \Delta', (\Delta)}$$

■

Finally, the 16 symbols for the connectives of Negation are defined via multiplicative Implication and constant for the False, while the 16 symbols for the connectives of Denegation are defined via multiplicative Disimplication and constants for the True as follows

$$\begin{array}{ccccccc}
A^{\bot} & \equiv & A \to \bot & \qquad & \top \leftarrow A & \equiv & {}^{\top}A \\
A^{|\bot} & \equiv & A \to |\bot & & \top| \leftarrow A & \equiv & {}^{\top|}A \\
A^{\bot|} & \equiv & A \to \bot| & & |\top \leftarrow A & \equiv & {}^{|\top}A \\
A^{|\bot|} & \equiv & A \to |\bot| & & |\top| \leftarrow A & \equiv & {}^{|\top|}A \\
\hline
A^{\underline{\bot}} & \equiv & A \to \underline{\bot} & & \overline{\top} \leftarrow A & \equiv & {}^{\overline{\top}}A \\
A^{|\underline{\bot}} & \equiv & A \to |\underline{\bot} & & \overline{\top}| \leftarrow A & \equiv & {}^{\overline{\top}|}A \\
A^{\underline{\bot}|} & \equiv & A \to \underline{\bot}| & & |\overline{\top} \leftarrow A & \equiv & {}^{|\overline{\top}}A \\
A^{|\underline{\bot}|} & \equiv & A \to |\underline{\bot}| & & |\overline{\top}| \leftarrow A & \equiv & {}^{|\overline{\top}|}A \\
\hline
A^{\overline{\bot}} & \equiv & A \to \overline{\bot} & & \underline{\top} \leftarrow A & \equiv & {}^{\underline{\top}}A \\
A^{|\overline{\bot}} & \equiv & A \to |\overline{\bot} & & \underline{\top}| \leftarrow A & \equiv & {}^{\underline{\top}|}A \\
A^{\overline{\bot}|} & \equiv & A \to \overline{\bot}| & & |\underline{\top} \leftarrow A & \equiv & {}^{|\underline{\top}}A \\
A^{|\overline{\bot}|} & \equiv & A \to |\overline{\bot}| & & |\underline{\top}| \leftarrow A & \equiv & {}^{|\underline{\top}|}A \\
\hline
A^{\overline{\underline{\bot}}} & \equiv & A \to \overline{\underline{\bot}} & & \overline{\underline{\top}} \leftarrow A & \equiv & {}^{\overline{\underline{\top}}}A \\
A^{|\overline{\underline{\bot}}} & \equiv & A \to |\overline{\underline{\bot}} & & \overline{\underline{\top}}| \leftarrow A & \equiv & {}^{\overline{\underline{\top}}|}A \\
A^{\overline{\underline{\bot}}|} & \equiv & A \to \overline{\underline{\bot}}| & & |\overline{\underline{\top}} \leftarrow A & \equiv & {}^{|\overline{\underline{\top}}}A \\
A^{|\overline{\underline{\bot}}|} & \equiv & A \to |\overline{\underline{\bot}}| & & |\overline{\underline{\top}}| \leftarrow A & \equiv & {}^{|\overline{\underline{\top}}|}A
\end{array}$$

4.3 Derivability between Conjunction connectives

Definition 4.3 *Inf is a rule among Weakening or Contraction (Left or Right). If a proof π uses Inf and at least one connective inducing (i.e.*

making derivable) Inf*, then we say that the use of* Inf *in* π *is* endogenous.

If a proof π *uses* Inf *but does not use at least one connective inducing* Inf*, then we say that the use of* Inf *in* π *is* exogenous.

In Figures below the derivability relation between connectives are showed by directed graphs in according to the conventions:

- a continuous arrow from A to B means that $A \vdash B$ is derivable without the use of Weakening or Contraction
- a dashed arrow from A a B means that $A \vdash B$ is derivable only with an exogenous use of Weakening
- a dotted arrow from A a B means that $A \vdash B$ is derivable only with an exogenous use of Contraction

The derivability relation between Conjunction connectives are determined by the following three theorems.

Theorem 4.4 *Each Conjunction connective of the cancellative family* $\circ$ *derives each Conjunction connective of the cancellative family and of west-over-, east-over- and west-east-over-adjunctive subfamilies* $\circ'$*, i.e.* $A \circ B \vdash A \circ' B$*. The use or not of* W *and* C *is specified in Figure 1.*

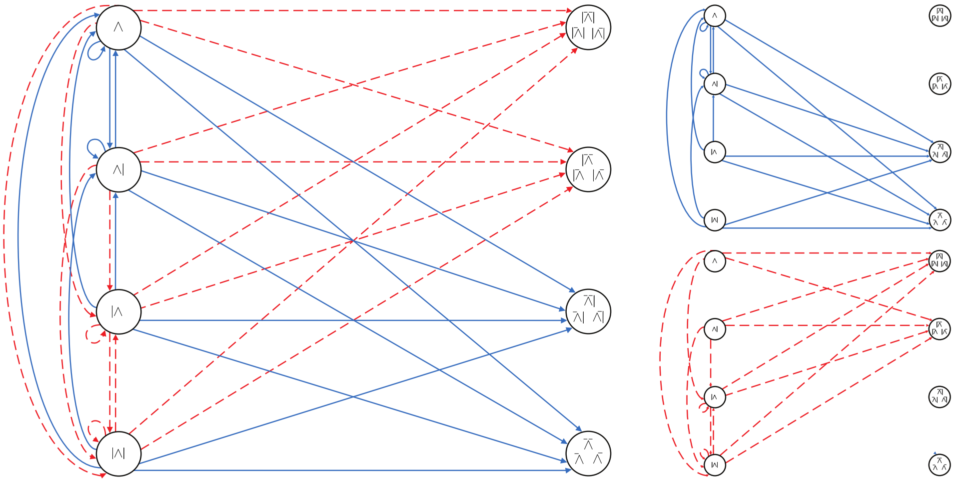

Figure 1: derivability between Cong. without subf. over-adjunctive

Note that, given an arbitrary pair of Conjunctions $(\circ, \circ')$ by excluding the over-adjunctive subfamily, if $A \circ B \nvdash A \circ' B$, then the sequent $\Gamma, A \circ B \vdash A \circ' B$ is always derivable where for all $C \in \Gamma, C = A$ or $C = B$.

Below there are some examples of the shorts proofs

Proof

$$\dfrac{\dfrac{W\,\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge\!| B \vdash A} \qquad \dfrac{W\,\dfrac{B \vdash B}{A, B \vdash B}}{A \wedge\!| B \vdash B}}{A \wedge\!| B \vdash A |\!\wedge\!| B}$$

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge\!| B}}{A |\!\wedge\!| B \vdash A \wedge\!| B}$$

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \,\bar{\wedge}\!| B}}{A |\!\wedge B \vdash A \,\bar{\wedge}\!| B}$$

$$\dfrac{\dfrac{W\,\dfrac{A \vdash A}{A, B \vdash A} \qquad W\,\dfrac{B \vdash B}{A, B \vdash B}}{A, B \vdash A |\!\wedge B}}{(A), A \,\bar{\wedge}\!| B \vdash A |\!\wedge B}$$

■

Theorem 4.5 *Each Conjunction connective of the over-adjunctive subfamily* $\circ$ *derives each Conjunction connective of the over-adjunctive and west-east-over-adjunctive subfamily* $\circ'$, *i.e.* $A \circ B \vdash A \circ' B$. *The use or not of* W *and* C *is specified in Figure* 2.

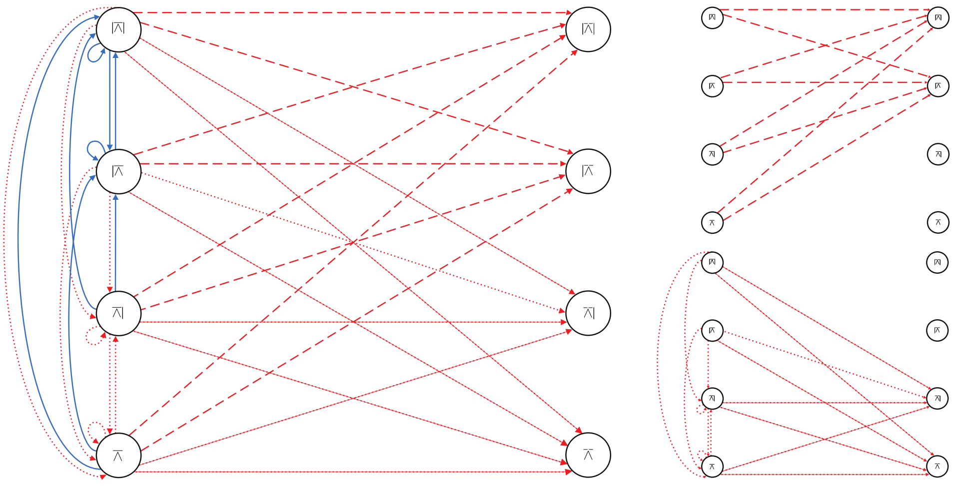

Figure 2: derivability between Cong. of subf. over- and west-east-over-adj.

Theorem 4.6 *Each Conjunction connective of the cancellative family or the over-adjunctive subfamily* $\circ$ *derives each Conjunction connective of the cancellative family or of the over-adjunctive subfamily* $\circ'$, *i.e.* $A \circ B \vdash A \circ' B$. *The use or not of* W *and* C *is specified in Figure* 3.

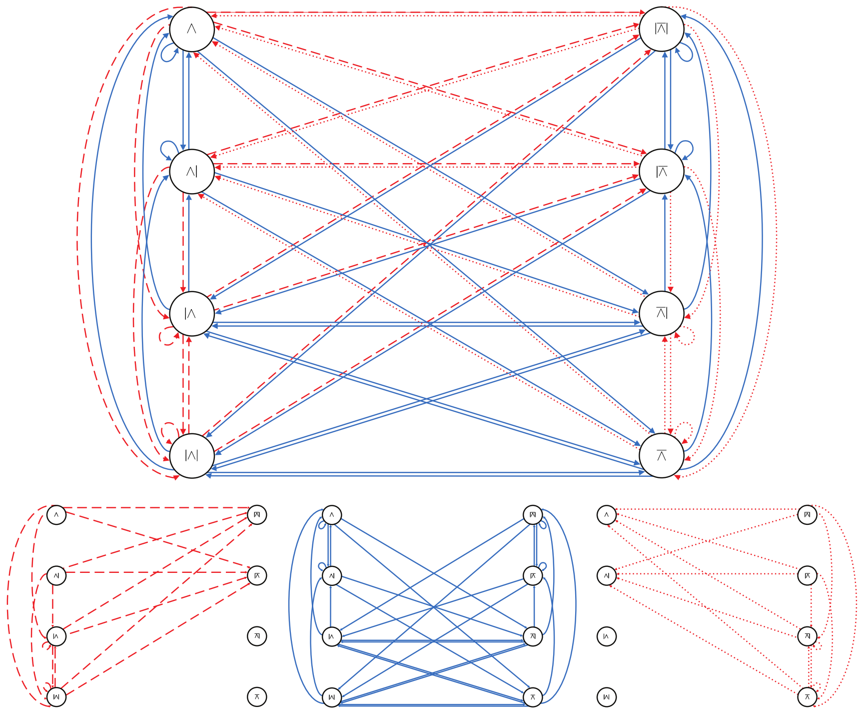

Figure 3: derivability between Cong. of (sub)f. canc. and over-adj.

5 Cut-tradeoff theorem for *AEM.LSat*

Usually the Cut-elimination theorem for a Logic L is formulated as: *If a sequent is derivable in the Logic L, then it is derivable in L without the use of Cut.*

From the Saturated Logic perspective, the Structural Rules different from Identity and Cut should not be considered among the primitive rules but they are, so to speak, second rules, and they are useful precisely because guarantee of 'circumscribing' the use of Cut rule in a proof. In this perspective, it makes sense to speak of a theorem of Cut-tradeoff for a Logic L that can be formulated as: *If a sequent is derivable in the Logic L, then it is derivable in L by restricting the possible use of Cut to subproofs of some Tradeoff (Structural) Rules.*

Note that the first formulation coincides with the particular case of the second formulation where the Tradeoff set is empty.

Now, we can state the following Cut-tradeoff theorem for the system considered:

Theorem 5.1 *If a sequent is derivable in AEM.LSat, then it is derivable in AEM.LSat by restricting the possible use of Cut to subproofs of* WL*,* WR*,* CL *or* CR*.*

The traditional Cut-elimination strategy devised by Gentzen is adaptable to *AEM.LSat*,[20] except for the need to consider a much larger number of subcases. Broadly, the strategy is simple and, given a proof π that contains an application of the rule *Cut* as the last rule, provides only two circumstances:

- if one of the Cut premises is an axiom, then we proceed to 'elimination' of Cut by the *containment procedure* [basic step], that is we exhibit a proof π' with the same terminal sequent and 'without' any *Cut* (in the sense that: when possible, we completely 'eliminate' the Cut, otherwise we 'circumscribe' the Cut by replacing it with the application of a Tradeoff Rule, understood as an abbreviation of the previous subproof with Cut);
- otherwise, we proceed to the 'upward shift' Cut by the *shifting procedure* [inductive step: by induction on the degree or the rank of Cut], that is we exhibit a proof π'' with the same terminal sequent and one or more Cut with lower degree - i.e. involving simpler formulas - or Cut with the same degree but lower rank - i.e. appearing at a lower height in the proof - (also, when necessary, we 'circumscribe' some Cut by replacing them with the application of one or more Tradeoff Rules).

The Cut-tradeoff proof proceeds by double induction on two parameters: the Cut degree and the Cut rank. We distinguish two cases:

1. $\rho = 2$: the Cut rank equals two. We distinguish two subcases:

   1a. if one of Cut premises is an axiom, then we can directly eliminate the Cut (containment procedure)

   1b. if each the Cut premises are the conclusion of a unary or binary connectives rule, then we proceed by *induction on degree*: assuming that the Cut elimination holds for proofs with lower Cut degree, we can transform the proof into another with lower Cut degree (shifting procedure)

2. $\rho > 2$: the Cut rank is greater than two. We want to reduce us to the previous case $\rho = 2$. Because of $\rho > 2$, it holds that the left rank $L\rho > 1$ or the right rank $R\rho > 1$, we distinguish two subcases:

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[20] In order to treat the case of Contraction in Classical Logic, Gentzen considers an equivalent system with the rule so-called *multiCut* $\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Pi, \Gamma \vdash \Delta', \Sigma}$ with $\Pi = \Gamma'$ without any occurrence φ, and $\Sigma = \Delta$ without any occurrence φ. As for the Classical Logic, it is immediate to show that a sequent is derivable in *AEM.LSat* iff the same sequent is derivable in *AEM.LSat* $\setminus$ *Cut* $\cup$ *multiCut*, because in either cases we have W and C (Left and Right). Note that if one considers subsystem of *AEM.LSat* without one or more of such Structural Rules, it is no longer possible to use *multiCut* with equal nonchalance.

2a. if the rank $L\rho > 1$ and $R\rho > 1$, then, by the Lemma on the history of an occurrence and the substitution Lemma, we can transform the proof into another with the same conclusion and $X\rho = 1$ with $X \in \{L, R\}$ (shifting procedure)

2b. if the rank $X\rho = 1$ and the rank $Y\rho > 1$ with $X, Y \in \{L, R\}$ and $X \neq Y$, then we proceed by *induction on the rank* $Y\rho$: assuming that the Cut-elimination holds for proof with the same Cut degree but lower Cut rank, we can transform the proof into another with the same Cut degree but lower Cut rank (shifting procedure)

In regard to the connectives, let us observe that for each connective we need to consider all the rules of introduction on the left and on the right, since to apply Cut we have to introduce the same connective both on the left and on the right and we have also to consider what happens to the contexts on each side of the involved sequents.[21]

Below there are few examples among the connectives rule subcases

Proof

$$\dfrac{\vdash \underline{\top}|, \Delta' \qquad \dfrac{\begin{array}{c}\vdots\ \pi \\ \Gamma \vdash \Delta, \Delta\end{array}}{\Gamma, \underline{\top}| \vdash \Delta}}{\Gamma \vdash \Delta, \Delta'} \qquad \Rightarrow \qquad \begin{array}{cl} \vdots\ \pi & \\ \Gamma \vdash \Delta, \Delta & \\ \hdashline \Gamma \vdash \Delta & C \\ \hdashline \Gamma \vdash \Delta, \Delta' & W \end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi \\ \Gamma' \vdash A, \Delta'\end{array} \qquad \begin{array}{c}\vdots\ \sigma \\ \Gamma'' \vdash B, \Delta''\end{array}}{\Gamma', \Gamma'' \vdash A \overline{\wedge} B, \Delta', \Delta''} \qquad \dfrac{\begin{array}{c}\vdots\ \tau \\ \Gamma, C \in \{A, B\} \vdash \Delta\end{array}}{\Gamma, A \overline{\wedge} B \vdash \Delta}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''}$$

$$\Downarrow$$

$$\begin{array}{lcl} & \dfrac{\begin{array}{c}\vdots\ \pi \\ \Gamma' \vdash A, \Delta'\end{array} \qquad \begin{array}{c}\vdots\ \tau \\ \Gamma, C = A \vdash \Delta\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} & \\ W & \hdashline \Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta' & \\ & \hdashline \Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta'' & W \end{array}$$

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[21] However, we can make some observations at least partially useful to limit the number of checks. We observe that: i) with regard to junctions and implicatures the distinction among over, over-west-, east-over-adjunctive and among under-, under-west-, east-under-adjunctive subfamilies loses relevance because the Cut-elimination proofs structure is substantially similar, so it is sufficient to check one of the above (remember that Exchange is a primitive rule of *AEM.LSat*); ii) conversely, with regard to all three kinds of connectives, if one has Weakening and Contraction both Left and Right, as in the case of full *AEM.LSat*, the distinction among the remaining subfamilies loses relevance for the Cut-elimination, but becomes important when one considers *AEM.LSat* subsystems that do not have some of these second Structural Rules (remember that Weakening and Contraction rules are not *AEM.LSat* primitives).

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi}{\Gamma' \vdash B, C, \Delta'}}{\Gamma' \vdash A \overline{\rightarrow}| B, C, \Delta'}}{\Gamma' \vdash C, A \overline{\rightarrow}| B, \Delta'}\,E \qquad \overset{\tau}{\Gamma, C \vdash \Delta}}{\begin{array}{c}\Gamma, \Gamma' \vdash \Delta, A \overline{\rightarrow}| B, \Delta' \\ \hdashline \Gamma, \Gamma' \vdash A \overline{\rightarrow}| B, \Delta, \Delta'\end{array}\,E} \quad \Rightarrow \quad \dfrac{\dfrac{\overset{\vdots\,\pi}{\Gamma' \vdash B, C, \Delta'}}{\Gamma' \vdash C, B, \Delta'}\,E \qquad \overset{\tau}{\Gamma, C \vdash \Delta}}{\dfrac{\begin{array}{c}\Gamma, \Gamma' \vdash \Delta, B, \Delta' \\ \hdashline \Gamma, \Gamma' \vdash B, \Delta, \Delta'\end{array}\,E}{\Gamma, \Gamma' \vdash A \overline{\rightarrow}| B, \Delta, \Delta'}}$$

■

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