Universal Algebra for Logics

Joanna GRYGIEL
University of Czestochowa
Poland
j.grygiel@ajd.czest.pl

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These notes form Lecture Notes of a short course which I will give at 1st School on Universal Logic in Montreux.

They cannot be recommended for self studies because, although all definitions and main ideas are included, there are no proofs and examples. I'm going to provide some of them during my lectures, leaving easy ones as exercises.

In the first part we discuss some of the most important notions of universal algebra. Then we concentrate on free algebras and varieties. Our main goal is to prove Birkhoff's Theorem, which says that a class of similar algebras is a variety iff it is definable by a set of equations. In the last part we say more about lattices and boolean algebras, as these algebraic structures which are especially important for logic.

Universal algebra is sometimes seen as a special branch of model theory, in which we are dealing with structures having operations only. However, it is only one of aspects of universal algebra, which appears to be a powerful tool in many areas. Universal algebra borrows techniques and ideas from logic, lattice theory and category theory. The connections between lattice theory and the general theory of algebras are particularly strong.

We assume that the reader has the basic knowledge of mathematics.

1 Algebras

We start with some very general ideas of universal algebra. Of course, the fundamental notion is that of an algebra. In general, algebra can be understood as a nonempty set together with some finitary operations defined on it. Let us make it precise.

Let A be a set and n be a natural number. An *operation* of rank (or arity) n on A is any total function from A^n into A. Thus, an operation of rank n assigns an element of A to every n-tuple of elements of A. Operations of rank 1 are called *unary operations* and those of rank 2 –*binary operations*. Operations of rank 0 on a nonempty set A have only one value. We call them *constants* and identify them with their values.

An *algebra* (or algebraic structure) is a pair $\mathcal{A} = \langle A, F \rangle$, where A is a nonempty set, called the *universe* (or a carrier set) of $\mathcal{A}$, and $F = \{f_t^A\}_{t \in T}$ is a sequence of operations on A. The operations from F are called the *basic* (or fundamental) operations of the algebra $\mathcal{A}$ and T is called the *index set* of $\mathcal{A}$. The *type* of $\mathcal{A}$ is the function $\tau : T \longrightarrow \mathbf{N}$, where $\tau(t)$ is equal to the rank of the operation f_t^A. The type of $\mathcal{A}$ is sometimes called the rank function of $\mathcal{A}$. If $T = \{1, ..., k\}$, then the type of $\mathcal{A}$ is usually written as the sequence $\langle n_1, ..., n_k \rangle$, where n_i is the arity of f_i^A for $i = 1, ..., k$.

As a general convention we use calligraphic letters $\mathcal{A}$, $\mathcal{B}$,... to denote algebras and the corresponding uppercase letters A, B,... to denote their universes.

Algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *similar* iff they have the same rank functions. Most of the time only algebras of the same similarity types will be considered.

Algebras can be regarded as special instances of some more general structures, so called relational structures, where instead of operations we talk about relations but we skip this point.

Now, we introduce shortly some known algebras, which play an especially important role in algebraic logic.

1.1 Semigroups

A *semigroup* is a nonempty set endowed with an associative binary operation, i.e. an algebra $\mathcal{A} = \langle A, \circ \rangle$, where

$$(a \circ b) \circ c = a \circ (b \circ c)$$

for all $a, b, c \in A$.

Let us observe that a semigroup is an algebra of type $\langle 2 \rangle$.

1.2 Monoids

A *monoid* is an algebra $\mathcal{A} = \langle A, \circ, e \rangle$, where $\langle A, \circ \rangle$ is a semigroup and e is a constant such that

$$a \circ e = a = e \circ a$$

for every $a \in A$. Thus, a monoid is an algebra of type $\langle 2, 0 \rangle$.

1.3 Groups

A *group* is an algebra $\mathcal{A} = \langle A, \circ, ^{-1}, e \rangle$, where $\mathcal{A} = \langle A, \circ, e \rangle$ is a monoid and

$$a \circ a^{-1} = e = a^{-1} \circ a$$

for every $a \in A$.

A group is *abelian* if it additionally satisfies the commutativity law, i.e. for all $a, b \in A$

$$a \circ b = b \circ a.$$

In this sense a group (and an abelian group, as well) is an algebra of type $\langle 2, 1, 0 \rangle$.

1.4 Rings

A *ring* is an algebra $\mathcal{A} = \langle A, +, \cdot, -, 0 \rangle$ such that $\langle A, +, -, 0 \rangle$ is an abelian group, $\langle A, \cdot \rangle$ is a semigroup and for all $a, b, c \in A$:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c);$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c).$$

1.5 Lattices

Lattices are algebras of a bit different nature. They are essentially an algebraic encoding of partially ordered sets which have the property that any pair of elements of the ordered set has the least upper bound and the greatest lower bound. We will discuss it more carefully in the last part of the lecture. At the moment we define a *lattice* as an algebra $\mathcal{A} = \langle A, \wedge, \vee \rangle$ of type $\langle 2, 2 \rangle$, where for all $a, b \in A$

$$a \wedge a = a; \quad a \vee a = a;$$

$$a \wedge b = b \wedge a; \quad a \vee b = b \vee a;$$

$$a \wedge (b \vee a) = a; \quad a \vee (b \wedge a) = a.$$

The operation $\wedge$ is referred to as meet and the operation $\vee$ is called join.

A lattice is *distributive* if it satisfies the distributivity laws:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c);$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b, c \in A$.

1.6 Boolean algebras

A *boolean algebra* is an algebra $\mathcal{A} = \langle A, \wedge, \vee, - \rangle$, where $\langle A, \wedge, \vee \rangle$ is a distributive lattice and $-$ is a unary operation such that for all $a, b \in A$

$$-(a \wedge b) = (-a) \vee (-b);$$

$$-(a \vee b) = (-a) \wedge (-b);$$

$$-(-a) = a;$$

$$(-a \wedge a) \vee b = b;$$

$$(-a \vee a) \wedge b = b.$$

The unary operation "$-$" is called a *complementation.* Let us notice that, according to that definition, every boolean algebra is of type $\langle 2, 2, 1 \rangle$.

2 Subalgebras, homomorphisms and direct products

In this chapter we present main tools which are used to construct new algebras from given ones.

2.1 Subalgebras

The simplest idea is just to take a "part" of the given algebra, i.e. the algebra which universe is a subset of the given one and which operations are "the same". Let us formalize this intuitive notion.

Let f be an operation of rank n on a nonempty set A and let $B \subseteq A$. We say that B is *closed* with respect to f (or f preserves B) iff $f(a_1, ..., a_n) \in B$ for all $a_1, ..., a_n \in B$.

If f is a constant, this means that B is closed with respect to f iff $f \in B$.

A *subuniverse* of an algebra $\mathcal{A}$ is a subset of A which is closed with respect to every basic operation of $\mathcal{A}$. $\mathcal{B}$ is a *subalgebra* of an algebra $\mathcal{A}$ if $\mathcal{B}$ is an algebra similar to $\mathcal{A}$, B is a nonempty subuniverse of $\mathcal{A}$ and for every operation symbol f_t of rank n in type τ of $\mathcal{A}$

$$f^B(a_1, ..., a_n) = f^A(a_1, ..., a_n)$$

for all $a_1, ..., a_n \in B$. Then f^B is called the *restriction* of f^A to B.

Let us denote by $Sub(\mathcal{A})$ the set of all subuniverses of $\mathcal{A}$. This set is naturally ordered by inclusion. Moreover,

Theorem 1 *Let $\mathcal{A}$ be an algebra and let S be a nonempty collection of subuniverses of $\mathcal{A}$. Then $\bigcap S$ is a subuniverse of $\mathcal{A}$.*

If we consider an arbitrary subset X of the universe of an algebra $\mathcal{A}$ then it is probably not a subuniverse of $\mathcal{A}$. However, we can always produce some subuniverses in which it is included. The subuniverse of $\mathcal{A}$ *generated* by $X \subseteq A$ is the set

$$\bigcap\{B : \quad X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathcal{A}\}.$$

The subuniverse of $\mathcal{A}$ generated by X will be denoted by $Sg^A(X)$. When the context is obvious, the index A will be omitted. The existence of $Sg^A(X)$ for every $X \subseteq A$ is ensured by Theorem 1.

X is called the *set of generators* of the subalgebra $Sg^A(X)$. Every subalgebra has at least one set of generators. If there is a finite set of generators of a subalgebra $\mathcal{B}$, then $\mathcal{B}$ is said to be *finitely generated.*

$Sg^A(X)$ can be regarded as a unary operation on the power set $P(A)$, which has the following properties:

Proposition 2 *For every $X, Y \in P(A)$:*

1. $X \subseteq Sg^A(X)$;
2. $Sg^A(Sg^A(X)) = Sg^A(X)$;
3. *If* $X \subseteq Y$ *then* $Sg^A(X) \subseteq Sg^A(Y)$;
4. $Sg^A(X) = \bigcup\{Sg^A(Z) : \; Z \subseteq X \text{ and } Z \text{ is finite }\}$.

Thus, the process of generating of subalgebras satisfies extensivity, idempotency and monotonicity (properties 1.-3.) — all properties of a closure operation. In other words, Sg^A can be regarded as a closure operator on $P(A)$.

Let us observe that the subuniverses of an algebra $\mathcal{A}$ are those subsets of A that $X = Sg^A(X)$.

Proposition 3 *For every algebra $\mathcal{A}$ the algebraic structure $\mathcal{S}ub(\mathcal{A}) = \langle Sub(\mathcal{A}), \wedge, \vee\rangle$, where*

$$B \wedge C = B \cap C;$$

$$B \vee C = Sg^A(B \cup C)$$

for any subuniverses B and C of $\mathcal{A}$, is a lattice, called the lattice of subuniverses of $\mathcal{A}$.

2.2 Homomorphisms

The notion of isomorphism is common not only in mathematics, but in philosophy, as well. The idea is following: two objects are isomorphic if they have the same structure. The notion of isomorphism is a special case of a more general notion of homomorphism.

A homomorphism of two similar algebras can be understood as a mapping which is compatible with all operations of the algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be similar algebras of type $\tau : T \longrightarrow \mathbf{N}$. A mapping $h : A \longrightarrow B$ is a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ iff for every $t \in T$, if $\tau(t) = n$, then for all $a_1, ...a_n \in A$

$$h(f_t^A(a_1, ...a_n)) = f_t^B(h(a_1), ..., h(a_n)).$$

If h is a surjective homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then $\mathcal{B}$ is called a *homomorphic image* of $\mathcal{A}$. An *isomorphism* is a homomorphism which is both bijective and surjective. In that case the algebras $\mathcal{A}$ and $\mathcal{B}$ are called *isomorphic.*

A homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is called an *endomorphism* of $\mathcal{A}$. An endomorphism of $\mathcal{A}$ which is an isomorphism is called an *automorphism* of $\mathcal{A}$. Clearly, the identity mapping id_A on the set A is always an automorphism of the algebra $\mathcal{A}$.

It is easy to see that the image $B_1 = h(A_1)$ of a subuniverse A_1 of the algebra $\mathcal{A}$ under the homomorphism $h : A \longrightarrow B$ is a subuniverse of $\mathcal{B}$ and the preimage $h^{-1}(B_2) = A_2$ of a subuniverse B_2 of the homomorphic image $h(\mathcal{A})$ of the algebra $\mathcal{A}$ is a subuniverse of $\mathcal{A}$.

2.3 Direct products

The constructions of subalgebras and homomorphic images do not lead to more complicated algebras then we started with. The direct product construction is different from this point of view.

Let us start, for simplicity, with the definition of the direct product of only two algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be similar algebras of type $\tau : T \longrightarrow \mathbf{N}$. Their *direct product* $\mathcal{A} \times \mathcal{B}$ is the algebra of the same similarity type with the universe being $A \times B$ (the Cartesian product of the universes of $\mathcal{A}$ and $\mathcal{B}$) and operations defined as follows: if $t \in T$ and $\tau(t) = n$ then

$$f^{A \times B}((a_1, b_1), ..., (a_n, b_n)) = (f^A(a_1, ..., a_n), f^B(b_1, ..., b_n))$$

for all pairs $(a_i, b_i) \in A \times B$. In other words, we do the operations of $\mathcal{A} \times \mathcal{B}$ coordinatewise.

To generalize this definition to any sequence of similar algebras we need the notion of a choice function and of a direct product of sets.

Let $\langle A_s \rangle_{s \in S}$ be a system of sets. By a *choice function* we mean a function $\alpha : S \longrightarrow \bigcup_{s \in S} A_s$ such that $\alpha(s) \in A_s$. The direct (or

Cartesian) product $\Pi_{s\in S}A_s$ of the system $\langle A_s\rangle_{s\in S}$ is the set of all choice functions for the system. Every A_s is called a *factor* of this product, the elements of $\Pi_{s\in S}A_s$ are called *S-tuples.* If $A_s = A$ for all $s \in S$ we write A^S instead of $\Pi_{s\in S}A_s$.

For every $s \in S$ we have s-th *projection* $\pi_s : \Pi_{s\in S}A_s \longrightarrow A_s$ such that $\pi_s(\alpha) = \alpha(s)$ for each $\alpha \in \Pi_{s\in S}A_s$.

Now, it is time to define the direct product of a system of similar algebras.

Let $\langle \mathcal{A}_s\rangle_{s\in S}$ be a system of algebras of the same type τ. The *direct product of the system* is the algebra $\mathbf{\Pi}_{s\in S}\mathcal{A}_s$ of type τ with universe $\Pi_{s\in S}A_s$ and such that, if f is an operation symbol of rank n in type τ, then

$$\pi_s(f^{\Pi_{s\in S}A_s}(\alpha_1, ..., \alpha_n)) = f^{A_s}(\alpha_1(s), ..., \alpha_n(s))$$

for all $\alpha_1, ..., \alpha_n \in \Pi_{s\in S}A_s$.

If $S = \{1, ..., n\}$ we will write $\mathcal{A}_1 \times ... \times \mathcal{A}_n$ instead of $\mathbf{\Pi}_{s\in S}\mathcal{A}_s$.

2.4 Varieties

Let $\mathcal{K}$ be a class of similar algebras. We shall denote:

$H(\mathcal{K})$ — the class of all homomorphic images of algebras from $\mathcal{K}$.

$I(\mathcal{K})$ — the class of all isomorphic copies of algebras from $\mathcal{K}$.

$S(\mathcal{K})$ — the class of all isomorphic copies of subalgebras of algebras from $\mathcal{K}$.

$P(\mathcal{K})$ — the class of all isomorphic copies of direct products of systems of algebras from $\mathcal{K}$.

It is said that $\mathcal{K}$ is closed under homomorphic images, taking subalgebras and constructing direct products if, respectively, $H(\mathcal{K}) \subseteq \mathcal{K}$; $S(\mathcal{K}) \subseteq \mathcal{K}$ and $P(\mathcal{K}) \subseteq \mathcal{K}$.

If a class $\mathcal{K}$ of similar algebras is closed under all these three operations then $\mathcal{K}$ is called a *variety.*

Let $V(\mathcal{K})$ denotes the smallest variety containing a class $\mathcal{K}$.

Theorem 4 *$V(\mathcal{K}) = HSP(\mathcal{K})$.*

Varieties are one of the central topics of universal algebra. As we are going to show later, varieties are exactly the classes of similar algebras, which can be defined by equations.

3 Congruences and quotient algebras

Let us recall that an equivalence relation on a set A is any binary relation which is reflexive, symmetric and transitive. Every equivalence relation on a set A determines a partition of A into mutually exclusive and jointly exhaustive subsets, called equivalence classes of the equivalence relation.

A *congruence relation* θ on an algebra $\mathcal{A}$ of type τ is any equivalence relation on the universe A which has the *substitution property* for $\mathcal{A}$, i.e. for every basic operation f_t^A of the algebra $\mathcal{A}$ and for all $a_1, ..., a_n, b_1, ...b_n \in A$, if $\tau(t) = n$ and $a_i \theta b_i$ for every $i = 1, ..., n$ then

$$f_t^A(a_1, ..., a_n)\theta f_t^A(b_1, ..., b_n).$$

If θ is a congruence on $\mathcal{A}$, we shall denote

$a|\theta = \{b \in A : \ a\theta b\}$ — the *congruence class* of an element $a \in A$ modulo θ

$A|\theta = \{a|\theta : \ a \in A\}$ — the set of all congruence classes of θ.

For every algebra $\mathcal{A}$ the trivial equivalence relations:

$$\Delta_A = \{(a, a) : \ a \in A\} \text{ — the equality relation on A}$$

and $\nabla_A = A \times A$ — the total relation on A are congruence relations. An algebra which has no congruence relations except trivial ones is called simple.

As every congruence θ on $\mathcal{A}$ is an equivalence relation on A then $A|\theta$ is a partition of A.

Let f_t^A be a basic operation of rank n on an algebra $\mathcal{A}$. Then we can define on $A|\theta$ the corresponding operation f_t^θ by

$$f_t^\theta(a_1|\theta, ..., a_n|\theta) = f_t^A(a_1, ..., a_n)|\theta,$$

since the substitution property provides that the operation f_t^θ is well-defined.

This means that for every congruence relation θ on an algebra $\mathcal{A}$ of type τ, we can define the similar algebra $\mathcal{A}|\theta$, called the *quotient algebra*, with the universe $A|\theta$ and basic operations f_t^θ corresponding to basic operations f_t^A.

Let us observe that if $h : A \longrightarrow B$ is a homomorphism of similar algebras $\mathcal{A}$ and $\mathcal{B}$ then

$$\ker h = \{(a, b) : \ h(a) = h(b)\}$$

is a congruence relation on $\mathcal{A}$. The congruence is called the kernel of h. Thus, every homomorphism of $\mathcal{A}$ determines a congruence on $\mathcal{A}$.

On the other hand, for every congruence θ on $\mathcal{A}$ there is a natural homomorphism

$$h(a) = a|\theta,$$

which maps A onto $A|\theta$.

Both these observations lead to the homomorphism theorem:

Theorem 5 *Let $\mathcal{A}$ and $\mathcal{B}$ be similar algebras, let h be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$, let θ be any congruence relation on $\mathcal{A}$. Then*

1. *the mapping $g : A \longrightarrow A|\theta$ defined by $g(a) = a|\theta$ for every $a \in A$ is a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{A}|\theta$ whose kernel is θ;*

2. *if $\theta = ker\ h$, then there is a unique isomorphism $f : A|\theta \longrightarrow B$ such that $h = f \circ g$.*

We shall denote by $Con(\mathcal{A})$ the set of all congruences of an algebra $\mathcal{A}$. It is easy to notice that the intersection of any nonempty collection of congruences on $\mathcal{A}$ is a congruence itself.

Let us define on $Con(\mathcal{A})$ two binary operations $\wedge$ and $\vee$ by

$$\theta \wedge \gamma = \theta \cap \gamma;$$

$$\theta \vee \gamma = \bigcap \{\phi \in Con(\mathcal{A}) :\ \theta, \gamma \subseteq \phi\}$$

for all $\theta, \gamma \in Con(\mathcal{A})$.

Theorem 6

1. *$\mathcal{C}on(\mathcal{A}) = \langle Con(\mathcal{A}), \wedge, \vee \rangle$ is a lattice called the congruence lattice of the algebra $\mathcal{A}$.*

2. *Congruences of $\mathcal{A}$ are exactly these subuniverses of the algebra $\mathcal{A} \times \mathcal{A}$ which are equivalence relations.*

3.1 Free algebras

Now, we are going to introduce notions of equations, algebras of terms and free algebras, which are essential tools of universal algebra and its applications.

Let $\mathcal{K}$ be a class of similar algebras and let $\mathcal{A}$ be an algebra of the same type with the set of generators X. It is said that $\mathcal{A}$ is *free for* $\mathcal{K}$ over X iff for every algebra $\mathcal{B} \in \mathcal{K}$ any mapping $g : X \longrightarrow B$ can be extended to a homomorphism $g^* : A \longrightarrow B$. If, in addition, $\mathcal{A} \in \mathcal{K}$ then we say that $\mathcal{A}$ is *free in* $\mathcal{K}$ over X. Then X is called a *free generating set* of $\mathcal{A}$ and it is said that $\mathcal{A}$ is *freely generated* by X.

Lemma 7

1. *If $\mathcal{A}$ is free for $\mathcal{K}$ over X then $\mathcal{A}$ is free over X for the variety $V(\mathcal{K})$ generated by $\mathcal{K}$.*

2. *If $\mathcal{A}$ and $\mathcal{B}$ are free in $\mathcal{K}$ over X and Y, respectively, and $|X| = |Y|$ then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

We are going to show that for every nontrivial variety V and every nonempty set X there is an algebra free in V over X. We start with constructing absolutely free algebras. An algebra is *absolutely free* if it is free in a variety of all algebras of a given type.

We need an appropriate language to describe classes of algebras of the same type by logical expressions. This formal language is built up by variables which will be taken from some set called an alphabet. We also need a set of operation symbols of type τ.

Let $\mathcal{K}$ be a class of all algebras of type $\tau : I \longrightarrow \mathbf{N}$, let X be a nonempty set. The *set of terms* of type τ over X is the smallest set $T(X)$ of finite strings such that

1. $X \subseteq T(X)$;

2. If $i \in I$ and $\tau(i) = 0$, then $f_i \in T(X)$;

3. If $i \in I$, $\tau(i) = n$ and $t_1, ..., t_n \in T(X)$, then $f_i(t_1, ..., t_n) \in T(X)$.

In other words, $T(X)$ is the set of words on the alphabet $X \cup \{f_i\}_{i \in I} \cup \{(,)\}$ fulfilling conditions 1.–3.. For convenience, we will assume that X and $\{f_i\}_{i \in I}$ are disjoint.

The *term algebra* of type τ over a nonempty set X is the algebra $\mathcal{T}(X)$ of type τ with universe $T(X)$ and such that

$$f^{T(X)}(t_1, ..., t_n) = f(t_1, ..., t_n)$$

for all $t_1, ..., t_n \in T(X)$ and every operation symbol f of rank n in type τ.

Theorem 8 *For every $\tau : I \longrightarrow \mathbf{N}$ and every nonempty set X the term algebra $\mathcal{T}(X)$ is absolutely free algebra of type τ over X.*

Thus, if $\mathcal{A}$ is an algebra of type τ, then $\mathcal{A}$ is a homomorphic image of the term algebra $\mathcal{T}(A)$.

Theorem 9 *Let $\mathcal{K}$ be a class of algebras of type τ, let X be a nonempty set. Let us define*

$$\theta = \bigcap\{\alpha \in Con(T(X)) : \ T(X)|\alpha \in S(\mathcal{K})\}.$$

Then $\mathcal{T}(X)|\theta$ is free for $V(\mathcal{K})$ over the set

$$X|\theta = \{x|\theta : \ x \in X\}.$$

We can notice that if all algebras in $\mathcal{K}$ are trivial (i.e. one-element algebras) then $\mathcal{T}(X)|\theta$ is also trivial. On the other hand, if $\mathcal{K}$ contains at least one non-trivial algebra then we can find in $V(\mathcal{K})$ an algebra which is isomorphic to $\mathcal{T}(X)|\theta$ and which is free for $V(\mathcal{K})$ over X.

4 Birkhoff's Theorem

We often assume that terms of type τ are the elements of one fixed absolutely free algebra $\mathcal{T}(\omega)$ with an infinite, countable free generating set. Elements of this set are called variables and denoted $x_1, x_2, ...$

We can observe that, for every term $t \in T(\omega)$, there is the unique smallest set of variables $\{x_1, ..., x_n\}$ such that $t \in T(\{x_1, ..., x_n\})$. We say then that variables $x_1, ...x_n$ occur in the term t and we write $t \in T_n$.

Every term of type τ can be regarded as a name for one or more term operations in every algebra of type τ. Strictly speaking, to every term $t \in T_n$ of type τ corresponds in an algebra $\mathcal{A}$ of type τ an operation t^A of rank n, which can be defined as follows:

- if $t = x_i$, then $t^A(a_1, ..., a_n) = a_i$;
- if $t = f_i$, where $\tau(f_i) = 0$ then $t^A(a_1, ..., a_n) = f_i^A$;
- if $t = f_i(t_1, ..., t_m)$, then

$$t^A(a_1, ..., a_n) = f_i^A(t_1^A(a_1, ...a_n), ..., t_m^A(a_1, ..., a_n)).$$

Now, we are ready to deal with the notion of equation.

An *equation* of type τ is a string of the form $t \approx s$, where s and t are terms of type τ. In other words, just as terms are taken to be certain words over an alphabet, then we can regard equations as words over the alphabet expanded by an additional letter $\approx$.

It is said that the n-tuple $(a_1, ..., a_n) \in A^n$ *satisfies* the equation $t \approx s$ iff

$$t^A(a_1, ..., a_n) = s^A(a_1, ..., a_n).$$

We say that the equation $t \approx s$ is *true in an algebra* $\mathcal{A}$ iff all n-tuples from A^n satisfy the equation (in other words, $t^A = s^A$). We denote this by

$$\mathcal{A} \models t \approx s.$$

Sometimes, we say that $t \approx s$ is an identity in $\mathcal{A}$ or is valid in $\mathcal{A}$.

If $\mathcal{K}$ is a class of algebras of type τ, then the equation $t \approx s$ of type τ is *true in the class* $\mathcal{K}$ iff it is true in every algebra from $\mathcal{K}$.

Let Σ be a set of equations of type τ. The class of all algebras of type τ which satisfy all the equations in Σ is denoted by $Mod(\Sigma)$. Such a class of algebras is called an *equational class* and it is said that Σ axiomatizes the class.

On the other hand, for any class $\mathcal{K}$ of similar algebras, we can define $\Theta(\mathcal{K})$ — the set of all equations which are true in every algebra of $\mathcal{K}$. A set of equations Σ is called an *equational theory* iff $\Sigma = \Theta(\mathcal{K})$ for some class $\mathcal{K}$.

It can be observed that $Mod\Theta$ and ΘMod are closure operators on the class of algebras and on the set of equations of the same type, respectively.

There is a one-to-one correspondence between varieties and equational theories of the same type. Now, we are ready to formulate the very important Birkhoff's Theorem, proved in 1935:

Theorem 10 *Let $\mathcal{K}$ be a class of similar algebras. Then*

$$HSP(\mathcal{K}) = Mod(\Theta(\mathcal{K})).$$

In other words, $\mathcal{K}$ is a variety iff $\mathcal{K}$ is an equational class.

We can notice that all algebraic structures introduced in Chapter 1 were defined (axiomatized) by some sets of equations. It means that, for example, all lattices form a variety of lattices and hence every subalgebra of a lattice is a lattice, every homomorphic image of a lattice is a lattice and every direct product of a collection of lattices is a lattice. The same concerns the variety of distributive lattices and the variety of boolean algebras. Of course, the same occurs for varieties of semigroups, monoids, groups, abelian groups and rings.

5 Some remarks on lattices

As we saw in previous Chapters, lattices often appear in algebraic investigations. In particular, $\mathcal{S}ub(\mathcal{A})$ and $\mathcal{C}on(\mathcal{A})$ are lattices for every algebra $\mathcal{A}$. It can be proved that the collection of all varieties of a given type forms a lattice. Furthermore, it turns out that lattices play an important role in the algebraic description of logic.

Up to now we have looked at lattices as algebras with two binary operations: meet and join, axiomatized by some set of equations. However, lattices can also be viewed as a kind of posets.

Let us recall that a binary relation $\leq$ is a *partial order* on a nonempty set A iff $\leq$ is reflexive, anti-symmetric and transitive, i.e.

1. $x \leq x$;
2. $x \leq y$ and $y \leq x$ imply $x = y$;
3. $x \leq y$ and $y \leq z$ imply $x \leq z$

for all $x, y, z \in A$.

Two elements of A which are in the relation $\leq$ are called *comparable*.

If $\leq$ is a partial order on A then $\langle A, \leq \rangle$ is called a *partially ordered set* or, simply, a *poset*.

Let $\langle A, \leq \rangle$ be a poset and let $X \subseteq A$. An element $a \in A$ is called an *upper (lower) bound* of X iff $x \leq a$ ($a \leq x$) for all $x \in X$. If $a \in X$ and

a is an upper (lower) bound of X then a is called the *greatest (least) element* of X. There exists at most one greatest (least) element of a given subset of a poset.

The least element (if there exists) in the set of upper bounds of $X \subseteq A$ is called the *supremum* of X and denoted $\sup X$. Dually, the greatest element of the set of all lower bounds of X is called the *infimum* of X and denoted $\inf X$.

If $\mathcal{A} = \langle A, \wedge, \vee \rangle$ is a lattice, then we can define a partial order on A by

$$x \leq y \Longleftrightarrow x \wedge y = x.$$

It can be proved that $\langle A, \leq \rangle$ is a poset and for every $x, y \in A$, $x \wedge y$ and $x \vee y$ are, respectively, the infimum and the supremum of the set $\{x, y\}$.

On the other hand, if $\langle A, \leq \rangle$ is a poset in which every two-element subset has the infimum and the supremum, then $\mathcal{A} = \langle A, \wedge, \vee \rangle$, where

$$x \wedge y = \inf\{x, y\};$$

$$x \vee y = \sup\{x, y\}$$

is a lattice.

Thus, lattices can be viewed as a poset in which every finite subset has the infimum and the supremum.

A lattice $\mathcal{A}$ is said to be *complete* iff for every subset X of A there exist in $\mathcal{A}$ the infimum and the supremum of X. Obviously, any finite lattice is complete. What is more, many lattices mentioned before, like $\mathcal{S}ub(\mathcal{A})$ and $\mathcal{C}on(\mathcal{A})$, for any algebra $\mathcal{A}$, or a lattice of all varieties of a given type are complete lattices.

A poset $\langle A, \leq \rangle$ is called a *chain* if for every $x, y \in A$ either $x \leq y$ or $y \leq x$ (i.e. every two elements of A are comparable). It is obvious that every chain is a lattice and, what is more, a distributive lattice.

The earliest lattices to be investigated were distributive lattices. Not only chains but lattices of all subsets of any given set, as well, are distributive lattices. Moreover, it was shown by Funayama and Nakayama that the congruence lattice of any lattice is distributive.

A lattice $\mathcal{A}$ is bounded if it has both the greatest and least elements. They are usually denoted 1 and 0, respectively.

Let $\mathcal{A}$ be a bounded lattice with the greatest element 1 and the least element 0. The element $b \in A$ is called a *complement* of $a \in A$ iff

$a \wedge b = 0$ and $a \vee b = 1$. The lattice $\mathcal{A}$ is said to be *complemented* iff every element of $\mathcal{A}$ has a complement.

A complemented distributive lattice is called a *boolean lattice*.

Let us observe that in the definition of a boolean algebra we have a basic unary operation of complementation which is defined by some equations. Thus, although boolean algebras form a variety, boolean lattices do not since a sublattice of a boolean lattice need not be a boolean lattice.

One of the most important results concerning boolean algebras is the following Representation Theorem, proved by Stone:

Theorem 11 *Every boolean algebra is isomorphic to a field of sets.*

There are many applications of lattices and boolean algebras to logic. In particular, the Lindenbaum algebra of classical propositional calculus can be shown (using some modifications) to be a boolean algebra.

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