New logics for quantum non-individuals?

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Abstract

According to a very widespread interpretation of the metaphysical nature of quantum entities, the so-called *Received View on quantum non-individuality*, quantum entities are non-individuals. Still according to this understanding, non-individuals are entities for which identity is restricted or else does not apply at all. As a consequence, it is said, such approach to quantum mechanics would require that classical logic be revised, given that it is somehow committed with the unrestricted validity of identity. In this paper we examine the arguments to the inadequacy of classical logic to deal with non-individuals, as previously defined, and argue that they fail to make a good case for logical revision. In fact, classical logic may accommodate non-individuals too. What is more pressing, it seems, is not a revision of logic, but rather a more adequate metaphysical characterization of such entities.

Key-words: non-individuals; logical revision; quantum mechanics; identity.

1 Introduction

Non-reflexive systems of logic are systems where some of the laws of identity are restricted or violated. In this sense, such systems are thought to revise classical logic with identity, where identity holds overall. The main motivation for advancing such systems comes from non-relativistic quantum mechanics. In a nutshell, the plan goes as follows: it is said that quantum entities may have a very distinctive metaphysical nature, they are *non-individuals*; this non-individuality, according to some of the proponents of non-reflexive systems, is cashed precisely in terms of a failure of identity, and this state of affairs is captured by non-reflexive systems (see French and Krause [10, chaps.7-8]). Classical logic, on its turn, deals only with individuals. As a result, the argument goes, classical logic must be substituted by non-reflexive systems if we are to assume a metaphysics of non-individuals.

What is most interesting about this story is that this would be a clear example of an empirical theory (plugged with a metaphysical understanding of its entities, of course) demanding a revision of logic. The idea is simple: if we assume that quantum mechanics deals with non-individuals, and that non-individuality must be cashed in terms of a failure of identity, then, classical logic with identity must be revised.[1] According to French and Krause [10, p.240], classical set theory and logic

> ... involve a theory of identity which takes the elements of a set (even the *Urelemente*, if they are admitted by the theory) to be *individuals* of a kind. In short, this 'theory of identity' contrasts with the Received View of quantum entities as absolutely or 'strongly' indistinguishable entities, as we have discussed, and cannot provide the grounds for treating 'truly' indistinguishable non-individuals.

The Received View (the RV from now on) mentioned here encompasses this specific understanding of the metaphysical nature of quantum entities that seems to lead us directly to the need of a revision of classical logic: classical logic is not able to adequately accommodate 'indistinguishable non-individuals'. And why does that happen? The main problem seems to be that classical logic cannot deal with such kind of entities. Notice that the problem, as it appears in the above quote, is two-folded: i) classical logic cannot deal with non-individuals, it deals only with *individuals*, and ii) classical logic cannot deal with indiscernible entities, it deals only with *discernible* entities.

If we are to make a good case for the change of logic, of course, we may proceed as French and Krause [10] do: show how to precisely understand the notions of 'indiscernibility' and 'non-individuality' that are at stake, and then, argue that classical logic fails to correctly account for them, while non-reflexive logics succeed in doing so. If everything goes well, there is a nice case for a change of logic when dealing with quantum non-individuals.

But revising logic is not always simple as that. It will be our contention in this paper that, under the specific understanding that the notions of 'indiscernibility' and 'non-individuality' receive by French and Krause [10], classical logic may be seen as treating them successfully too! Indeed, due to arguments by French and Krause themselves, we shall propose that i) first-order and higher-order logic are able to accommodate those notions, and ii) classical set theory (Zermelo-Fraenkel kind of system, for definiteness), may also do so, with some provisos. In fact, as we shall see, French and Krause are sometimes ambiguous on the specific meaning of non-individuality, leaving us some choices as to what kind of approach is taken and how to represent those entities. As a byproduct of our discussion, we shall get a better understanding of non-individuality.

This paper is divided as follows: we begin in section 2 by presenting the main contentions by French and Krause that a revision of logic is required if we are to represent more adequately the metaphysical nature of quantum non-individuals. In section 3 we examine how first-order classical logic deals

[1] Logical revision is now returning to the center of the stage among philosophers of logic; see Hjortland [11] for discussion and references.

with the concept of identity. We argue that it is possible to avoid the criticism advanced by French and Krause in this case by focusing in the so-called non-normal models of identity. In section 4 we deal with classical second-order logic. Again, there are non-standard models which prevent the arguments from French and Krause to have the expected effect. In section 5 we deal with set theory. We argue that the criticism to pure set theory (set theory without Urelemente) is rather inappropriate, while the criticism to set theory with atoms may be softened if we just take into account the appropriate view on non-individuals. In section 6 we present our concluding remarks. The result seems to be that, while we may use non-individuality as a motivation to change logic, there is no need to do so: classical logic is compatible with such unorthodox metaphysics; what is still lacking is a better comprehension of the *metaphysics* of non-individuals.

2 Problems with classical logic

As we have already mentioned, French and Krause [10, p.240] claim that classical logic involves a theory of identity that prevents it from dealing with 'truly indistinguishable non-individuals'. There are two sources of the difficulties for the adequacy of classical logic: i) it deals only with individuals, and ii) it deals only with discernible objects. Non-reflexive logics, on the other hand, should provide solutions to both problems, allowing for indiscernible non-individuals.

Let us begin with the claim that classical logic deals only with discernible things, being unable to represent legitimate indiscernible items. The problem with *classical set theory*, it is claimed, is that "[t]he language of standard mathematics . . . is based on the assumption that a set is a collection of distinguishable objects" (French and Krause [10, p.248]). So, in this sense, what must be revised in the new system, it seems, is the fact that every item in classical set theory may be distinguished from every other item. Discernibility is the problem here: every item has a property that allows it to be distinguished from every other item in set theory.

Quantum entities of the same kind (e.g. electrons) are typically considered as indiscernible. Their indiscernibility is a result of the permutation symmetry present in the theory: particles of the same kind may be permuted without the permutation giving rise to a distinct state. The idea is that the expectation of the result of the measurement of an observable is the same before and after a permutation of the particles. No property whatever is able to discern the entities. Now, the way this is achieved in the formalism of quantum mechanics is not particularly faithful to the phenomena: one first labels the particles, (thus discerning them by their labels), and then postulates that permutations of the labels are not observable. Heinz Post demanded that indiscernibility be attributed right at the start, without such *ad hoc* manoeuvres. It is to such a plea for coherence that French and Krause urge us to change classical logic for a non-reflexive system. As French and Krause [10, p.318] put it:

> [...] the basic quantum entities of the same kind may be indistinguishable, and so it is a pertinent question to look for the kind of 'logic' they obey. In such a logic, of course, we would be able to talk of indistinguishability, and to consider that some entities may have all their relevant properties in common without turning out to be the very same entity, as implied by Leibniz's Law.

Leibniz's law here is understood as the Principle of Identity of Indiscernibles (PII): numerically distinct entities must be discernible by at least one property. So, given that there are situations in which quantum entities are qualitatively indiscernible while being numerically distinct, Leibniz's Law fails. The correct representation of this would require that we allow in our logic entities that violate Leibniz's Law. To do such a thing, French and Krause propose that we follow Heinz Post's suggestion that 'indiscernibility should be attributed right at the start':

> [...] we might be interested in following Post's suggestion that the indistinguishability of quantum entities should be something attributed to them *right at the start*, as a primitive notion, and not obtained *a posteriori*, by *ad hoc* devices [...]. This is where quasi-set theory helps. It enables us to consider quantum objects as really and truly indistinguishable entities *right from the start*.

Non-reflexive logics deal with this issue by the introduction of a primitive relation of indiscernibility. This relation does not collapse on identity, that is, items may be indiscernible without being the same (violating thus Leibniz's Law). Now, from a metaphysical point of view, Leibniz's Law is typically associated with bundle theories of individuation, the view according to which an individual is nothing over and above the properties that it bears. In this sense, the numerical difference of individuals is accounted for by the differences in the bundles forming these individuals, which amounts to a difference in the properties the individuals instantiates (*i.e.*, a qualitative difference). Given that quantum entities seem to fail the PII, it could well be suggested that the entities failing this law could be seen as non-individuals of a kind. In this sense, the problems of indiscernibility and non-individuality would have the same kind of solution: non-individuals are the entities that may be indiscernible without being identical to other entities of the same kind (see Arenhart [2]). However, while the failure of PII is explicitly called to explain the indiscernibility, this is not how French and Krause officially approach non-individuality. Let us check.

The next point concerns the claim that classical logic deals with individuals. French and Krause [10, p.250] claim that "...classical languages ... talk essentially about individuals". So, in this sense, the problem is individuality, but now, this notion is not understood in terms of a bundle theory of individuality; now, individuality is attributed by self-identity, which represents a form of haecceity. In fact, from a metaphysical point of view, haecceities are non-qualitative properties that confer individuality to concrete particulars. For instance, Socrates' individuality is conferred by the property of 'being Socrates', which is his haecceity, a property that only Socrates has. Haecceities work in the same lines as bare particulars or substrata, with the main difference that while the latter are thought of as particulars too, the haecceity is a non-qualitative property. Anyway, what is relevant to us is the fact that a haecceity is a kind of individuating mechanism that grants individuality independently of the qualitative properties instantiated by the particular. In this sense, two particulars could well share every qualitative property and still be numerically distinct, due to the fact that they bear distinct haecceities. For this reason, a haecceity is called also a form of 'Transcendental Individuality', due to the fact that the individuality conferring principle transcends the qualities of the entity. As French and Krause [10, pp.13-14] put it:

> . . . the idea is apparently simple: regarded in haecceistic terms, "Transcendental Individuality" can be understood as the identity of an object with itself; that is, '$a = a$'. We shall then defend the claim that the notion of non-individuality can be captured in the quantum context by formal systems in which self-identity is not always well-defined, so that the reflexive law of identity, namely, $\forall x(x = x)$, is not valid in general.

In this sense, individuality is related with self-identity, not with discernibility. Non-individuality, on its turn, is related not with indiscernibility, but rather with lack of self-identity (which means, in metaphysical terms, lack of a haecceity). Now identity is at the center of the stage:

> We are supposing a strong relationship between individuality and identity . . . for we have characterized 'non-individuals' as those entities for which the relation of self-identity $a = a$ does not make sense. (French and Krause [10, p.248])

What results from all this is that classical logic, according to this claim, endows everything with individuality through haecceity by the very fact that it has the reflexive law of identity. Everything is identical to itself, so everything is an individual.

Non-reflexive logics attempt to solve this problem precisely by restricting formulas using the relation of identity. In this sense, there will be no haecceities for some of the entities dealt with by these systems. These systems originated with Newton da Costa [6, pp.138-141] (first edition of 1980), who baptized them 'Schrödinger logics'; that is, systems of logic attempting to encapsulate the claim by Schrödinger that identity does not make sense for quantum entities. The often repeated quote is:

> I beg to emphasize this and I beg you to believe it: it is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning. (Schrödinger [16, pp.121-122]).

So, as understood by da Costa and by French and Krause,[2] what Schrödinger is claiming is that identity, as a relation, does not make sense when its relata are quantum entities. A solution, as non-reflexive logics were advanced to be, consists in eliminating the concept of identity of the language (more on this soon). Metaphysically, as we have seen, this is tied to individuality and non-individuality in haecceistic terms (again, see Arenhart [2] for a discussion). Individuals have haecceities, represented by self-identity, while non-individuals have no haecceities, represented by the fact that identity makes no sense for them. In this sense, on what concerns individuality the complaint about classical logic is simple: it fails because it attributes identity to everything.

For definiteness, let us label the two distinct sources of problems for classical logic that arise from quantum mechanics:

Post problem: quantum entities may be indiscernible without being numerically identical.

Schrödinger problem: identity does not apply meaningfully to quantum entities.

Let us see how classical logic is presumed to deal with the core concepts of individuality and discernibility. We shall focus on French and Krause's arguments to those effects.

3 Problems with first-order classical logic

To begin with, let us consider first-order classical logic with identity. As we have mentioned, according to French and Krause [10, chap.6], classical logic commits us with a theory of individuals and with full discernibility. Both problems may be dealt with by considering the proper workings of the relation of identity in first-order systems. From a syntactical point of view, the relation of identity may be defined in languages with a finite non-logical vocabulary, or else be a primitive symbol. Given that the problems are the same in both cases, we discuss mainly the case in which there is a primitive symbol, which is characterized by two postulates (when the axiomatic method is employed as the proof method, but identical remarks hold for other proof systems such as natural deduction):

Reflexivity $\forall x(x = x)$

Substitution $\forall x \forall y(x = y \rightarrow (\alpha \rightarrow \alpha[y/x]))$, with the usual restrictions.

Now, what is the problem with these postulates from the point of view of the Received View? That is, what is wrong with them when considered along the general thesis that quantum entities are indiscernible non-individuals? The answer should be clear: these postulates allow us to grant that everything is

[2]We by no means imply that this is precisely what Schrödinger was claiming originally.

self-identical (possessing thus a haecceity) and that a version of the PII holds, so that every item is discernible from every other item.

We begin by considering the claim that the entities being dealt with are individuals, that is, that they possess a form of haecceity. This could be thought to follow straightforwardly from the first postulate for identity. However, we shall argue, there is a problem with this conclusion. In fact, from a purely syntactical point of view, for any individual constant c in the language, it holds that $c = c$. However, that can only mean that every thing is identical to itself once it is granted that '$=$' really denotes the relation of identity. It is here that problems appear, given that it is widely known that first-order languages fail in characterizing identity (and French and Krause explicitly address this point when considering identity in first-order languages in [10, chap.6]). As we shall argue, it is only through some kind of equivocation that these languages may be said to deal with individuals without further considerations; that is, we have to consider that it is the mere identity sign that is doing the whole job, irrespectively of its interpretation. In this sense, the claim that classical logic is committed to a theory of individuality, is not completely justified, — at least in its first-order version.

In fact, instead of committing us with individuals, these postulates seem to be able to commit us with non-individuals too. There are two main reasons for that. First of all, as French and Krause [10, pp.252-254] point out, first-order languages fail to capture the intended meaning of identity, when it is understood as the diagonal of the domain of interpretation. That is, with the traditional first-order postulates for identity, it is not guaranteed that the interpretation of the symbol '$=$' will always be

$$Diag(D) = \{\langle x, x\rangle | x \in D\},$$

where D is the domain of the interpretation. In more technical terms: not every interpretation of the first-order symbol of identity is a *normal* interpretation (see also the discussion in Béziau [3] and Mendelson [13, p.93]). In this sense, there are interpretations of first-order languages in which the meaning of '$=$' is simply not identity.

This happens precisely because there are *non-intended interpretations* of the symbol of identity; this symbol may be interpreted as a kind of indiscernibility relation in the first-order language. Indeed, even supposing that the relation $\sim$ over D that is the denotation of the sign of identity '$=$' is not the diagonal, one still grants that $\sim$ is a congruence relation; in other words, it is an equivalence relation (reflexive, symmetric, and transitive) and it is compatible with every other relation in the domain of interpretation: whenever the elements a and b of D are such that $a \sim b$, and P^I is any property over D (which is the denotation of a corresponding predicate sign P in the language), we are granted that if $P^I a$, then also $P^I b$ (again, see the details in Béziau [3]). A similar compatibility holds for relations, so that, for instance, if $R^I ac$, then $R^I bc$, and if $R^I ca$, then $R^I cb$, for any $c \in D$, and so on for relations of other weights. The relation $\sim$ may be understood as partitioning the domain D into equivalence classes of indiscernible objects (nota bene: indiscernible by the properties and relations over D that are extensions associated with the predicate signs of the language). Of course, by taking the quotient of this domain by $\sim$ we obtain another set, $D/_{\sim}$, which may then be converted into the domain of a normal model for the same language which is elementarily equivalent to the former. That, however, does not preclude $\sim$ from being a congruence relation that is not the diagonal (for the technical details see Mendelson's book [13, p.93]; one may also go the other way around and convert any normal model into a non-normal one; see da Costa and Bueno [7, pp.186-187]).

Now, when these non-intended interpretations are taken into account, the identity sign represents an indiscernibility relation, not identity, and one cannot claim that we have granted that every thing is self-identical according to the first-order language in consideration. So, identity does not hold overall, given that it may well be the case that *numerically distinct* entities of the domain may be related by the relation $\sim$ playing the role of the denotation of the identity sign.

This is evidence against both claims that classical logic cannot deal with the Schrödinger problem and with the Post problem. Indeed, under such non-normal interpretations the identity sign is not identity over the domain of interpretation, so identity does not apply overall, unless one is strictly speaking about formulas where the symbol $'='$ occurs, not its intended meaning (which clearly is not the case). So, the Schrödinger problem is not a problem after all, at least in these cases. Also, the relation $\sim$ is an indiscernibility relation, so that indiscernible items need not be the same. In this case, the Post problem may be addressed too in these non-normal interpretations.

But there is more to say about indiscernibility. In fact, the first-order version of the PII, which may only be formulated through a schema,

$$\forall x \forall y((\alpha \leftrightarrow \alpha[y/x]) \to x = y)$$

is not a theorem of first-order logic. It is quite simple to find counter-models for it (for extensive discussion on many distinct versions of the PII in first-order logic, see [5]).[3] For a quick example, consider a first-order language with a single unary predicate symbol P. We define an interpretation I such that its domain is $D = \{0, 1\}$, with $P^I = \{0, 1\}$. It is easy to check that there are valuations where $\forall x \forall y(Px \leftrightarrow Py)$ is satisfied in this interpretation, without $x = y$ being satisfied. As a result, again, items may be indiscernible without being identical. That is, the claim that first-order logic obeys a version of the Principle of the Identity of Indiscernibles does not hold, even when the interpretation of identity is normal.

[3] There is a whole discussion about weak discernibility in quantum mechanics: quantum entities are said to be weakly discernible by symmetric and irreflexive relations (see Muller and Saunders [15], Bigaj [4]). Here, we shall not enter into such discussions, which would certainly have an impact on the way French and Krause address indiscernibility. We just follow their usage of indiscernibility to grant that there is a sense in which classical logic may deal with indiscernible entities.

Both of the mentioned reasons for the compatibility of first-order logic with the failure of identity are, of course, related, and stem from the fact that first-order languages have limited expressive power. Furthermore, notice that a model may be normal and even so fail to satisfy the first-order version of the PII. The only way to avoid the possibility of non-normal models is to explicitly mention in the metalanguage that the interpretation of the identity sign is provided by the diagonal of the domain. But that is an imposition from the metalanguage, and is not guaranteed by the postulates of identity. So, in the end, for first-order languages we have it that indiscernible quantum non-individuals are compatible with the approach to identity furnished by such a logic! It can only be avoided by employing the resources of the metalanguage, which is typically classical set theory, but we shall discuss set theory later. If we restrict our discussion to this level, the introduction of non-reflexive logics would not be clearly motivated.

This expressive limitation reaches very far, infecting even non-reflexive systems of logic such as Schrödinger logics (thus threatening its metaphysical adequacy). Indeed, suppose that one adopts first-order Schrödinger logics. This system is developed from classical first-order logic with identity by restricting the relation of identity in the language. To do so, one may simply employ a two-sorted language with two kinds of terms, let us say, without individual constants or function symbols for the sake of simplicity,

micro terms $u, v, x, y, z, w, u_1, v_1, x_1, y_1, \ldots$ and

macro terms $U, V, X, Y, Z, W, U_1, V_1, X_1, Y_1, \ldots$.

Micro terms are intended to range over quantum entities, while macro terms range over usual (individual) objects. Identity forms well-formed formulas only for macro terms, but not for micro terms; this is the strategy behind the restriction of identity in Schrödinger Logic (see French and Krause [10, chap.8] for technical details and further references). As a result, $t = k$ is not a formula when either t or k is a micro term. In this sense, there is no reflexive law of identity for micro terms, and they may be said to represent legitimate non-individuals, in the sense that identity does not apply to them (as Schrödinger seems to have required, according to the interpretation of da Costa, French and Krause). For macro terms identity is governed by the two axioms of identity already presented.

Now, while the goal of Schrödinger logics is to represent non-individuals, it seems that the micro terms clearly do that. Identity fails by default for them, given that there is no identity there at all. However, the same problem of characterization of identity we have just discussed for classical logic presents itself for macro terms in Schrödinger logic. There is no way to grant that the interpretation of $' =' $ for macro terms will be the diagonal of the domain (even if we employ a non-reflexive logic in the metalanguage; see Arenhart [1]). By allowing that some of the models of the macro terms may be non-normal with respect to the symbol of identity, we are left with another kind of terms that represent entities without identity. So, we have two distinct kinds of terms to represent non-individuals. In this sense, the Schrödinger logics, as opposed to what it was assumed to do, may deal only with non-individuals in some (non-intended) interpretations, just as classical logic does, when identity has a non-normal interpretation.

The main difference between classical logic and Schrödinger logic is that while the former deals only with one kind of non-individuals, the latter may deal with two kinds, the ones that are represented by the micro terms, and those represented by the macro terms, under a non-normal interpretation of identity. However, classical logic seems to have the advantage that its non-individuals are all related by the congruence relation $\sim$, which is the denotation of the identity sign, so that everything enters the indiscernibility relation. In the case of Schrödinger logic, this only happens for the macro terms, under the non-normal model of identity. Non-individuals of the kind represented by micro terms have no indiscernibility relation, so, when restricted to first-order, they fail in addressing the Post problem!

As a conclusion, it seems, classical logic seems to confer a pleasant uniformity to the treatment of non-individuals, solving both the Post and the Schrödinger problem at once. Schrödinger logics also solve both problems, but only in a more limited and rather unintended way: we can only solve the Post and the Schrödinger problems if we allow a non-normal model of identity. This is due to the fact that first-order Schrödinger logics did not originally have any indiscernibility relation for the micro terms. So, revision of classical logic, in this case, is not only unmotivated, but it would be better to keep classical logic if we restrict ourselves to first-order languages.

4 Problems with higher-order logic

Higher-order classical logic is also said to present problems to the correct representation of non-individuals. Here we shall confine ourselves mostly to second-order logic, but our discussion easily generalizes to higher-order systems.

Due to the expressive power of second-order languages, identity needs not to be a primitive symbol. The reason is that such systems allow for a direct definition of identity in terms of Leibniz's Law:

$$x = y \leftrightarrow \forall F(F(x) \to F(y)).$$

It is easy to prove that such a definition, with a conditional in the definiens, works only provided we allow in the scope of the variable F the property "being identical with x". If that is the case, then there is nothing that can be done to avoid identity in the language, and *nota bene*, identity works then as identity. If, however, we rule out such a property from the scope of the quantifier, then the definition is typically stated with a biconditional in the definiens. Anyway, whatever our decision as to whether self-identity is an allowable property in the scope of the quantifier, identity is definable in the object language.

Let us begin with the claim by French and Krause that this language is Leibnizian, so that it cannot account for indiscernible non-individuals. This seems to be the problem with second-order languages according to French and Krause, if we consider their claim that classical logic deals with individuals: given that identity is definable, there is no hope for representing non-individuals when these languages are employed. But is it true? It all depends on how one is going to interpret the language. On some interpretations, it is possible to have almost the same kind of argument that we had for first-order languages, and hold that second-order languages are also compatible with indiscernible non-individuals. The main reason is similar to the case of first-order languages: there are interpretations of the symbol of identity that are not the diagonal of the domain; in other words, identity works as an indiscernibility relation, which is compatible with the existence of qualitatively indiscernible non-identical items.

The main line of the argument for that claim is rather simple, and is related to the semantics of second-order languages. As it is well known, second-order logic has mainly two kinds of interpretations: *principal interpretations* and *secondary interpretations*. Let us check very briefly their differences. In order to interpret a second-order language, we must provide for a structure of interpretation:

$$\langle D, R_i \rangle,$$

where D is a non-empty domain, and where we associate each $n-$ary predicate symbol of the language to a subset R_i of D^n, which acts as the extension for the predicate symbol of the language (for simplicity, we leave out function symbols and individual constants).

In order to provide the truth conditions for open formulas and quantified formulas, there must also be evaluation functions s attributing values to variables. For first-order variables, the evaluation function attributes a member of D (that is, $s(x) \in D$). The distinction between principal and secondary interpretations comes when second-order variables are involved. What is the range of second-order variables? Properties and relations, of course. In *principal interpretations*, evaluations may attribute to second-order variables any property or relation available in D; that is, every subset of D^n is available to be the value of an $n-$ary predicate variable. In particular, for property variables ($1-$ary predicate variables), the unitary subsets of each element of the domain is available, so that for each $a \in D$, the property 'being identical to a' is available, the variables range over it (indeed, it is the set $\{a\}$). For *secondary interpretations*, on the other hand, this is not so. One selects a subset of the available relations and allow the variables to have their values only from a restricted range of possible values.

In the case of secondary interpretations, then, it is a simple task to select properties and relations such that they are unable to discern the elements of the domain (see French and Krause [10, p.257]). In this case, it is possible that $\forall X(X(a) \leftrightarrow X(b))$ holds in the interpretation, so that we would also have by definition that $a = b$, without it being the case that a and b denote the same entity in the domain. So, even if this is not a violation of the letter of Leibniz's Law, this is a violation of its intentions! This clearly cannot be called a Leibnizian logic. There are entities numerically distinct, but which are not discernible by the properties allowed in the range of the quantifiers. In this sense, dealing with indiscernible items is not really a problem for second-order logic, provided that one is willing to assume that secondary interpretations are allowed. This would account for the Post problem in second-order logic.

Also, again there is a sense to be made that identity does not hold overall. In fact, even if we grant that $x = y$ holds, this fact by itself does not mean that the denotations of those items are the same. Identity sign is not accompanied by a corresponding semantics attributing identity as the denotation for it. So, in this sense, we do not have identity in this logic, but rather an indiscernibility relation. That would account for the Schrödinger problem (at the same time that it accounts for the Post problem). It is very much the same reasoning as the one we considered for first-order logics.

Of course, if those arguments are to have any impact on the consideration of non-individuals, there should be a reason for one to prefer secondary interpretations instead of principal interpretations. But that is not really difficult to provide, at least not when we consider that we are in the context of logical choice: we are considering whether it is reasonable to keep classical logic or to shift to non-reflexive logics, in the hypothesis that quantum entities behave in a determinate way that should be accounted for by the logic. So, given that we are considering a move to another kind of logic, it is not unreasonable to put second-order logic with secondary models among the options. In order to motivate this specific choice, it could be argued that it is quantum mechanics itself that imposes constraints on the properties and relations available to each item, so that most of the properties that could count as discerning properties would not count as physical properties, but rather as hidden logical properties, properties with no physical meaning that must be discarded and kept out of the scope of the second-order quantifiers (it seems that French and Krause are willing to consider that the choice of logic is not made purely on *a priori* grounds, see [10, p.317], so that a justification like this one is not to be easily dismissed). That is, properties like 'being identical to a' should not count as doing the discernibility work because they are simply not physical properties. In this case, then, this could count as a motivation for the preferability of secondary interpretations restricted to physically meaningful properties.

In this sense, then, second-order logic with secondary interpretations may be seen as being compatible with indiscernible entities, dealing well with the Post problem, and, again, given that the identity sign is not interpreted as identity over the domain of interpretation, it is also possible to claim that it deals with the Schrödinger problem. So, a shift to a non-reflexive approach is not mandatory.

In fact, we could go even farther and argue again that the problem of identity seems to contaminate even second-order Schrödinger logics. Let us see.

In order to keep Schrödinger's intuitions, French and Krause present a higher-order system of logic, Schrödinger's logic, which is a higher-order version of the first-order system we sketched before. The system is two-sorted, and for one kind of terms, identity is not well-formed, just as in the previous case (see French and Krause [10, chap.8]). There are two main difficulties with this approach on what concerns its goals, and they are related with the interpretation of identity.

The first problem concerns indiscernibility. Again, granted that identity is not present for one kind of terms, we still cannot have *indiscernible* non-individuals. In fact, by eliminating identity, we have only solved the problem posed by Schrödinger, but not the problem posed by Post. The obvious option to grant indiscernibility would be to follow, again, the typical intuition behind indiscernibility and allow that entities are indiscernible when they share every property and relation. That would give us $x \equiv y \leftrightarrow \forall X(X(x) \leftrightarrow X(y))$, where $\equiv$ is the relation of indiscernibility being defined. But that, notice, is just how identity is defined for higher-order languages. So, that move would re-introduce identity.

Of course, now the impact of such a definition really depends on how one interprets the identity sign. Given that it was expected that one kind of terms represent indiscernible non-individuals, one could insist that the sign $\equiv$ thus defined be interpreted in secondary models that do not attribute it the meaning of identity. That would prevent us form having identity re-introduced, and would give us a kind of indiscernibility. However, notice, that is precisely what happens in the classical case, where secondary interpretations violating identity are taken into account. So, if this option is taken seriously, there is no need to change logic: classical logic already does the job. In this case, a shift for non-reflexive logics would be unmotivated.

Another option would be to restrict the quantifier to a given kind of property when it comes to deal with indiscernibility for the terms representing non-individuals. So, for instance, let us say that P is a third-order property, which is instantiated by those properties that are somehow allowed to count as discerning the entities in question (assume we are able to select these properties in a third-order language, or even a higher level). Then, there is a relative notion of discernibility, relative to properties that have P, that is introduced as follows (see French and Krause [10, p.327]):

$$x \equiv_P y \leftrightarrow \forall X(P(X) \rightarrow (X(x) \leftrightarrow X(y))).$$

In set theoretic language: entities are indiscernible relative to a given set of properties if and only if they share every property in that set.

This solves the problem of having discernibility collapsing with identity. Of course, provided that P is not the collection of every second-order property, then $\equiv_P$ will not coincide with $=$. However, that opens a very unpleasant possibility: recall that the main objection against classical logic was that it is unable to deal with *absolutely* indiscernible entities. The relative indiscernibility that one introduces in higher-order Schrödinger logics is not really absolute indiscernibility, but only indiscernibility relative to a given collection of properties P. It could well be the case that distinct indiscernibility relations $\equiv_P$ and $\equiv_Q$ could disagree on whether two entities are indiscernible, that is, perhaps we could have $x \equiv_P y$ while not having $x \equiv_Q y$ (here, Q could well be a more restrict set than P, or a wider set, including possibly the set of every property). The result would be that x and y are after all discernible. But the initial idea was that nothing whatsoever may discern quantum particles of the same kind.[4] So, distinct discernibility relations would be responsible for discerning indiscernible entities!

What results from this is that, as a solution, introducing relative discernibility is not the best option. In any case, it seems that higher-order Schrödinger logic will have trouble with identity and indiscernibility. On the one hand, if we grant the relation of full indiscernibility, as we have seen, then either we re-introduce identity, or else, by allowing only secondary interpretations, we employ the same strategy that is already available in the classical case. On the other hand, if we grant a relation of relative discernibility, while this is distinct from the relation of identity, there is the difficulty that distinct such relations may disagree on whether two items are indiscernible, allowing us, by that very fact, to discern entities that were supposed indiscernible.

5 Problems with set theory

Classical set theory is also said to present challenges for the correct representation of indiscernible non-individuals. In fact, the *Manin problem* could be added to the Post problem and the Schrödinger problem. In a symposium evaluating the heritage of Hilbert's problems — a list of mathematical problems presented by David Hilbert in 1900 at the International Congress of Mathematicians, held in Paris —, with the preparation of a new list of Problems of Mathematics, Yuri Manin suggested that quantum entities require a new theory of sets. Classical set theory, he says, is

> . . . an extrapolation of commonplace physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking glass box, or of electrons in a nickel piece are much less Cantorian then the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'. (as quoted in French and Krause [10, p.240])

Manin then goes on to suggested that it would be fruitful to consider the development of new formalisms for set theory closer to physical theories. French and Krause attempt to provide for non-reflexive logics as an answer to the Manin problem too (besides the Post and the Schrödinger problem). They seem to believe that both of the previous problems are related to Manin's problem. In fact, in presenting an alternative system of set theory (quasi-set theory) to

[4] Recall, we are not taking into account the claims by some authors according to whom quantum particles are weakly discernible. Our point is to address the discussion by French and Krause in their own terms.

replace classical set theory when it comes to deal with quantum entities, they claim that [10, p.245]:

> Our main goal is to pursue Post's suggestion, which is obviously linked to the Manin problem mentioned above, and to follow Schrödinger's intuitions regarding the lack of sense in applying the concept of identity (as given by the 'classical' theory) to these entities. It is our view that the quantum realm still lacks an adequate mathematical framework which enables us to deal with entities which are treated as non-individuals *right at the start* for ... standard set theories do not provide such tools, and the standard ways of dealing with this concept within the usual set theoretical frameworks only mask the basic philosophical problem.

So, the solution to the Manin problem involves the recognition that standard set theories, encompassing the classical relation of identity, are inadequate to deal with indiscernible non-individuals right at the start. It will be required that the new formalism encompasses both indiscernibility as well as Schrödinger's intuitions concerning lack of identity.

The greatest part of the discussion on set theory by French and Krause [10, chap.6] concerns discernibility. The claim is that set theory — and we are talking about ZFC, no Urelemente included — is unable to allow for difference *solo numero*; every two entities are always qualitatively discernible. As French and Krause [10, p.261] put it,

> [a]ccording to the theory ... there is no place for 'indistinguishable' objects, that is, for entities that differ *solo numero*: if they differ, there exists a set — which corresponds to a property — to which one of them belongs, while the another one does not.

This conclusion is grounded on a discussion of the treatment given to indiscernibility inside ZFC. In this theory, indiscernibility may be accomplished provided we restrict ourselves to a specific structure. Let us briefly present the idea.

Consider a given set theoretical structure $\mathfrak{A} = \langle D, R_1, \ldots R_n \rangle$ existing within the set theoretical universe V (the von Neumann universe). Objects a and b in the domain of $\mathfrak{A}$ are indiscernible (relatively to that structure) if and only if there is an automorphism f of $\mathfrak{A}$ such that $f(a) = b$. For a very simple example, consider the group structure $\langle \mathbb{Z}, + \rangle$. The function $f(x) = -x$ is an automorphism for this structure, so that each integer is indiscernible from its additive opposite. Something similar happens, for instance, in the field of the complex numbers $\mathbb{C} = \langle C, +, \cdot, 0, 1 \rangle$, where a function $f(x) = \overline{x}$ mapping each complex x to its conjugate $\overline{x}$ is also an automorphism for this structure, so that each complex number is indiscernible from its complex conjugate.

According to French and Krause, as we have seen, the major obstacle to this approach to indiscernibility is that it is not a 'legitimate' indiscernibility relation, it only 'masks the basic philosophical problem'. What is wrong? The main problem is that in standard set theory every structure may be extended to a *rigid structure*, that is, a structure in which the identity function is the only automorphism. An extension of a structure, roughly speaking, is obtained by the introduction of new properties or relations in that structure. In order to make any structure rigid, for instance, it is enough that we add to it a well-ordering relation R of the elements of the domain D. Any two items a and b that are numerically different will be such that aRb or bRa, but not both. In this sense, they are discernible by R. Of course, the existence of R for any structure is granted by the axiom of choice.

The first consequence is that in a rigid structure each object is indiscernible only from itself, so that there are no indiscernible entities differing *solo numero*. Numerically different items are always discernible in such structures. The second and most pressing problem is that entities that were considered indiscernible in a non-rigid structure are in fact discernible. It is only a limitation of the structure that prevents us from grasping the discerning relations that are available. This is the idea behind the claim that classical set theory only masks the problem. The entities are discernible; we make them indiscernible by employing a rather impoverished amount of the resources available.

If that were not bad enough, the same phenomenon is present not only on a 'local' scale, relative to structures, but also on a general scale, for the whole set theoretical universe: the well-founded universe $\langle V, \in \rangle$, model of the axioms of ZFC, is itself rigid. There is only one automorphism for this structure, the identity function. In fact, every item is discernible from every other by its singleton $\{x\}$. As a result, set theory is inadequate to deal with quantum non-individuals because of its failure in accommodating numerically distinct indiscernible entities.

However, the existence of a rigid model for set theory depends essentially on the fact that such a set theory is a pure theory, that is, that there are no Urelemente (or atoms). As it is known, for a set theory with a enumerable collection of Urelemente there are models where every set is invariant under a permutation of the atoms. This kind of model was used by Fraenkel originally to prove the independence of the axiom of choice in the presence of atoms (see Fraenkel, Bar-Hillel and Levy [9, pp.58-59]). So, it could well be used to represent the indiscernible quantum particles, given that the rigidity of the global model (and on local structures) fails in this case.

So, is it the case that ZFU is more adequate than ZFC to represent quantum particles? That could well be the case, but we believe that this is not a matter of the dispute between the most appropriate between the two. The issue may be better understood if we put in the most appropriate perspectives the roles of these different theories in the formalization of physical theories such as quantum mechanics. We are assuming that theories are represented as a set theoretical structure, as recommended by some versions of the semantic approach to scientific theories (see Krause and Arenhart [12]). Both set theories may be employed, and their adequacy or inadequacy should be judged having in mind some important facts. Let us get these issues clear.

In order to represent physical objects in the domain of a structure, a pure theory of sets, such as ZFC, has only pure sets available; so, the theory must somehow employ some of the pure sets to play the role of the physical objects. This task could be performed by a collection of finite ordinals, for instance, so that a collection of four particles would have to be represented by a set such as $\{1, 2, 3, 4\}$. One could complain, as French and Krause do, that this fails to take the indiscernibility of quantum particles into account: it is really simple to discern those sets. However, in a pure set theory we should not take those representations at 'face value': physical objects are not the same thing as abstract sets. The theory would fail to represent faithfully any physical entity by the simple fact that the sets there are abstract; they are surrogates for the real entities. So, no wonder that they have some pathological properties, due to their set theoretical nature, that are not reproduced in the real thing, that is, the physical object. Given that we are working within pure set theory, we should recognize that the representation is less than ideal, and admit that some features of the set theory will not perfectly match features of the physical objects, with full discernibility among them when rigid structures are taken into account (see also Muller [14] on the "Problem of Lost Beings" and Krause and Arenhart [12]). In that sense, all that the case of rigid structures would show is that sets are discernible, and if we use sets to represent quantum particles, then they end up being discernible too, but we should not confuse those things.

As a result, the criticism directed at ZFC is rather misguided. Indeed, it attains itself to what may be considered an artifact of the system. In fact, given that we are not willing to confuse abstract sets with physical entities, we should not also think that the representation in set theoretical terms should be free of idiosyncrasies. On the other hand, when the particles are assumed as atoms, then, the set theory that arises, ZFU, is allowed to treat them as indiscernible entities — indiscernible by automorphisms. So, in this sense, while we are taking the physical entities seriously, the indiscernibility also results well motivated and represented in the resulting system.

Let us be more specific about ZFU and the indiscernibility for its atoms. Let us suppose that we take as atoms a collection $m = m_1 \cup m_2$ (which may be finite) of quantum particles; m_1 as a collection of electrons, and m_2 as a collection of protons, for the sake of definiteness. As a result of our discussion above, of first-order logics, identity may have non-standard models, that is, models where identity is interpreted as a congruence relation. Given that ZFU is a particular case of a first-order theory, it too may have non-standard models where identity is not identity for those objects of m, but rather a weaker congruence relation.[5] In this sense, even set theory may be understood as allowing for indistinguishable non-individuals, in the same vein as in first-order languages in general, provided that we interpret identity in a non-normal model.

This claim may be backed more rigorously by the presentation of an inner model of ZFU in which the elements of m are indiscernible in the relevant

[5]Of course, due to Gödel's second incompleteness theorem, there are important issues about the metatheory for those models, but we may safely leave those issues aside.

sense. We shall do even more, presenting a model in which some entities are indiscernible but identity does not hold for them. Here, we follow the presentation of da Costa and de Ronde [8] of one such model. We employ the two finite sets m_1 and m_2 we have already presented (more than two collections could be employed here, of course), with the proviso that $m_1 \cap m_2 = \emptyset$, and we define $m = m_1 \cup m_2$. We also employ a set M of classical objects. Informally, the items in m will be the quantum objects for which identity makes no sense, while the items in M are the classical objects, for which identity must hold. By transfinite induction we define a universe $\mathcal{V}$ as follows:

Definition 5.1 $V_0 = m \cup M$

$\vdots$

$V_{n+1} = \mathcal{P}(V_n)$

$\vdots$

$V_\omega = \bigcup_{b<\omega} V_\beta$

$\vdots$

$V_{\omega+n+1} = \mathcal{P}(V_{\omega+n})$

$\vdots$

$\mathcal{V} = \bigcup_{\alpha \in Ord} V_\alpha$

It is possible to prove that $\mathcal{V}$ is an inner model of ZFU (see da Costa and de Ronde [8]). Also, it is possible to define a classical sub-model $\mathcal{V}'$ of $\mathcal{V}$ by restricting the above definition to M. It provides for a model of sets of ZFU formed using only the elements of M, which shall be our classical elements.

Definition 5.2 $V_0' = M$

$\vdots$

$V_{n+1}' = \mathcal{P}(V_n')$

$\vdots$

$V_{\omega'} = \bigcup_{b<\omega} V_\beta'$

$\vdots$

$V_{\omega+n+1'} = \mathcal{P}(V_{\omega+n}')$

$\vdots$

$\mathcal{V}' = \bigcup_{\alpha \in Ord} V_\alpha'$

Inside $\mathcal{V}'$ it is possible to define the classical notion of cardinal number. Now, by using the universe $\mathcal{V}$, we may define identity and indiscernibility $\equiv$ in the whole universe:

Definition 5.3 *For $\equiv$, we define:*
Basic clauses for elements of V_0. If x, y are elements of the same m_i (with $1 \leq i \leq 2$),

then $x \equiv y$ and $y \equiv x$. If $x, y \in M$, then $x \equiv y$ if and only if $x = y$. If $x \in m_1$ and $y \in m_2$, then $\neg(x \equiv y)$ and $\neg(y \equiv x)$. Also, if $x \in m_i$ (with $1 \leq i \leq 2$) and $y \in M$, then $\neg(x \equiv y)$ and $\neg(y \equiv x)$. If $x \in V_0$ and y is a set, $\neg(x \equiv y)$ and $\neg(y \equiv x)$.

Inductive clauses. If x and y are sets such that $x \in \mathcal{V}'$ and $y \notin \mathcal{V}'$, then $\neg(x \equiv y)$ and $\neg(y \equiv x)$. If If x and y are sets such that $x \in \mathcal{V}'$ and $y \in \mathcal{V}'$, then $(x \equiv y)$ if and only if $x = y$. If If x and y are sets such that $x \notin \mathcal{V}'$ and $y \notin \mathcal{V}'$, then $(x \equiv y)$ if and only if
$\forall z(z \in x \rightarrow \exists w(w \equiv z \wedge w \in y)) \wedge \forall w(w \in y \rightarrow \exists z(z \equiv w \wedge z \in x)) \wedge card(x) = card(y)$ where card is the concept of cardinal number.

The relation $\equiv$ is defined for the whole universe $\mathcal{V}$, being an indiscernibility relation for elements of m, and coinciding with identity for elements of M. The basic idea is very similar to what is actually used in quasi-set theory, but now using a classical set theory. Indiscernibility $\equiv$ should hold for elements of m_1 and of m_2, with elements of m_1 indiscernible among themselves, and elements of m_2 indiscernible among themselves, and only then. Elements from m_1 are discernible from elements of m_2 and vice-versa, and elements form the m_i are discernible from the elements of M. It is also defined that elements of M are related by $\equiv$ if and only if they are the same. In this sense, $\equiv$ reduces to identity for classical objects. The relation $\equiv$ then is extended to the whole set theoretical universe by a transfinite induction, but the details shall not concern us here (see da Costa and de Ronde [8, sect.2]). What is relevant is that one may provide for an inner model of ZFU that obeys a kind of permutation symmetry: elements of m that are indiscernible end up belonging to indiscernible collections, so that they are indiscernible by properties, and identity is not defined for elements of m and collections made of such elements. In this sense, one may emulate the very ideas of non-reflexive logics inside classical set theory, and as a result, the Schrödinger problem and the Post problem may be addressed classically, by the presentation of a non-normal model of ZFU.

So, we have seen, a very similar strategy that was employed in first- and higher-order logics may be employed for set theory. Classical set theory may well accommodate indiscernible non-individuals, in the senses required by French and Krause. As a result, it seems to us, the classical apparatus is not so completely committed to the theory of identity they so much despise. This may be seen as a result of the fact that identity facts cannot be completely characterized in formal languages, with the possibility always being open that the symbol of identity does not really mean identity.

6 Conclusion

It is time to recap what has been achieved. We have seen that French and Krause propose that quantum mechanics, under the interpretation furnished by the Received View, requires a change in logic. Classical logic, it is said, in its many versions (first-order, second and higher-order, set theory), involves direct violations of two main features of quantum entities: they deal with discernible objects, and with individuals. As we have seen, the issues are kept separated by French and Krause: while indiscernibility is understood in terms of sharing every property of the relevant kind, individuality is understood as possession of a haecceity, which is formally represented by self-identity. Accordingly, the solution to the two problems must come in two stages: there must be a new relation of indiscernibility and a dropping of identity. Putting both together, we have indiscernible non-individuals.

We have argued that the alleged shortcomings of classical logic are not so dramatic. In fact, first and second-order languages may be provided with unintended interpretations of identity which may do the job of failing identity and play the role of an indiscernibility relation. In these cases, at least, classical logic needs not to be revised, it seems. It could, but it needs not. As a byproduct of our discussion about these systems, we have seen that even non-reflexive Schrödinger logics will have unintended interpretations of identity, so that the lack of identity may be understood in distinct ways even in those systems.

In the most dramatic case of set theory, as we have seen, some important issues must be considered. The case of pure set theory, ZFC without atoms, is certainly one where the critique by French and Krause is on the right tracks, but, we argued, it misfires. In fact, it somehow misses the target because such a system is not to be understood as representing physical systems at face value. There is a representation of physical systems inside the theory, which deals only with abstract entities, the pure sets. So, we should already expect that pure set theory is a representational device, which certainly introduces some of its idiosyncrasies into the representation. On the other hand, a set theory employing atoms, ZFU, does not suffer from such problems: it is possible to find inner models of ZFU accounting for both the lack of identity required by Schrödinger as well as for the indiscernibility required by Post. In this case too, it seems, one needs not revise logic in order to account for such features.

What this discussion has allowed us to clarify is the urgent need to address the metaphysical status of non-individuality. In fact, it seems clear that both classical logic as well as non-reflexive logics may deal with entities indiscernible and for which identity somehow fail. The underlying system of logic, in this case, is not responsible for the metaphysical nature of the entities involved. What is required is that we first advance more on the metaphysical side, and, in cooperation with quantum mechanics, determine more precisely the most adequate characterization of quantum non-individuals and their difference from typical individuals, if there are any. This characterization is still to be done, in large measure.

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