K3, Ł3, LP, RM3, A3, FDE, M:
How to Make Many-Valued Logics Work for You

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Abstract

We investigate some well-known (and a few not-so-well-known) many-valued logics that have a small number (3 or 4) of truth values. For some of them we complain that they do not have any *logical* use (despite their perhaps having some intuitive semantic interest) and we look at ways to add features so as to make them useful, while retaining their intuitive appeal. At the end, we show some surprising results in the system FDE, and its relationships with features of other logics. We close with some new examples of "synonymous logics." An Appendix contains a Fitch-style natural deduction system for our augmented FDE, and proofs of soundness and completeness.

1 Introduction

This article is about some many-valued logics that have a relatively small number of truth values – either three or four. This topic has its modern (and Western) start with the work of (Post, 1921; Łukasiewicz, 1920; Łukasiewicz and Tarski, 1930), which slowly gained interest of other logicians and saw an increase in writings during the 1930s (Wajsberg, 1931; Bochvar, 1939, 1943; Słupecki, 1936, 1939a,b; Kleene, 1938). But it remained somewhat of a niche enterprise until the early 1950s, when two works brought the topic somewhat more into the "mainstream" of formal logic (Rosser and Turquette, 1952; Kleene, 1952). The general survey book, Rescher (1969), brought the topic of many-valued logics to the attention of a wider group within philosophy. Also in the 1960s there was an interest in free logics, where singular terms were not presumed to designate existing entities (Lambert, 1967; van Fraassen, 1966, 1969). The works of van Fraassen in particular promoted the notion of "supervaluations", which were seen (by some) as a way to retain two-valued logic while allowing for a third value of a "gap" – no truth value at all. More recently there was a spate of work ... still continuing ... that seems to have begun in the 1980s (da Costa, 1982; Urquhart, 1986; Rozoner, 1989; Avron, 1986, 1991; Arieli and Avron, 1998; Bolc and Borowick, 1992).

Our purpose is not to recount the development of this type of many-valued logic in its generality, but instead to focus on one specific group of these logics and show some surprising connections amongst them. Of course, in the totality of the writings on the various logics, many – perhaps even most (although we think not all) – of these connections have been noted. We will remark on some of these notices as we go. But we think these connections have not been put together in the same way as we do. Most of the work on these logics with small numbers of truth values have focused on the three-valued case, adding or comparing different connectives that different authors have proposed. In contrast, we will start with a four-valued logic and add one connective to it. And using that we will show how it might be applied to various three-valued logics. A further difference between our approach and many of the others in the area is that we mostly concentrate on the truth-valued semantic descriptions of the logics. Other approaches tend to emphasize boolean or de Morgan lattice-theoretic approaches for their semantics. Related to this last, these other approaches tend to develop a syntactic theory that supports their lattice-theoretic semantics by favouring sequent calculus formulations or sometimes (especially in older works) a Hilbert-style axiomatic syntax. Even ones that mention natural deduction are pretty much uniformly working within a Gentzen-style formulation, whereas we provide pedagogically more attractive Fitch-style natural deduction systems for all the logics under consideration (in an Appendix).

In particular, we try to show how one interesting four-valued logic can absorb one new connective and that this can lead us to see many of the previously-noted connections in a new light, namely, as the natural application of this connective to differing background logics. In this way we think of our development as "bottom up", starting with a particular logic and adding a specific connective. And that this can be extended in various ways to other specific logics. This is in contrast to works that proceed in a "top down" manner by considering whole classes of logics and considering all the possible connectives that might have some properties in common, and then remarking that some of these combinations have been employed by previous authors. It may well be that the conclusions reached by the two ways are identical, but we think ours might be found to be more intuitive and memorable, and easier for students (and non-logicians) to grasp, even if not so wide-reaching as what the top-down method might provide.

2 Truth-Values: Many, versus Gaps and Gluts

Everyone knows and loves the two "classical" truth-values, **T**rue and **F**alse. In "classical" logic[1], every sentence is either **T** or **F** but not both. Before getting to the detailed issues of this paper, we pause to make four comments of a "philosophical nature" with the intent of setting the issues aside. First, there is the long-standing issue of whether a logic contains sentences, statements, or propositions. We take no position on this issue: what we think of as the *logical* point is neutral on this issue. Related to this, perhaps, is the issue of whether a logic should be considered a set of its theorems or defined instead by the valid inferences it accepts. This topic *does* mark a distinction in the three-valued cases we consider, and we will remark on this as the occasion arises. Second,

[1]There are those (e.g., Priest, 2006) who claim that this common terminology of 'classical' betrays a bias and is in fact incorrect as an account of the history of thought about logic. We won't attempt to deal with this, and will just use the common terminology. For a more comprehensive review of topics and issues relevant to truth values see Shramko and Wansing (2017)

there are logics – perhaps not "classical" ones – that would prefer to claim the objects of interest are (e.g.) obligations, or commands, or questions, or ethical/epistemic items, rather than **T** and **F**. (E.g., Rescher, 1966; Adler, 1980; Searle and Vanderveken, 1985; Vanderveken, 1990; Hale, 1993; Wiśniewski, 1995; Vranas, 2008. See Schroeder (2008) for a discussion of the logical issues concerning non-truth-valued arguments.) Perhaps the items that have such values are not to be considered 'sentences' but rather exhortations or promises or. . . ? If so, then we say that we are restricting our attention to (declarative) sentences. Third, while 'sentence' seems appropriate for propositional logic, some further accommodation is required for formulas of predicate logic. The present paper is mostly concerned with propositional logic, and so we will simply use 'sentence'. And finally, there is the philosophical issue of whether **T** and **F** (and any other truth-values) should be considered to be *objects*– perhaps existing in a "third realm" – or as *properties* of sentences. Our view is that this distinction – which is maybe of considerable philosophical interest – does not affect any of the claims we will be making, and so we just will pass over it in silence.

So, given that **T** and **F** are so revered, where does this leave many-valued logic? One intuition that has been held by very many theorists is that, despite the fact that **T** and **F** are clearly excellent truth-values, there simply are more than just the **T** and **F** truth values. Some of these theorists hold that various phenomena show that there is another category beyond **T** and **F**: perhaps we should call it **I** for **I**ndefinite. Presupposition failures in a formula might determine that it is **I**; vagueness might determine that a formula is **I**; the unactualized (as yet) future might determine that some formulas are **I**; fictional discourse seems neither straightforwardly true or false, so perhaps sentences about fictional objects also should designate a third truth-value, **I**. And there no doubt are other types of formulas that could intuitively be thought to be "indeterminate".

(1) a. The present Queen of Germany is happy.
    b. That 45-year-old man is old.
    c. There will be a Presidential impeachment in the next 30 years.
    d. James Tiberius Kirk is captain of the starship *Enterprise*.

Some theorists, agreeing with the just-expressed intuition that the sentences in (1) are neither **T** nor **F**, nonetheless wish to deny that they designate a third value. Rather, they say, such sentences *lack* a truth-value (i.e., are *neither* **T** nor **F**; nor are they **I**, since they deny such a value). Such sentences *have no truth-value*: they express a truth-value *gap*. And logics that have this understanding are usually called "gap logics."

Formally speaking, there is little difference between the two attitudes of expressing **I** or being a gap (in a simple three-valued logic where there is just one non-classical truth value). In both cases one wishes to know how to evaluate complex sentences that contain one of these types of sentences as a part. That is, we need a way to express the truth-table-like properties of this third option. In the remainder of this paper we will use **N** for this attitude, whether it is viewed as expressing **I** or as being a gap. (The **N** can be viewed as expressing the notion "neither **T** nor **F**", which is neutral between expressing **I** or expressing a gap.) And we will call the logics generated with either one of these understandings of **N**, "gap logics."

Gap logics – whether taken to allow for truth values other than **T** and **F**, or instead to allow for some sentences not to have a truth value – are only one way to look at some recalcitrant natural language cases. Other theorists point to some different phenomena in their justification of a third truth value. The semantic paradoxes, such as the Liar Paradox (where the sentence "This sentence is false" is false if it is true, and is true if it is false) suggest that some statements should be treated as *both* true and false. It is difficult to deal with this paradox in a gap logic, because saying that "This sentence is false" is neither true nor false leads immediately to the strengthened Liar Paradox: "Either this sentence is false or else it is neither true nor false". Here it seems that claiming that the sentence expresses a gap leads to truth, while saying that it is true leads to falsity and saying that it is false leads to truth. In contrast, to say that the sentence "This sentence is false" is *both* true and false does not lead to this regress: The sentence "This sentence is either false or else is both true and false" leads only to the conclusion that it is both true and false. This *dialetheic* theory – that some sentences are both true and false – is called a "glut theory" because of the presence of both truth values characterizing some sentences. The intermediate value, on this conception, can be called **B** (for "both"), and we will use this. Again, there is perhaps a distinction between some sentence's having a third value **B**, or of partaking of both truth and falsity. Again we remain neutral and employ **B** for this intermediate possibility.

There are also examples from naïve set theory such as the Paradox of Well-Founded sets, the Paradox of a Universal Set, Russell's Paradox, and Richard's Paradox (see Cantini (2014) for a historical overview of these and others). A desire on the part of some theorists in the philosophy of mathematics is to reinstate naïve set theory and use it in the development of mathematics, instead of (for example) Zermelo-Frankel set theory. Such theorists perhaps will feel encouraged in their belief that the glut-style of resolution of the Liar Paradox might be usable in the case of these other set theoretic paradoxes, and that naïve set theory can once again be used.

Various theorists have pointed out that even vagueness needn't be viewed as *missing* a truth-value; it is equally plausible to think of vagueness as manifesting *both* the positive character and its absence. A middle-height person can be seen as *both* short and tall, equally as plausibly as being *neither* short nor tall. (See Hyde (1997); Beall and Colyvan (2001); Hyde and Colyvan (2008).) As well, certain psychological studies seem to indicate that the "dialetheic answer" is more common in general conversation than the "gap" answer, at least in a wide variety of cases (see Alxatib and Pelletier (2011); Ripley (2011) for studies that examine people's answers to such cases).

There are also other categories of examples that have been discussed in philosophy over the ages: Can God make a stone so heavy that He can't lift it? Perhaps the answer is, "Well, yes and no." There are cases of vague predicates such as *green* and *religion*, e.g., an object can be both green and not green, and a belief system such as Theravada Buddhism or Marxism can be both a religion and not a religion. There are legal systems that are inconsistent, in which an action is both legal and illegal. In the physical world, there is a point when a person is walking through a doorway at which the person is both in and not in the room. And so on. As a result, the glut theory has its share of advocates.

A way to bring the gap and glut views of the truth values under the same conceptual roof is to think of the method that assigns a truth value to formulas as a *relation* rather than a *function*. A function, by definition, assigns exactly one value to a formula – either **T** or **F** or, in the context of our 3-valued logics, it may also assign **B** or **N**. A relation, however, can assign more than one value: we may now think that there are exactly two truth values, **T** and **F**, but the assignment *relation* can assign subsets of $\{\mathbf{T}, \mathbf{F}\}$. A gap logic allows assignment of $\{\mathbf{T}\}$, $\{\mathbf{F}\}$, and $\emptyset$. A glut logic allows $\{\mathbf{T}\}$, $\{\mathbf{F}\}$, and $\{\mathbf{T}, \mathbf{F}\}$.

But both the gap and the glut logics seem to have difficulties of a logical nature that we will examine in §5.

3 K3 and LP: The Basic 3-Valued Logics

In all of the 3-valued logics we consider, the 3-valued matrices for $\wedge, \vee, \neg$ are the same, except for what the "third value" is called: in K3 (Kleene, 1938; Kleene, 1952, §64) we call it **N** but in LP (Asenjo, 1966; Asenjo and Tamburino, 1975; Priest, 1979) we call it **B**.

One way to understand why the K3 truth values for these connectives are correctly given in the following tables is to think of **N** as meaning "has one of the values **T** or **F**, but I don't know which one". In this understanding, a negation of **N** would also have to have the value **N**; a disjunction with one disjunct valued **N** and the other **T** would as a whole be **T**... but if one were valued **N** and the other **F**, then as a whole it would have to be **N**. Similar considerations will show that the $\wedge$ truth table is also in accord with this understanding of the **N** value. (This interpretation of the truth values presumes that one pays no attention to the *interaction* of the connectives: $(p \wedge \neg p)$ might seem to be false and $(p \vee \neg p)$ seem true, but that's because of the interaction of two connectives and their arguments.)

On the other hand, if the middle value is understood as being *both* true and false as in LP, we will still get those same truth tables, with only a change of letter from **N** to **B**: negating something that is both true and false will yield something that is both false and true. If a disjunction had one disjunct valued **B** and the other **T**, then the entire disjunction would be valued **T**. But if that second disjunct were valued **F** instead, then the entire disjunction would be valued **B**. Similar considerations will show that the $\wedge$ truth table is also in accord with this understanding of the **B** value.

$\wedge$	**T**	**N/B**	**F**
T	**T**	**N/B**	**F**
N/B	**N/B**	**N/B**	**F**
F	**F**	**F**	**F**

$\vee$	**T**	**N/B**	**F**
T	**T**	**T**	**T**
N/B	**T**	**N/B**	**N/B**
F	**T**	**N/B**	**F**

$\neg$	
T	**F**
N/B	**N/B**
F	**T**

Despite the apparent identity of K3 and LP (other than the mere change of names of **N** and **B**), the two logics are in fact different by virtue of their differing accounts of what semantic values are to be considered as privileged. That is, which values are to be the *designated values* for the logic – those values that are semantically said to be the ones that the logic is concerned to manifest in a positive light. In both logics this positivity results from the value(s) that exhibit truth, or at least some degree of it: but in K3 only **T** has that property whereas in LP both **T** and **B** manifest some degree of truth. And therefore LP treats the semantically privileged values – the *designated values* – to be both **T** and **B**, while in K3 the only designated value is **T**.

Although both K3 and LP have no formula whose value is always **T**, LP (unlike K3) does have formulas whose semantic value is always designated. In fact, the class of propositional LP formulas that are designated is identical to that of propositional classical 2-valued logic, a fact to which we return in §5.

Note that these truth tables give the classical values for compound formulas with only classically-valued components, and give the intermediate value for formulas all of whose components have that value. Thus neither logic is functionally complete, that is, neither logic can describe all the possible relations that their semantics allows to hold among their truth-values. For, no truth function giving a non-classical (classical) value for uniformly classical (non-classical) arguments can be represented by them.

4 FDE: A Four-Valued Logic

The logic FDE was described in (Belnap, 1992) and (Dunn, 1976). It is a four-valued system: the values are **T**, **F**, **B**, and **N**... the four values we have already encountered (although the intuitive semantics behind these values, as given by Belnap, are perhaps somewhat different than we encountered above). K3 and LP agree with each other (and with classical logic) as regards the values **T** and **F**, so in combining them as a way to form FDE, it is natural to identify FDE's **T** and **F** with those of K3 and LP. FDE then agrees with K3 about **N**: agrees that it is not a designated value, and agrees on the values of combinations of **N** with **T** and **F**. FDE also agrees with LP about **B**: agrees that it is a designated value, and agrees on the values of combinations of **B** with **T** and **F**. Neither of the three-valued logics allow combinations of **B** with **N**, of course, so these combinations need a bit more thought. The choice made is perhaps most easily described by using the "relational" account of truth assignment, where the four values are *sets* of the two classical values: **T**={T}, **B**={T,F}, **N**={ }, **F**={F} (where of course the bold-face values stand for FDE values and the normal text values stand for classical ones!). A given classical value then goes into the set of values of a compound formula just in case the component formulas have values containing classical values which, by the classical rules, would yield the given value for the compound. (This way of thinking also makes the designation status of **B** and **N** seem natural: an FDE value is designated just in case it, thought of as a set, contains the classical value T.)

This treatment of the four values and their ordering is perhaps best visualized by the diagram:

$$
\begin{tikzcd}
 & \mathbf{T} \arrow[dl] \arrow[dr] & \\
\mathbf{B} \arrow[dr] & & \mathbf{N} \arrow[dl] \\
 & \mathbf{F} &
\end{tikzcd}
$$

The four values form a lattice, with the value of a conjunction (disjunction) being the meet (join) of the values of its conjuncts (disjuncts), and with negation interpreted as inverting the lattice order.

Pictorially, a join operation of two elements finds the least upper bound of the ordering that contains both values while a meet operation finds their greatest lower bound. In the diagram, the only novel features of these operations occurs when the values **B** and **N** are joined or meeted. Their join is **T** while their meet is **F**.

More formally, we take the basic truth-tables for $\wedge$, $\vee$ and $\neg$ to be straightforwardly taken from K3 and LP, for the values that do not involve only **B** and **N** together. For those values we interpolate, using meet and join, as given in the truth tables in Table 1 for the FDE connectives.

φ	ψ	$\neg\varphi$	$(\varphi \wedge \psi)$	$(\varphi \vee \psi)$
T	**T**	**F**	**T**	**T**
T	**B**		**B**	**T**
T	**N**		**N**	**T**
T	**F**		**F**	**T**
B	**T**	**B**	**B**	**T**
B	**B**		**B**	**B**
B	**N**		**F**	**T**
B	**F**		**F**	**B**
N	**T**	**N**	**N**	**T**
N	**B**		**F**	**T**
N	**N**		**N**	**N**
N	**F**		**F**	**N**
F	**T**	**T**	**F**	**T**
F	**B**		**F**	**B**
F	**N**		**F**	**N**
F	**F**		**F**	**F**

Table 1: Four-valued truth-tables for the basic connectives of FDE

Most, if not all, of the concerns mentioned in §3 as motivating interest in LP or K3 can be elaborated in ways to motivate the use of FDE. A number of authors have argued specifically for the appropriateness of FDE for some applications: for example, Belnap 1992, which first appeared as two articles in 1977, has argued that FDE is appropriately used in connection with some kinds of data base (or better: knowledge base) systems, where different sources of data entry might contain contradictory information, and where some information is missing but the knowledge base is required nevertheless to be able to reason about such cases; and Camp (2002) has argued that it might be useful in argumentation where the references of our interlocutors' (or our own!) terms are indeterminate, and for that reason can generate statements that are **N** and others that are **B**. Arieli and Avron (2017) refer to a number of other areas, such as knowledge-base integration, fuzzy logic, relevance logics, self-reference, and preferential modelling.

5 What's Wrong with K3, LP, and FDE

We start with K3 and LP. Note that the $\wedge, \vee, \neg$ truth-functions always yield a **N** (or a **B**) whenever all the input values are **N** (or **B**, respectively). This means that there are no formulas that always take the value **T**. In these logics, those three truth functions are all of the primitive functions, and so that same fact also implies that not every truth function is definable in K3 (or LP). And if φ's being semantically valid means that φ always takes the value **T** for all input values, as it does in K3, then there are no semantically valid formulas in K3.

And given that these are all the primitive truth functions, then the only plausible candidate for being a conditional in these logics comes by way of the classical definition, $(\varphi \supset \psi) =_{df} (\neg\varphi \vee \psi)$, whose truth tables in K3 and LP are given in Table 2:

φ	ψ	$(\varphi \supset_{K3} \psi)$	$(\varphi \supset_{LP} \psi)$
T	**T**	**T**	**T**
T	**N/B**	**N**	**B**
T	**F**	**F**	**F**
N/B	**T**	**T**	**T**
N/B	**N/B**	**N**	**B**
N/B	**F**	**N**	**B**
F	**T**	**T**	**T**
F	**N/B**	**T**	**T**
F	**F**	**T**	**T**

Table 2: Conditionals available in K3 and LP, using the classical definition

The only difference between $\supset_{K3}$ and $\supset_{LP}$ is that in K3 the "third value" is understood as indicating a truth-value gap, i.e., as *neither* **T** nor **F**, whereas whereas in LP it is understood as indicating a truth-value glut, i.e., as *both* **T** and **F**.

One notes that the rule of inference, Modus Ponens (MP), $(\varphi \supset_{K3} \psi), \varphi \vDash \psi$, is a valid rule in K3: if both premises are **T**, then the conclusion will also be, as can be seen from the truth table for $\supset_{K3}$ in Table 2. Despite this, the *statement* of MP, $((\varphi \supset_{K3} \psi) \wedge \varphi) \supset_{K3} \psi$, does *not* always exhibit the value **T** (since no formula of K3 always takes the value **T**). Thus the deduction theorem [if $\Gamma, \varphi \vDash \psi$ then $\Gamma \vDash (\varphi \supset_{K3} \psi)$] does not hold for K3. Matters are different for the logic LP: the rule of MP is *invalid*, as can be seen by making φ be **B** and ψ be **F**. In that case both $(\varphi \supset_{LP} \psi)$ and φ have designated values (both are **B**), but yet the conclusion, ψ, is **F**. On the other hand, the *statement* of MP, $((\varphi \supset_{LP} \psi) \wedge \varphi) \supset_{LP} \psi$, is always designated (either **T** or **B**). (One can see that the counterexample to the *rule* MP makes both conjuncts of the antecedent be **B** and hence that antecedent is **B**. But a **B** antecedent with a **F** consequent is evaluated **B**. Hence the statement is designated.)

Not having a conditional that enforces MP (not having what is often called "a detachable conditional") is a serious drawback. It means that the language cannot state what inferences it takes to be designated, or at least, if it does claim some conditional to be designated then it cannot apply MP to that conditional, even when the antecedent is designated. This has the consequence that one cannot state what inferences can be chained together, that is, the formula $(((\varphi \supset \psi) \wedge (\psi \supset \theta)) \supset (\varphi \supset \theta))$ is not designated in K3 (suppose all its parts are **N**); and the argument $((\varphi \supset \psi) \wedge (\psi \supset \theta)) \vDash (\varphi \supset \theta)$ is invalid in LP (let $[\![\varphi]\!] = \mathbf{T}$, $[\![\psi]\!] = \mathbf{B}$, $[\![\theta]\!] = \mathbf{F}$). In essence, these sorts of shortcomings mean that no reasoning can be carried out *within* such logics – although perhaps some reasoning *about* the logics can be carried out.[2]

As we remarked above, the theorems of LP are identical to those of classical logic – LP does nothing more than divide up the classical notion of truth into two parts: the "true-only" and the "true-and-also-false". As both types of truth are designated values in the logic, the theorems of the two are the same, and hence LP is not different from classical logic so far as logical truth goes. Where it does differ is in the class of valid rules of inference, as we have noted. A different description applies to K3: whereas the deduction theorem holds and modus ponens fails for LP, in K3 the deduction theorem fails but modus ponens holds.

So, it seems that both K3 and LP are not very useful as guides to reasoning, despite their apparent (to some) virtues in accounting for intuitive semantic values of sentences that exhibit some troublesome features (such as vagueness or semantic paradox). After discussing the logical state of K3 and LP, and the disappointing properties of the conditional operators available in them, Soloman Feferman put it:

> Multiplying such examples, I conclude that *nothing like sustained ordinary reasoning can be carried on in either logic.* (Feferman, 1984, p. 264, italics in original.)

On the topic of non-classical logics more generally, van Fraassen (1969) expresses the worry in more picturesque terminology. New logics, he worries, along with "the appearance of wonderful new 'logical' connectives, and of rules of 'deduction' resembling the prescriptions to be read in *The Key of Solomon*" will make "standard logic texts read like witches' grimoires".

Turning now to FDE, it is clearly a merger of K3 and LP, so shares all their shortcomings. Since FDE includes the feature of K3 that there are no truth functions from all-atomics-having-the-value-**N** to any other truth value, it follows that there are no formulas that always take one or the other of the designated values **T** or **B**. And similarly, FDE is functionally incomplete since there is no formula that will yield one of **T**, **B**, or **F** if all its atomic letters are assigned **N**. (A similar situation holds holds for LP and FDE: Although LP does have formulas that always take designated values, there is no truth function in FDE that can generate **T**, **N**, or **F**, when all the atomic letters are assigned **B**.) Since these logics fail to be functionally complete, there are features of the semantic model that cannot be expressed in the language – such as that there are possible functions that take sets of **N**-valued (or **B**-valued) atomics to classically-valued formulas. Surely a logic has a weakness if it can't express what its semantic metatheory says is a feature of its domain. And so far, all the logics we have discussed contain this weakness. Like K3 and LP, FDE does not have

[2]For example, perhaps the pair of inferences in one or another of these logics is that $\varphi \vDash \psi$ is semantically valid, and so is $\psi \vDash \theta$. And therefore the inference $\varphi \vDash \theta$ must also be valid. But note that this would be a judgement made *about* the logic and not *within* the logic. The logic itself can't be used in this or any equivalent way.

a usable conditional and hence there is no sense in which it is a logic that one can reason with. Although Feferman (1984) didn't include FDE in his disparaging remarks we just quoted, since FDE is simply a "gluing together " of K3 and LP – the two logics Feferman did complain about – it is clear that he would have held the same opinion about FDE.

We wish to show that matters are not so dire as Feferman seems to think for the FDE-related logics we are investigating. And so we now turn to possible ways to fix these shortcomings. We start with the topic of adding an appropriate conditional; later we consider the issue of functional completeness.

6 An FDE Fix?

Various researchers have added conditionals to K3 and LP, with the thought of being able to do "sustained reasoning" in these logics. We might mention here Łukaseiewicz's conditional added to K3[3], with the intent of accommodating "future contingents." This yields a logic that is usually called Ł3. And conditionals can be added to LP with the intent of allowing reasoning to take place in the context of dialetheism. For this we will look here at the more closely at the logics RM3 and a logic we will call $LP^{\rightarrow}$ that are generated by adding different conditionals to LP.[4]

But rather than starting with these 3-valued logics and try to extend one or another of the conditionals to the 4-valued FDE, our strategy here will be to start with the general case of FDE, propose our conditional $\rightarrow$ for it, which generates the logic $FDE^{\rightarrow}$, and to look at how that conditional behaves when applied to the K3 and LP subparts of FDE. We find many interesting and unexpected connections with previous work.[5]

Here is the conditional we propose: we call it the "classical material implication" and we symbolize it $\rightarrow$.[6]
This conditional has been suggested before – indeed, (Omori and Wansing, 2017, p. 1036) say it is "the most well-known, as well as well-studied, implication". Be that as it may, we propose to further investigate the resulting $FDE^{\rightarrow}$ logic, as well as some further possible additions to it. But prior to that we will look at extending this logic by restricting the possible truth-values to three.

In §4 we remarked that the natural understanding of FDE's semantic values is that the designated ones are **T** and **B**, while the undesignated ones are **N** and **F**. As can be seen in Table 3, any conditional statement that takes a designated value has the feature that if its antecedent is also designated, then its consequent will be. It can also be seen that if the values are restricted to the

[3]Although of course this is not the historical reason he came up with his conditional, since he didn't have K3 before him at the time.

[4]Tedder (2015) uses the name A3, in honour of Avron (1991), for what we call $LP^{\rightarrow}$. Since we are comparing a number of logics, we prefer our more systematic nomenclature of adding the relevant conditional as a superscript.

[5]For a survey of work done on describing and enhancing FDE, including various possibilities of adding conditionals, see (Omori and Wansing, 2017). This is also an overview of the work on FDE that is reported in this special issue of *Studia Logica*, titled "40 Years of FDE".

[6](Omori and Wansing, 2017, p. 10) call it an extensional operator, saying it can be "seen as a material implication defined in terms of exclusion negation and disjunction".

$\rightarrow$	T	B	N	F
T	T	B	N	F
B	T	B	N	F
N	T	T	T	T
F	T	T	T	T

$$\llbracket \varphi \rightarrow \psi \rrbracket = \llbracket \psi \rrbracket, \text{ if } \llbracket \varphi \rrbracket \text{ is designated}$$
$$= \mathbf{T}, \text{ otherwise}$$

Table 3: Truth matrix for our conditional, $\rightarrow$.

classical **T** and **F**, then the conditional mirrors the classical $\supset$. We also obtain the truth table for the classical $\supset$ if we look at this with blurred vision, so that the two designated values blur into one and the two undesignated values similarly coalesce, showing that the valid formulas (i.e., those taking a designated value on every assignment of truth values to their atoms) and inferences (i.e., those preserving designation on every assignment) are precisely those of the pure $\supset$ fragment of classical logic. Combined with a similarly blurred view of the truth tables for $\wedge$ and $\vee$ we can extend this observation to formulas having these connectives as well as $\rightarrow$, and we can extend it further to quantifiers if we think of them as generalized conjunctions and disjunctions. Thus:

Proposition 1. *For* positive *formulas (i.e., those not containing $\neg$), the valid formulas and inferences of FDE$^{\rightarrow}$ are exactly those of classical logic.*

The 4-valued logic FDE$^{\rightarrow}$ has two obvious 3-valued extensions, defined semantically by restricting the set of truth values allowed: K3$^{\rightarrow}$ defined by reference to {**T**,**N**,**F**} and LP$^{\rightarrow}$ defined by reference to {**T**,**B**,**F**}. These logics result from the addition of a conditional connective, $\rightarrow$, defined by the relevant rows/columns of Table 3, to the conditional-free logics K3 and LP.

6.1 FDE$^{\rightarrow}$

The classical nature of the conditional in FDE$^{\rightarrow}$ makes it a much more pleasant item to work with than most previously proposed conditionals in many-valued logics: something like sustained ordinary reasoning *can* be carried out in FDE$^{\rightarrow}$and in its 3-valued extensions.[7] However, although the positive logic is completely classical, there are some surprises in the interaction of $\rightarrow$ with negation. One unpleasant one is that the principle of Contraposition fails:

$$\nvDash (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$$

if $\llbracket \varphi \rrbracket = \mathbf{T}$ and $\llbracket \psi \rrbracket = \mathbf{B}$, then $\llbracket \varphi \rightarrow \psi \rrbracket$ takes the designated value **B**, but $\llbracket \neg\psi \rightarrow \neg\varphi \rrbracket$ is **F**. And also, if $\llbracket \varphi \rrbracket = \mathbf{N}$ and $\llbracket \psi \rrbracket = \mathbf{F}$, then $\llbracket \varphi \rightarrow \psi \rrbracket$=**T**, but $\llbracket \neg\psi \rightarrow \neg\varphi \rrbracket$= **F**. Since a counterexample can

[7](Omori and Wansing, 2017, p.1036) mention three other conditionals that each have their own interesting properties, although these properties detract from the classical nature of these conditionals. Sutcliffe et al. (2018) give a different four-valued conditional using some "intuitive" considerations of "meaning", and employ an automated theorem proving system to contrast the different conclusions that can be drawn from each.

be obtained using either of the nonclassical values, Contraposition is invalid not only in $\mathrm{FDE}^{\rightarrow}$ but also in both of its 3-valued extensions $\mathrm{K3}^{\rightarrow}$ and $\mathrm{LP}^{\rightarrow}$.

We can, of course, define a new conditional connective for which Contraposition holds:

$$(\varphi \Rightarrow \psi) =_{df} ((\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi))$$

This is a useful connective![8] We will make use of the corresponding biconditional below, in §7.

$$(\varphi \Leftrightarrow \psi) =_{df} ((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$$

takes a designated value if and only if $[\![\varphi]\!] = [\![\psi]\!]$. (Note that $[\![(\varphi \Leftrightarrow \psi)]\!]$=**B** if $[\![\varphi]\!] = [\![\psi]\!] =$ **B**, and =**T** if $[\![\varphi]\!]$ and $[\![\psi]\!]$ have the same one of the other three values.) As a result, it supports a principle of Substitution: formulas of the form

$$(\varphi \Leftrightarrow \psi) \rightarrow ((\cdots\varphi\cdots) \Leftrightarrow (\cdots\psi\cdots))$$

are valid. In contrast, the biconditional similarly defined in terms of our basic conditional yields a designated value just in case either both of its terms have designated values or both have undesignated values. As a result, it does not license substitution: If φ and ψ have *different* designated values, and also if they have different undesignated values, $(\varphi \leftrightarrow \psi)$ will have a designated value, but $(\neg\varphi \leftrightarrow \neg\psi)$ will not. However, $\leftrightarrow$ can be added to the list of "positive" connectives, $\wedge, \vee, \rightarrow$, whose logic is exactly classical. We will return to the usefulness of $\Rightarrow$ for FDE in §7.

On the other hand, the $\Rightarrow$ connective has certain undesirable features which militate against its adoption as the basic conditional operator of a logic designed for use. Principles analogous to some of the *structural* rules of (Gentzen, 1934), easily derivable by the conventional natural deduction rules for the conditional, fail for it. Some fail for $\Rightarrow$ in $\mathrm{K3}^{\rightarrow}$, and others fail for $\Rightarrow$ in $\mathrm{LP}^{\rightarrow}$, as we shall see.

6.2 On the $\mathrm{K3}^{\rightarrow}$ Side

The truth tables for $\rightarrow$ and $\Rightarrow$ differ on only one of the nine lines for $\mathrm{K3}^{\rightarrow}$, as indicated in Table 4: if $[\![\varphi]\!]$=**N** and $[\![\psi]\!]$=**F**, then $[\![\varphi \rightarrow \psi]\!]$=**T** but $[\![\varphi \Rightarrow \psi]\!]$=**N**.

This is enough, however, to invalidate the principle of Contraction for $\Rightarrow$:

$$\nvDash (\varphi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \psi)$$

This should be old news! The derived truth table for the $\Rightarrow$ of $\mathrm{K3}^{\rightarrow}$ in Table 4 is exactly that of the conditional of Łukasiewicz's 3-valued logic, for which Contraction failure is familiar. Since the interpretations of the other connectives are the same for $\mathrm{K3}^{\rightarrow}$and Ł3, we have

[8](Omori and Wansing, 2017, p.1036) cite this conditional as one of their four "interesting" conditionals that could be added to FDE.

φ	ψ	$(\varphi \to \psi)$	$(\varphi \Rightarrow \psi)$	
T	**T**	**T**	**T**	
T	**N**	**N**	**N**	
T	**F**	**F**	**F**	
N	**T**	**T**	**T**	
N	**N**	**T**	**T**	
N	**F**	**T**	**N**	$\Leftarrow$
F	**T**	**T**	**T**	
F	**N**	**T**	**T**	
F	**F**	**T**	**T**	

Table 4: Comparison of two conditionals in $K3^{\to}$

Proposition 2. *Łukasiewicz's 3-valued logic Ł3 is faithfully interpretable in* $K3^{\to}$.

In fact, we have the converse as well[9]: our $\to$ can be defined in Ł3 by

$$(\varphi \to \psi) =_{df} (\varphi \Rightarrow (\varphi \Rightarrow \psi)),$$

and hence

Proposition 3. $K3^{\to}$ *is faithfully interpretable in Ł3.*

$K3^{\to}$ and Ł3 can thus be seen as, in effect, alternative formulations of a single logic (see further discussion in Section 8).[10] We think the classical nature of $\to$ (other than some of its interaction with $\neg$, such as the failure of contraposition) makes it easier to use than $\Rightarrow$. It also has a straightforward natural deduction formulation (see Appendix I), and we recommend translation into $K3^{\to}$ to anyone interested in actually proving theorems in Ł3.

6.3 On the $LP^{\to}$ Side

Now looking at the logic with truth values {**T**,**B**,**F**}, we see in Table 5 that $\Rightarrow$ again differs from $\to$ on only one of the nine lines: but this time it is the line where $[\![\varphi]\!]$=**T** and $[\![\psi]\!]$=**B**. Here $[\![\varphi \to \psi]\!]$=**B** but $[\![\varphi \Rightarrow \psi]\!]$=**F**.
This difference in truth table is enough to invalidate the principle of Thinning:

[9](Nelson, 1959) should be credited with the observations of the failure of contraposition, its brute force restoration, and contraction failure in his system of constructible falsity... which is essentially the addition of intuitionistic implication to K3. What our observation adds to this is that these phenomena do not depend on the intuitionistic nature of Nelson's implication, but arise already in the 3-valued and 4-valued logics. It might also be noted, as pointed out in Kamide 2017, p. 1168 fn1, that viewing Nelson's work as a three- four-valued logic, or as a paraconsistent logic, is rather a later imposition on his actual axiomatic system. It was not introduced with a many-valued semantics at all.

[10]Mutual faithful interpretability is not in general sufficient to establish identity of logics, but the particulars of our cases will add on what further is required, as we discuss in §8.

φ	ψ	$(\varphi \to \psi)$	$(\varphi \Rightarrow \psi)$	
T	**T**	**T**	**T**	
T	**B**	**B**	**F**	$\Leftarrow$
T	**F**	**F**	**F**	
B	**T**	**T**	**T**	
B	**B**	**B**	**B**	
B	**F**	**F**	**F**	
F	**T**	**T**	**T**	
F	**B**	**T**	**T**	
F	**F**	**T**	**T**	

Table 5: Comparison of two conditionals in LP

$\nvDash \varphi \Rightarrow (\psi \Rightarrow \varphi)$

For, if $[\![\varphi]\!] = \mathbf{B}$ and $[\![\psi]\!] = \mathbf{T}$, then $[\![\psi \Rightarrow \varphi]\!]$ and hence $[\![\varphi \Rightarrow (\psi \Rightarrow \varphi)]\!]$ will be **F**. As Tedder (2015) notes[11], $\Rightarrow$ in $\text{LP}^{\to}$ has exactly the truth table of the logic RM3 (and, since RM3 is a cousin of the relevance family of logics, failure of Thinning in it is just what one would expect). RM3 and LP agree on $\{\wedge, \vee, \neg\}$, and so – as in the analogous case on the K3 side – we have

Proposition 4. *RM3 can be faithfully interpreted in* $LP^{\to}$*.*

Again, the converse is also true: $\to$ can be defined in terms of $\Rightarrow$ in RM3 by

$$(\varphi \to \psi) =_{df} ((\varphi \Rightarrow \psi) \vee \psi)$$

Hence

Proposition 5. $LP^{\to}$ *can be faithfully interpreted in RM3.*

So in the same sense as with Ł3 and $\text{K3}^{\to}$, RM3 and $\text{LP}^{\to}$ can be thought of as alternative formulations of a single logic (again, see discussion in Section 8), and again, we think adoption of $\to$ as primitive is likely to be more convenient. For, although the metamathematical study of RM3 has been fruitful in the study of relevance logics, it seems to us that $\text{LP}^{\to}$ is likely to be a more convenient to *use* in the task of actually trying to prove theorems of RM3 – particularly in light of Tedder's 2015 employment of it in formulating mathematically interesting axiomatic theories.

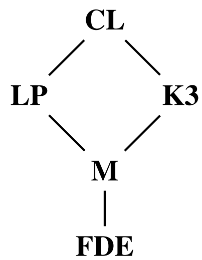
6.4 Putting the Two Sides Together

6.4.1 About $\text{M}^{\to}$

Given that K3 and LP are obtained semantically from FDE by adding different conditions on valuations, and syntactically by adding different rules, one might incautiously conjecture that the FDE

[11] Citing Avron (1991).

consequence relation is simply the intersection of the K3 and LP relations. Not so: $(\varphi \wedge \neg\varphi)$ implies $(\psi \vee \neg\psi)$ in both 3-valued logics, but not in FDE.[12] There is thus a fifth logic in the neighbourhood, which is sometimes[13] called *Mingle* and which we will unimaginatively call M.



It is characterized semantically as the set of inferences preserving designation on every FDE valuation which does not assign **B** to one formula and **N** to another, and syntactically by adding the above-mentioned inference as a general rule to a formulation of FDE. Since $M^{\rightarrow}$ contains $\rightarrow$, it of course has a deduction theorem: if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash (\varphi \rightarrow \psi)$. The structural rules of thinning and contraction fail for $\Rightarrow$: counterexamples to thinning need only gluts (it fails in RM3), and counterexamples to contraction require only gaps (it fails in Ł3). So both fail in $M^{\rightarrow}$ – but not in the same model! We note in passing that this means that the following formula is a theorem:

$$(C \Rightarrow (D \Rightarrow C)) \vee ((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$$

Note that this is a provable disjunction with no variable in common between the two disjuncts, yet neither disjunct by itself is provable![14] Some might think this shows that $M^{\rightarrow}$ is silly and unreasonable.

We too think M seems like a rather silly logic (why should the presence of a single truth-value glut rule out the existence of any truth-value gaps, or vice versa?[15]), but record here that $M^{\rightarrow}$ can be formulated in the obvious way, and that our completeness proof in Appendix II for $FDE^{\rightarrow}$, using the natural deduction system of Appendix I, extends without difficulty to cover it.

Proposition 6. *$M^{\rightarrow}$ and $M^{\Rightarrow}$ are mutually faithfully interpretable in each other*

[12]In K3, $(\varphi \wedge \neg\varphi)$ is never designated, so $(\varphi \wedge \neg\varphi) \vDash (\psi \vee \neg\psi)$ will (vacuously) hold; in LP, $(\psi \vee \neg\psi)$ is always designated, so $(\varphi \wedge \neg\varphi) \vDash (\psi \vee \neg\psi)$ always holds. However, in FDE, if $[\![\varphi]\!] = \mathbf{B}$ then $(\varphi \wedge \neg\varphi) = \mathbf{B}$ and hence is designated; but if $[\![\psi]\!] = \mathbf{N}$, then $[\![(\psi \vee \neg\psi)]\!] = \mathbf{N}$ and is undesignated. So $(\varphi \wedge \neg\varphi) \nvDash (\psi \vee \neg\psi)$.

[13]See (Humberstone, 2011, p. 334) for a discussion of Mingle in the context of relevant logic and why the present logic might also be named Mingle.

[14]The (very lengthy) proof in our Fitch-system of the Appendix, uses our "classicalizing" principle (in Appendix 1) that a conclusion follows *tout court* if it follows from ϕ and also from $\phi \rightarrow \psi$, for any instances of ϕ and ψ. The actual proof requires two further embeddings of different instances of the classicalizing rule. It is quite a tedious proof!

[15]Well, perhaps there is someone who is convinced that a three-valued logic is the right response to the paradoxes, but is not sure whether it should go the glut route or the gap route. But this person is sure that only one of these ways will be correct and that having both would be philosophically distasteful.

6.4.2 About FDE$^{\rightarrow}$

We have not explored the behaviour of $\Rightarrow$ in FDE$^{\rightarrow}$, and, in contrast to the situation with the 3-valued logics, do not know of a historically proposed equivalent for it.[16] Contraction and Thinning will, of course, both fail for $\Rightarrow$ in the 4-valued logic. With no relevant insight, we have resorted to construction of truth tables to verify that $\rightarrow$ can be recovered from $\Rightarrow$ in the 4-valued environment by the double-barrelled definition.

$$(\varphi \rightarrow \psi) =_{df} ((\varphi \Rightarrow (\varphi \Rightarrow \psi)) \vee \psi)$$

Proposition 7. *FDE$^{\rightarrow}$ and FDE$^{\Rightarrow}$ are mutually faithfully interpretable in each other*

Thus showing, as in the 3-valued extensions of FDE, that adding either of the 4-valued conditionals $\rightarrow$ or $\Rightarrow$ to basic FDE generates the identical logic as adding the other.

7 More and Less Drastic Expansions

7.1 Functional Completeness by Adding Constants and Parametric Operators

In classical two-valued logic, a standard way to prove functional completeness (i.e., that every two-valued truth function can be described by a given set of connectives. . . we will restrict our attention to the functions $\wedge, \vee, \neg$) is to show that any arbitrary truth table can be described by a formula using those connectives. Using these connectives, we can generate a formula in disjunctive normal form. In textbooks this is normally "proved" by providing an example that seems clearly to be generalizable, rather like this: Our example will have three atomic letters, *p, q, r*. There are two cases. Either the resulting truth table has at least one **T** value or it doesn't. In the latter case the formula is a contradiction and that truth table can be described by the formula $(p \wedge \neg p \wedge q \wedge r)$. Otherwise the example truth table will look something like Table 6. From this table we construct a formula as follows. For each row where the value is **T** – as for example in the first row – write a formula that "describes" the values of the three atomic arguments. If an atomic argument is **T** in that row, simply use the letter itself; if it is **F** in that row, use its negation; and conjoin the results. So, the formula for the first row would be $(p \wedge q \wedge r)$. The formula for the fifth row would be $(\neg p \wedge q \wedge r)$, etc. After constructing these "descriptive" conjunctions, the final formula is the disjunction of them all. In this example case, it is

$$(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

This formula clearly has the same truth table as that of Table 6. And the student is now expected to generalize this example.

A generalization of this method can be employed for many-valued logics, as described in Rosser and Turquette (1952). We might extend the language of FDE$^{\rightarrow}$ by adding constants (0-ary operators that always take a fixed value) for each of the four truth values. We use the names **t**,

[16](Omori and Wansing, 2017, p. 1036) list it as one of four possible conditionals to add to FDE, alongside $\rightarrow$, but our result shows that this does not yield a different logic.

p	q	r		
T	**T**	**T**	**T**	$(p \wedge q \wedge r)$
T	**T**	**F**	**F**	
T	**F**	**T**	**F**	
T	**F**	**F**	**F**	
F	**T**	**T**	**T**	$(\neg p \wedge q \wedge r)$
F	**T**	**F**	**T**	$(\neg p \wedge q \wedge \neg r)$
F	**F**	**T**	**F**	
F	**F**	**F**	**T**	$(\neg p \wedge \neg q \wedge \neg r)$

Table 6: Example of 2-valued Functional Completeness

b, **n**, and **f** for these four operators whose fixed values are **T**, **B**, **N**, and **F** respectively. We can also add four unary operators, $\mathrm{J}_{\mathbf{t}}(\phi)$, $\mathrm{J}_{\mathbf{b}}(\phi)$, $\mathrm{J}_{\mathbf{n}}(\phi)$, $\mathrm{J}_{\mathbf{f}}(\phi)$, whose values are **T** just in case the argument ϕ takes the value **t**, **b**, **n**, **f**, respectively, and otherwise takes the value **F**. In two-valued logic, ¬ takes the place of $\mathrm{J}_{\mathbf{f}}$, while an unadorned formula takes the place of $\mathrm{J}_{\mathbf{t}}$. But now that we have further values, we need a more generalized set of parametric operators. In a somewhat similar fashion, the "descriptive formula" indicates that it is **T** in that row, in the two-valued case, and not having a descriptive formula for a row means that it is **F** in that row. Now that we have further truth values we require the four constants **t**, **b**, **n**, **f**.

Even a simple example in FDE will be difficult to display: with three atomic letters we have 4^3 = 64 rows of a truth table. And each row needs to be "described", not just the **T** rows. As in the two-valued case, each row is a conjunction, and if we have just three atomic letters, there are just three conjuncts. Each row of the truth table in the FDE case will have the form

$v_a \quad v_b \quad v_c \quad | \quad v_k$

where the v_a, v_b, v_c are the values of the atomics p, q, r respectively and the v_k is the value assigned to the to-be-described formula in that row. The "descriptive formula" for that row is

$(J_{v_a}(p) \wedge J_{v_b}(q) \wedge J_{v_c}(r) \wedge \mathbf{k})$

where **k** is the constant that names the value v_k. The resultant formula that has the identical truth table as the arbitrary example given, is the disjunction of all the descriptive formulas for every row (not just the **t** rows, as in the two-valued case). Again, the example is supposed to be generalized and cover any possible FDE truth table. Since the method uses only the $\wedge, \vee, \neg$ of basic FDE plus the four constants and four parametric operators, this shows that such a logic is functionally complete for the given four-valued semantics.

Note that this method works for any of the logics under discussion, assuming that the appropriate J-operators and constants for the relevant truth values are present. (And also note that ¬ is not required in any of them.)

We turn our attention now to other ways to generate functional completeness in our logics of interest. That was a topic of interest in the 1930s, especially for the 3-valued logics, and especially for the particular case of Łukasiewicz Ł3. (See in particular Wajsberg (1931), who added to Ł3 a one-place operator that yielded the value **N** no matter what the truth value of the formula that was its argument.) However we will follow a different path, and show how to alter the Rosser-Turquette method just employed, by using features of our four-valued $\mathrm{FDE}^{\rightarrow}$; we do this by showing how to define or otherwise simplify the J-operators and truth-value constants in some different ways.

7.2 Some Definitional Possibilities

As we remarked above, adding $\rightarrow$ to FDE, yielding $\mathrm{FDE}^{\rightarrow}$, does not yield a functionally complete logic. But we can use this modified FDE that contains our $\rightarrow$ to lessen the requirement of needing all eight of the new primitives we added to basic FDE in the previous subsection.

We start by retaining the four truth-constants, **t**, **b**, **n**, **f** as a primitive. Now consider the formula

$(\varphi \rightarrow \mathbf{f})$

If $[\![\varphi]\!]$ is designated, i.e., is either **T** or **B**, then this formula is will be **F**. And if $[\![\varphi]\!]$ is undesignated, i.e., is either **N** or **F**, then this formula is will be **T**, as can be checked by truth tables.

Recall from §6.1 that we used $\rightarrow$ and $\neg$ to define $\Rightarrow$, and then used $\Rightarrow$ and $\wedge$ to define $\Leftrightarrow$. Finally, recall also that $[\![(\varphi \Leftrightarrow \psi)]\!]$ is designated if and only if $[\![\varphi]\!] = [\![\psi]\!]$. Putting this together, the biconditional $(\varphi \Leftrightarrow \mathbf{t})$ defines $J_{\mathbf{t}}(\varphi)$: it takes the value **T** if $[\![\varphi]\!] = \mathbf{T}$, and takes the value **F** otherwise. Similarly, $(\varphi \Leftrightarrow \mathbf{b})$, $(\varphi \Leftrightarrow \mathbf{n})$ and $(\varphi \Leftrightarrow \mathbf{f})$ define $J_{\mathbf{b}}(\varphi)$, $J_{\mathbf{n}}(\varphi)$, $J_{\mathbf{f}}(\varphi)$ respectively.

So, by adding the $\rightarrow$ conditional to FDE, we have thus eliminated four of the eight primitives we added to FDE in the previous subsection by using the Rosser-Turquette method. We could also eliminate one or the other of **t** and **f**, since $[\![\mathbf{t}]\!] = [\![\neg\mathbf{f}]\!]$ and $[\![\mathbf{f}]\!] = [\![\neg\mathbf{t}]\!]$. (However, this would make $\neg$ be a required primitive, whereas the Rosser-Turquette method did not employ $\neg$.)

7.3 Employing Second-Order Logic

In Hazen and Pelletier (2018) it is shown that a Second Order logic based on LP was surprisingly weak. This was due to the limited expressive power of the language with no conditional operator. In contrast, $\mathrm{FDE}^{\rightarrow}$, with its classical conditional, is expressively strong enough to provide a full base for classical Second Order Logic. Using propositional quantification, we can define a *falsum* propositional constant:

$$\mathbf{f} = \forall p(p)$$

(Obviously set/property quantification would do as well: $\mathbf{f} = \forall X \forall x(X(x))$.) This constant and the conditional give us in effect an ersatz "classical" negation: the conditional $(\varphi \rightarrow \mathbf{f})$ takes the value **F** when $[\![\varphi]\!]$ is designated and the value **T** when $[\![\varphi]\!]$ is undesignated, as we remarked in the previous subsection. Call a predicate (monadic or relational) *classical* just in case no atomic formula in it, on any assignment to the individual variables, takes either of the intermediate truth values **N** or **B**. **N** can be ruled out by the First Order formula $\forall x(\psi(x) \vee \neg\psi(x))$, which will not take a designated value if the predicate yields the value **N** for any individual in the domain. The possibility that ψ somewhere yields the value **B**, however, cannot be excluded by a purely First Order formula: if

ψ yields **B** for every individual, then every First Order formula in which ψ is the only bit of non-logical vocabulary occurring will have the designated value **B**. Using our ersatz classical negation, however, we can say $((\exists x(\psi(x) \wedge \neg\psi(x)) \rightarrow \mathbf{f})$. Finally, then, the classicality of a predicate, can by expressed by the conjunction of these two formulas. And so we may then interpret classical Second Order Logic in Second Order FDE$^{\rightarrow}$ by simply restricting all Second Order quantifiers to classical predicates.

Coming back down to the fragment of Second Order FDE$^{\rightarrow}$ with only propositional quantification, we can define the other three propositional constants. *Verum* is easy:

$$\mathbf{t} = \exists p(p)$$

Defining a constant for **B** is perhaps less obvious. $\exists p(p \wedge \neg p)$ would work in the 3-valued logics LP and LP$^{\rightarrow}$, but when **N** is also available it fails: $[\![p]\!] = \mathbf{B}$ gives $[\![p \wedge \neg p]\!] = \mathbf{B}$, $[\![p]\!] = \mathbf{N}$ gives $[\![p \wedge \neg p]\!] = \mathbf{N}$, and the existential quantification will have as value the join of these in the truth-value lattice: **T**![17] However, $[\![p \rightarrow p]\!]$=**B** when $[\![p]\!]$=**B**, and is **T** otherwise, so we may define

$$\mathbf{b} = \forall p(p \rightarrow p)$$

Defining a constant for **N** in FDE$^{\rightarrow}$is altogether harder (though $\exists p(p \wedge \neg p)$ would work in the 3-valued logic K3$^{\rightarrow}$). Indeed, it can be shown that

Theorem 1. *No* definiens *with, in prenex form, a single block of propositional quantifiers (all universal or all existential) is able to define a constant for* **N**.

Proof. Consider a purely propositional formula of FDE$^{\rightarrow}$. An assignment giving the value **B** to all of its atomic variables will give the formula the value **B**. Thus the set of values assumed by the formula on different assignments to its propositional variables will include **B**. The value of the sentence formed by binding its variables by existential quantifiers will be the lattice join of the values in this set, and so must be either **B** or **T**. The value of the sentence formed by binding its variables by universal quantifiers will be the lattice meet of the values in this set, and so must be either **B** or **F**. In neither case will the quantified formula serve as a definiens for the constant **n**. □

The propositional constant **n** can, however, be defined by a sentence of more complicated quantificational structure. Recall that $[\![(\varphi \Leftrightarrow \psi)]\!]$ is designated if and only if $[\![\varphi]\!] = [\![\psi]\!]$. It can thus be thought of as expressing identity of truth value. The conditional $((\varphi \Leftrightarrow \mathbf{t}) \rightarrow \mathbf{f})$, then, says that the value of φ is *not* **T**, and similarly for $((\varphi \Leftrightarrow \mathbf{b}) \rightarrow \mathbf{f})$ and $((\varphi \Leftrightarrow \mathbf{f}) \rightarrow \mathbf{f})$. So we may define **n** by

$$\mathbf{n} =_{df} \exists q(((q \Leftrightarrow \mathbf{t}) \rightarrow \mathbf{f}) \wedge ((q \Leftrightarrow \mathbf{b}) \rightarrow \mathbf{f}) \wedge ((q \Leftrightarrow \mathbf{f}) \rightarrow \mathbf{f}) \wedge q)$$

For, if q in the matrix is assigned one of the values **T**, **B**, or **F**, one of the first three conjuncts, and so the whole, will have the value **F**. If $[\![q]\!]$=**N**, however, the first three conjuncts will all have the value

[17] The existentially quantified formula takes the join of the values **N**, **B**, and **F**. . . which, perhaps surprisingly, is **T**. One doesn't expect a disjunction to take a higher value than any of its disjuncts, but in this case, because the four values are not linearly ordered, it does.

T, but the fourth, q, and so the whole conjunction, will have the value **N**. Since the join of **F** and **N** is **N**, the value of the full, quantified, sentence is **N**. By extracting the quantifiers concealed in the propositional constants in the contained biconditionals in the right order, we can put the definiens into prenex form with only one alternation of quantifiers (so, it will be Σ_2).

8 Synonymous Logics

In Pelletier (1984); Pelletier and Urquhart (2003), the notion of "translational equivalence" was introduced and defined, and used to describe a concept of *synonymous logics*. This concept was intended to describe cases where two logic systems were "really the same system" despite having different formulations, different vocabulary, and possibly having such different formulation that it would not be at all obvious that the logics were "really the same." This notion was shown to be different from various other conceptions in the literature, such as *mutual interpretability* and having *exact translations between logics*, which were shown to be weaker; other notions, such as having identical definitional extensions, were shown to be sufficient conditions for synonymy.[18] Two logics are *mutually interpretable* iff valid formulas and valid arguments get mapped to valid formulas and arguments in each other. They are *faithfully interpretable* if in addition invalid formulas and arguments get mapped to invalid formulas and arguments. Note that the definitions of mutual interpretability and even mutual faithful interpretability do not in any obvious way imply synonymy of the logics, although the counterexamples generally seem very complex and "artificial". (See French 2010, Chapter 3, for a categorization of the different types of interpretability.)

Two logics, $\mathcal{L}_1$ and $\mathcal{L}_2$, are translationally equivalent if and only if there are translation schemes t_1 from $\mathcal{L}_1$ into $\mathcal{L}_2$ and t_2 from $\mathcal{L}_2$ into $\mathcal{L}_1$ such that

1. if $\vdash_{\mathcal{L}_1} \varphi$ then $\vdash_{\mathcal{L}_2} \varphi^{t_1}$
2. if $\vdash_{\mathcal{L}_2} \varphi$ then $\vdash_{\mathcal{L}_1} \varphi^{t_2}$
3. for any formula φ in $\mathcal{L}_1$, $(\varphi^{t_1})^{t_2}$ is equivalent to φ (in $\mathcal{L}_1$)
4. for any formula φ in $\mathcal{L}_2$, $(\varphi^{t_2})^{t_1}$ is equivalent to φ (in $\mathcal{L}_2$)[19]

[18] Further aspects of the notion, as well as formal details, are in Pelletier and Urquhart (2003).

[19] In Pelletier and Urquhart (2003) it was assumed that the logics in question had a "biconditional equivalence connective" and the third and fourth conditions were expressed in terms of the biconditional being a theorem in the appropriate logics. In the context of that paper, the logics were classical except for modal operators, and so there were always such equivalence operators in each logic. In the present context, we cannot assume that the biconditionals of the various logics will operate in the same way, and so we envisage checking the "equivalent to" conditions semantically, by simply looking at the relevant truth tables. Although defining translational equivalence in full generality for non-truth-functional logics is quite difficult (as Nelson 1959, p.216, remarks), since the only logics we are discussing have a finite number of truth values, the relevant kind of equivalence of formulas is that of having (on each assignment) the same truth value. And in fact, our logics allow us to define a connective, $\Leftrightarrow$, which can reasonably be taken to express the relation of having the same truth value.

We are in a position to show some new results of synonymity of logics. Since the languages of our different logics are identical except for their conditionals (and biconditionals), we will employ the following translations for all the portions of the logics involved except the differing (bi)conditionals in the logics:

- For φ an atomic sentence, $(\varphi)^t$ is φ
- For negated formulas, $(\neg\varphi)^t$ is $\neg(\varphi)^t$
- If $\circ$ is any binary operator other than a (bi)conditional, $(\varphi \circ \psi)^t$ is $((\varphi)^t \circ (\psi)^t)$

In §6.2 we showed that systems K3$^{\rightarrow}$ and Ł3 could be faithfully interpreted in each other by means of the following "translations". It should be understood that these translation functions apply recursively, to any embedded formulas that also have the relevant conditionals.[20] (As for example in Case 2 of the proof of Theorem 2.)

- $t_1 : (\varphi \rightarrow \psi)^{t_1} =_{df} ((\varphi)^{t_1} \Rightarrow ((\varphi)^{t_1} \Rightarrow (\psi)^{t_1}))$
- $t_2 : (\varphi \Rightarrow \psi)^{t_2} =_{df} (((\varphi)^{t_2} \rightarrow (\psi)^{t_2}) \wedge ((\neg\psi)^{t_2} \rightarrow (\neg\varphi)^{t_2}))$

We can now prove a stronger result. The earlier proof of mutual interpretability showed that each of K3$^{\rightarrow}$ (whose conditional is $\rightarrow$) and Ł3 (whose conditional is $\Rightarrow$) could define a formula that had the same truth table as the conditional of the other (and since all other connectives were identical, that is all that is required to establish mutual interpretability). What is further needed to show synonymy is that applying the definition in one logic to generate the conditional of the other logic yields a result such that we can apply the other logic's definition to it and generate a formula semantically equivalent to the conditional in the first logic. (And vice versa, of course). That is to say: where t_1 and t_2 are the translations we used in §6.2, we need to be able to prove

- $((\varphi \Rightarrow \psi)^{t_2})^{t_1}$ is semantically equivalent to $(\varphi \Rightarrow \psi)$ in Ł3
- $((\varphi \rightarrow \psi)^{t_1})^{t_2}$ is semantically equivalent to $(\varphi \rightarrow \psi)$ in K3$^{\rightarrow}$

Theorem 2. K3$^{\rightarrow}$ and Ł3 are synonymous logics

Proof
Case 1:

$$
\begin{aligned}
((\varphi \Rightarrow \psi)^{t_2})^{t_1} &= [(\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi)]^{t_1} && (1)\\
&= ((\varphi \Rightarrow (\varphi \Rightarrow \psi)) \wedge (\neg\psi \Rightarrow (\neg\psi \Rightarrow \neg\varphi)) && (2)\\
&= (\varphi \Rightarrow \psi) && (3)
\end{aligned}
$$

[20] This follows from what (Kuhn and Weatherson, 2018) call the "compositionality" requirement of (Pelletier and Urquhart, 2003)'s definition of translational equivalence.

Case 2:

$$((\varphi \to \psi)^{t_1})^{t_2} = (\varphi \Rightarrow (\varphi \Rightarrow \psi))^{t_2} \tag{4}$$

$$= (\varphi \to (\varphi \Rightarrow \psi)^{t_2}) \wedge (\neg(\varphi \Rightarrow \psi)^{t_2} \to \neg\varphi) \tag{5}$$

$$= [[\varphi \to (((\varphi \to \psi) \wedge (\neg\psi \to \varphi))] \wedge [\neg((\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)) \to \neg\varphi]] \tag{6}$$

$$= (\varphi \to \psi) \tag{7}$$

It can be seen from the truth table in Table 7 that formulas (2) and (3) are semantically the same, and from the truth table in Table 8 that formulas (6) and (7) are semantically the same.

φ	ψ	$\varphi \Rightarrow \psi$	$\neg\psi \Rightarrow \neg\varphi$	$[(\varphi \Rightarrow (\varphi \Rightarrow \psi)))$	$\wedge$	$(\neg\psi \Rightarrow (\neg\psi \Rightarrow \neg\varphi))]$
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**N**	**N**	**T**	**N**	**N**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**
N	**T**	**T**	**F**	**T**	**T**	**T**
N	**N**	**T**	**T**	**T**	**T**	**T**
N	**F**	**N**	**N**	**T**	**N**	**N**
F	**T**	**T**	**T**	**T**	**T**	**T**
F	**N**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Table 7: $\varphi \Rightarrow \psi$ of Ł3 (left column) is equivalent to its "double translation" (the $\wedge$ column)

φ	ψ	$\varphi \to \psi$	$\neg\psi \to \neg\varphi$	$[[\varphi \to (((\varphi \to \psi) \wedge (\neg\psi \to \varphi))]$	$\wedge$	$[\neg((\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)) \to \neg\varphi]]$
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**N**	**N**	**N**	**N**	**N**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**
N	**T**	**T**	**T**	**T**	**T**	**T**
N	**N**	**T**	**T**	**T**	**T**	**T**
N	**F**	**T**	**N**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**
F	**N**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Table 8: $\varphi \to \psi$ of $K3^{\to}$ (left column) is equivalent to its "double translation" (the $\wedge$ column)

In §6.3 we showed that systems LP$^{\rightarrow}$ and RM3 could be faithfully interpreted in each other by means of the following "translations".

- $t_1 : (\varphi \rightarrow \psi)^{t_1} =_{df} (((\varphi)^{t_1} \Rightarrow (\psi)^{t_1}) \vee (\psi)^{t_1})$
- $t_2 : (\varphi \Rightarrow \psi)^{t_2} =_{df} (((\varphi)^{t_2} \rightarrow (\psi)^{t_2}) \wedge ((\neg\psi)^{t_2} \rightarrow (\neg\varphi)^{t_2}))$

Again we can now prove a stronger result. The earlier proof of mutual interpretability showed that each of LP$^{\rightarrow}$ (whose conditional is $\rightarrow$) and RM3 (whose conditional is $\Rightarrow$) could define a formula that had the same truth table as the conditional of the other (and since all other connectives were identical, that is all that is required to establish mutual interpretability). What is further needed to show synonymy is that applying the definition in one logic to generate the conditional of the other logic yields a result such that we can apply the other logic's definition to it and generate a formula semantically equivalent to the conditional in the first logic. (And vice versa, of course). That is to say: where t_1 and t_2 are the translations we used in §6.3, we need to be able to prove

- $((\varphi \Rightarrow \psi)^{t_2})^{t_1}$ is semantically equivalent to $(\varphi \Rightarrow \psi)$ in RM3
- $((\varphi \rightarrow \psi)^{t_1})^{t_2}$ is semantically equivalent to $(\varphi \rightarrow \psi)$ in LP$^{\rightarrow}$

Theorem 3. LP$^{\rightarrow}$ and RM3 are synonymous logics

Proof

Case 1:

$$((\varphi \Rightarrow \psi)^{t_2})^{t_1} = [(\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi)]^{t_1} \tag{8}$$
$$= (((\varphi \Rightarrow \psi) \vee \psi) \wedge ((\neg\psi \Rightarrow \neg\varphi) \vee \neg\varphi)) \tag{9}$$
$$= (\varphi \Rightarrow \psi) \tag{10}$$

Case 2:

$$((\varphi \rightarrow \psi)^{t_1})^{t_2} = ((\varphi \Rightarrow \psi) \vee \psi)^{t_2} \tag{11}$$
$$= (((\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi)) \vee \psi) \tag{12}$$
$$= (\varphi \rightarrow \psi) \tag{13}$$

It can be seen from the truth table in Table 9 that formulas (9) and (10) are semantically the same, and from the truth table in Table 10 that formulas (12) and (13) are semantically the same.

These instances of synonymous logics strike us as both somewhat unexpected and also as "cleaner" versions of synonymy of logics than the one(s) displayed in (Pelletier and Urquhart, 2003), which employed a propositional constant in one of the logics. As remarked in that article (following De Bouvère, 1965), two logics are translationally equivalent in this way if they have a common definitional extension. Note then that for each of our pairs of synonymous logics, the appropriate 3-valued logic with two conditional operators, $\rightarrow$ and $\Rightarrow$, is a definitional extension of both members of the pair. And thus the two logics are translationally equivalent to each other.

Lastly, and as mentioned in §6.4.2, we can state a final theorem concerning the addition of $\rightarrow$ or $\Rightarrow$ to FDE itself:

φ	ψ	$\varphi \to \psi$	$\neg\psi \to \neg\varphi$	$\varphi \Rightarrow \psi$	$[(\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)]$	$[((\varphi \Rightarrow \psi) \vee \psi)$	$\wedge$	$((\neg\psi \Rightarrow \neg\varphi) \vee \neg\varphi)]$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**B**	**B**	**F**	**F**	**F**	**B**	**F**	**F**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
B	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
B	**B**	**B**	**B**	**B**	**B**	**B**	**B**	**B**
B	**F**	**F**	**B**	**F**	**F**	**F**	**F**	**B**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**B**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**

Table 9: $\varphi \Rightarrow \psi$ of RM3 (third column) is equivalent to its "double translation" (the $\wedge$ column)

φ	ψ	$\varphi \to \psi$	$\neg\psi \to \neg\varphi$	$\varphi \Rightarrow \psi$	$[(\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)]$	$[((\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)) \vee \psi]$
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**B**	**B**	**F**	**F**	**F**	**B**
T	**F**	**F**	**F**	**F**	**F**	**F**
B	**T**	**T**	**T**	**T**	**T**	**T**
B	**B**	**B**	**B**	**B**	**B**	**B**
B	**F**	**F**	**B**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**	**T**	**T**
F	**B**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Table 10: $\varphi \to \psi$ of $LP^{\to}$ (left column) is equivalent to its "double translation" (the rightmost column)

Theorem 4. *$FDE^{\to}$and $FDE^{\Rightarrow}$ are synonymous logics.*

9 Concluding Remarks

We have discussed a family of logics that are related to FDE, showing how they are related to each other and also to some other logics that have populated the literature (such as Ł3 and RM3). We also identified a 4-valued logic $M^{\to}$ that is stronger than $FDE^{\to}$ but weaker than each of the two 3-valued extensions of $FDE^{\to}$. (That is, weaker than both $K3^{\to}$ and $LP^{\to}$.) Surprisingly, perhaps, we are also able to show some new and "cleaner" examples of synonymous logics (in the sense of Pelletier and Urquhart, 2003). Deductive systems for the various logics, as well as soundness and completeness proofs are in the Appendices.

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Appendix I: Deductive Systems

One direction that has been taken in the literature is the development of "super-Belnap" logics, that is, extensions of FDE. See Rivieccio (2012); Přenosil (2017) for examples and elaboration. One development in this tradition is the generation of proof systems for such logics, which have pretty much been in the tradition of Gentzen-style sequent calculi, usually with multiple conclusion sequents although some single conclusion systems have also been given. Sometimes this is paired with a comparison of a Hilbert-style axiomatic development of the same logics, and sometimes a natural deduction system of the Gentzen type is presented. We think that various natural deduction systems of the Jaśkowski style are preferable for a number of "philosophical" reasons (many recounted in Hazen and Pelletier, 2014) as well as for pedagogical clarity, and in particular we will present a Fitch-style version (Fitch, 1952) of the Jaśkowski method.

Tedder (2014) contains a Hilbert-style axiomatic system, and (more interestingly) a (multiple succedent) sequent calculus for $LP^{\rightarrow}$. (Tedder, 2015, presents only the Hilbert system.) The sequent calculus closely follows (Gentzen, 1934)'s system LP for classical logic, with the following changes:

(i) Gentzen's rules for negation ("change the side and change the sign") are dropped,

(ii) double negation rules are added, allowing a sequent to be followed by one like it except that one of its formulas–on either side–is doubly negated,

(iii) negative rules for conjunction and disjunction (and, analogously, for quantifiers) are added, allowing the insertion of a negated conjunction or disjunction under the same conditions as allow the insertion of its De Morgan equivalent disjunction or conjunction,

(iv) negative conditional rules are added, allowing $\neg(\varphi \rightarrow \psi)$ to be inserted under the same conditions as $(\varphi \wedge \neg\psi)$, and

(v) Gentzen's identity axioms, $\varphi \vdash \varphi$, are supplemented with "Gap excluding" axioms of the form $\vdash \varphi, \neg\varphi$.

Cut elimination is proven by a straightforward adaptation of Gentzen's method.

Dropping the Gap exclusion axioms from this system yields a sequent calculus for $FDE^{\rightarrow}$. Replacing them with "Glut excluding axioms" of the form $\varphi, \neg\varphi \vdash$ gives one for $K3^{\rightarrow}$, and replacing them instead with Mingle axioms of the form $\varphi, \neg\varphi \vdash \psi, \neg\psi$ gives one for $M^{\rightarrow}$.

A natural deduction system, either in the style of Gentzen's NK or in the Fitch-style presentation of many American textbooks, for any of these logics will include

(i) the standard Introduction and Elimination rules for $\wedge, \vee, \rightarrow$ (and for the quantifiers in a First Order system),

(ii) Double Negation Introduction and Elimination rues, by which a formula and its double negation may each be inferred from the other,

(iii) Negative Introduction and Elimination rules for $\wedge$ and $\vee$ (and, analogously for the quantifiers), as in (Fitch, 1952) enforcing the interdeducibility of negated conjunctions and disjunctions with their De Morgan equivalent disjunctions and conjunctions,

(iv) Negative Introduction and Elimination rules for $\rightarrow$, enforcing the interdeducibility of $\neg(\varphi \rightarrow \psi)$ with $(\varphi \wedge \neg\psi)$.

As is well-known, the standard Introduction and Elimination rules for $\rightarrow$ give only the Intuitionistic logic of the conditional, so these have to be supplemented with some classicizing postulate to give the full classical logic of the positive connectives. Addition of Peirce's Law, $(((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$ as an axiom scheme is one conventional way of doing this, but an alternative axiom scheme, $(\varphi \vee (\varphi \rightarrow \psi))$ seems a good deal easier to work with, and can be converted into a moderately elegant natural deduction rule:

(v) A formula, χ, may be asserted if it is derivable both from the hypothesis φ and from the hypothesis $(\varphi \rightarrow \psi)$.

These rules suffice for the propositional logic FDE$^{\rightarrow}$. A First Order system would have to supplement the standard and negative Introduction and Elimination Rules for the quantifiers with something to guarantee the "constant domain" inference

$\forall x(\varphi \vee \Psi(x)) \vdash (\varphi \vee \forall x(\Psi(x))$

(see Fitch 1952, §21.31.) Systems for the other logics are obtained by adding rules to this basic system:

1) Ex falso quodlibet ("explosion"), for K3$^{\rightarrow}$;

2) Excluded middle: χ may be asserted if it is derivable both from the hypothesis φ and from the hypothesis $\neg\varphi$, for LP$^{\rightarrow}$;

3) Mingle: $(\psi \vee \neg\psi)$ may be inferred from $(\varphi \wedge \neg\varphi)$, for M$^{\rightarrow}$.

More visually: Natural deduction rules for FDE$^{\rightarrow}$will be double negation and both the positive and negative IntElim rules for $\wedge$ and $\vee$. Additionally, there is a series of rules for our $\rightarrow$ operator.

Double Negation:

$$\begin{array}{l|ll} 1 & A & \\ \hline 2 & \neg\neg A & \neg\neg\ \text{Int} \end{array} \qquad\qquad \begin{array}{l|ll} 1 & \neg\neg A & \\ \hline 2 & A & \neg\neg\ \text{Elim} \end{array}$$

Standard rules for lattice connectives:

$$\begin{array}{l|ll} 1 & A \wedge B & \\ \hline 2 & A & \wedge\ \text{Elim} \end{array} \qquad \begin{array}{l|ll} 1 & A \wedge B & \\ \hline 2 & B & \wedge\ \text{Elim} \end{array} \qquad \begin{array}{l|ll} 1 & A & \\ 2 & B & \\ \hline 3 & A \wedge B & \wedge\ \text{Int} \end{array}$$

$$
\begin{array}{r|lll}
1 & A \vee B & & \\ \hline
2 & \quad \mid A & & \\
3 & \quad \mid \ldots & & \\
4 & \quad \mid C & & \\
5 & \quad \mid B & & \\
6 & \quad \mid \ldots & & \\
7 & \quad \mid C & & \\
8 & C & & \vee \text{ Elim}
\end{array}
\qquad
\begin{array}{r|ll}
1 & A & \\ \hline
2 & A \vee B & \vee \text{ Int}
\end{array}
\qquad
\begin{array}{r|ll}
1 & B & \\ \hline
2 & A \vee B & \vee \text{ Int}
\end{array}
$$

Negative rules for the lattice connectives[21]:

$$
\begin{array}{r|ll}
1 & \neg(A \vee B) & \\ \hline
2 & \neg A & \neg\vee \text{ Elim}
\end{array}
\qquad
\begin{array}{r|ll}
1 & \neg(A \vee B) & \\ \hline
2 & \neg B & \neg\vee \text{ Elim}
\end{array}
$$

$$
\begin{array}{r|ll}
1 & \neg A & \\
2 & \neg B & \\ \hline
3 & \neg(A \vee B) & \neg\vee \text{ Int}
\end{array}
$$

$$
\begin{array}{r|ll}
1 & \neg A & \\ \hline
2 & \neg(A \wedge B) & \neg\wedge \text{ Int}
\end{array}
\qquad
\begin{array}{r|ll}
1 & \neg B & \\ \hline
2 & \neg(A \wedge B) & \neg\wedge \text{ Int}
\end{array}
$$

$$
\begin{array}{r|ll}
1 & \neg(A \wedge B) & \\ \hline
2 & \quad \mid \neg A & \\
3 & \quad \mid \ldots & \\
4 & \quad \mid C & \\
5 & \quad \mid \neg B & \\
6 & \quad \mid \ldots & \\
7 & \quad \mid C & \\
8 & C & \neg\wedge \text{ Elim}
\end{array}
$$

[21]In the context of the positive rules, these negative rules are equivalent to Fitch's (1952) original negation IntElim rules.

Rules for Conditional[22]

$$\begin{array}{lll}
1 & (A \to B) & \\
2 & A & \\
\hline
3 & B & \to \text{Elim}
\end{array}$$

$$\begin{array}{llll}
1 & & A & \\
\cline{3-3}
2 & & \ldots & \\
3 & & B & \\
4 & A \to B & & \to \text{Int}
\end{array}$$

$$\begin{array}{lll}
1 & \neg(A \to B) & \\
\hline
2 & A & \neg \to \text{Elim}
\end{array}$$

$$\begin{array}{lll}
1 & \neg(A \to B) & \\
\hline
2 & \neg B & \neg \to \text{Elim}
\end{array}$$

$$\begin{array}{llll}
1 & & A & \\
\cline{3-3}
2 & & \ldots & \\
3 & & B & \\
4 & & A \to C & \\
\cline{3-3}
5 & & \ldots & \\
6 & & B & \\
7 & B & & \text{Dilemma}
\end{array}$$

$$\begin{array}{lll}
1 & A & \\
2 & \neg B & \\
\hline
3 & \neg(A \to B) & \neg \to \text{Int}
\end{array}$$

[22]The Dilemma rule, if added to, say, Intuitionistic Logic, would yield a formulation of full classical logic. It does not collapse FDE$^{\to}$into classical logic because the usual ¬-Introduction rule (*reductio*) of Intuitionistic Logic is absent.

Appendix II: Soundness and Completeness

The natural deduction system is provably sound and complete, where soundness is taken to mean that if all the premisses of a derivation have designated values on an assignment, the conclusion will as well.

Soundness can be verified in the usual way, arguing by induction on the size of the derivation after establishing that each rule is sound. For rules in which a conclusion is inferred directly from one or two premisses, this is immediate, by inspection of the truth tables. A rule in which a hypothesis is discharged (that is, in the terminology of Fitch-style natural deduction, a rule involving one or two *subordinate proofs*) is considered sound just in case, if all the undischarged hypotheses above the conclusion of the rule (in Fitch-style: the formulas *reiterated* into subordinate proofs) have designated values, the conclusion also has a designated value. It is easy to see that the subproof rules of the system will be sound *provided that* the reasoning within the subordinate proofs is sound. The full soundness proof, then, will take the form of a double induction, on the length, and on the depth of nesting of subordinate proofs within, a proof. The overall strategy is perfectly standard for soundness proofs of natural deduction systems.

Completeness can be proven by a variant of Henkin's method, similar to that used in Priest (2018). We desire to show that, if a formula A is not derivable from a set of premisses Γ, then there is an assignment on which A takes an undesignated value but every member of Γ is designated. In a Henkin-style proof this is done in two stages. In the first, it is shown that Γ can be extended to an eligible set Γ^* which still does not (syntactically) imply A, where a set of formulas is said to be eligible if it has some of the formal characteristics of the set of formulas taking designated values on some assignment. In the second it is shown that the eligible set is actually elected: an assignment is defined on which all and only the members of the set take designated values.

In applying Henkin's method to a classical system, eligible sets are simply *complete theories*, a.k.a. *maximal consistent sets* of formulas. In logics tolerating contradictions, consistency is obviously not a requirement, and in logics tolerating truth value "gaps" maximality is also not to be hoped for! An appropriate notion of eligibility for our purposes counts a set of formulas as eligible if and only if (i) it is deductively closed (and so, in particular, contains a conjunction if and only if it contains both conjuncts, and contains a disjunction if it contains either disjunct), and (ii) contains at least one of the disjuncts of each disjunction it contains. Given A not derivable from Γ, it is readily seen that a set maximal with respect to the property of containing Γ but not implying A will be eligible in this sense, and the existence of such a maximal set follows from standard set-theoretic considerations (Teichmüller-Tukey lemma). (In general these maximal sets will not be the only eligible supersets of Γ not implying A, and an alternative proof adding formulas to Γ only if they are required by clause (ii) may yield a smaller eligible set.)

Given an eligible set Γ^*, we define an assignment to propositional variables by setting

- $v(p) = \mathbf{T}$ iff p is a member but $\neg p$ is not a member of Γ^*,
- $v(p) = \mathbf{B}$ iff p and $\neg p$ both belong to Γ^*,

- $v(p) = \mathbf{N}$ iff neither p nor $\neg p$ belongs to Γ^*, and
- $v(p) = \mathbf{F}$ iff $\neg p$ but not p is a member of Γ^*.

(Note that a variable takes a designated value if and only if it belongs to Γ^*.) It remains to verify (by induction on formula complexity) that arbitrary formulas take values on this assignment under the same conditions of their, and their negations', membership in Γ^*. None of the cases are hard; those not immediately obvious usually become obvious when one remembers that formulas of the form $A \vee (A \to B)$ are provable in the system.

In a bit more detail:

Soundness:

The notion of validity we want is:

Definition 1. Validity: *A system of rules is sound for one of our logics if and only if: If all the assumptions*[23] *of a (possibly subordinate) proof have designated values, then every formula occurring diretly as an item of that derivation (as opposed to occurring in a subordinated derivation subordinate to it) will also have a designated value.*

One could easily check that all the rules are classically valid, and conclude that *of course* the FDE$^{\to}$system is sound. But perhaps some of the *negative* rules are worth checking.

$\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

Table 11: $\wedge$ truth-table

A quick check of the $\neg\wedge$-Int rules against this truth table makes it clear that these rules are sound. The presence of **B** means that $\neg\wedge$-Elim takes a bit longer, but is clearly correct also.

In full formal detail, the soundness proof (as is usual for a soundness proof of Fitch-style natural deduction systems) is a double induction on

1. Length: the number of non-assumption formula items in the derivation.
2. Depth: the depth of nesting of subproofs.

The induction step of the Depth induction goes:

[23] Where formulas reiterated into a subproof are counted among that subproof's assumptions.

Suppose A comes by a subproof-using rule. (Now, a number of cases.) By hypothesis of induction on Length, anything reiterated into the subproof is ok (since it would occur in a main subproof above this step). By hypothesis of induction on Depth [we call this the "key step"], the subproof is sound. So, if the hypothesis of the subproof is ok, so is its "active" last item. So – now verifying each case – A is ok.

This gives us the cases for subproof-involving rules in the inductive step of the Length induction for derivations of depth $\leq n$. So derivations at this depth are sound. In a derivation of depth $n + 1$, every subproof is of depth $\leq n$, so the "key step" is guaranteed.

Completeness Construction:

Define a set of formulas to be *saturated* if and only if

1. It is consistent in the sense of not being the set of all formulas, and
2. It is *deductively closed*, and so automatically contains
   (a) A conjunction if and only if it contains both conjuncts, and
   (b) A disjunction if it contains at least one disjunct
3. It contains at least one disjunct of each of its disjunctions

Note that for *classical* logic, in which by deductive closure $(A \vee \neg A)$ belongs to every saturated set, saturation amounts to *maximal consistency*: saturated sets are the generalization for non-classical logics of the "maximal consistent sets" familiar from Henkin proofs for classic(-ally based) logic(s).

What we have to prove is:

Lemma 1. *For any set of formulas,* Γ*, and for any formula, A, not derivable from* Γ*, there is a* saturated *superset of* Γ*,* Γ^**, not containing A. (Since this specification includes the requirement that* $A \notin \Gamma^*$*, the requirement of consistency, (1), doesn't* have *to be made part of the* definition *of saturated set.*

Proof. (By a version of the standard Lindenbaum construction.)

For classical logic, where we want *maximal* consistent sets anyway, it is normal to consider *all* the formulas of the language (in some order), tossing each one in if its addition doesn't permit the derivation of THE BAD THING. But this can tend to *undesirably stuffed* sets of formulas! For example, the *maximal* consistent sets of intuitionistic propositional logic are *classically* maximal! So we prefer a more cautious addition of formulas.

Definition 2. *An ordinally indexed series of sets of formulas:*

Let $\Gamma_0 = \Gamma$

Assume some fixed well-ordering of the formulas of the language:

For odd successors*,* α*,*

- *If* Γ_α *is saturated, stop*
- *if not, pick the* first *(in the assumed ordering) disjunction in* Γ_α *for which neither disjunct is in* Γ_α*. By the* $\vee$*-Elim rule, if* Γ_α *does not imply A, at least one of these disjuncts can be added to* Γ_α *without permitting the derivation of A. Let* $\Gamma_{\alpha+1}$ *be the result of adding the first such disjunct (or the only one, if only one can be added) to* Γ_α*.*

For even successors *(and for 0), let* $\Gamma_{\alpha+1}$ *be the deductive closure of* Γ_α*.*

For limit ordinals λ*, let* $\Gamma_\lambda = \bigcup_{\alpha<\lambda}(\Gamma_\alpha)$.

By cardinality considerations (there are more ordinals than formulas, so eventually we will run out of formulas to add), this sequence must reach a fixed point, Γ_Ω. By the usual Henkin arguments, Γ_Ω is saturated and does not imply A.

Now, for *atomic* p, define

- p has the value **T** iff p does and $\neg p$ does not belong to Γ_Ω
- p has the value **B** iff both p and $\neg p$ belong to Γ_Ω
- p has the value **N** iff neither p nor $\neg p$ belong to Γ_Ω
- p has value **F** iff $\neg p$ but not p belongs to Γ_Ω

We now in a position to *verify* that for arbitrary formulas A, the same correlation holds between value and status with respect to membership in Γ_Ω

□

Completeness Verification:

Theorem 5. *Given a* saturated *set S, if we define, for atomic p:*

$p \rightsquigarrow \mathbf{T}$ *iff* $p \in S$ *and* $\neg p \notin S$
$p \rightsquigarrow \mathbf{B}$ *iff* $p \in S$ *and* $\neg p \in S$
$p \rightsquigarrow \mathbf{N}$ *iff* $p \notin S$ *and* $\neg p \notin S$
$p \rightsquigarrow \mathbf{F}$ *iff* $p \notin S$ *and* $\neg p \in S$
we will have the same coincidence of values and S-status for all formulas

Proof. By induction on formula structures. We omit proofs for the $\neg, \vee, \wedge$ connectives, which are obvious. So we consider $\rightarrow$:
The way up Suppose equivalence holds for A and B, we show it holds for $A \rightarrow B$.

There are 16 combinations of truth-like values for A, B. For each, by hypothesis of induction, assume the right S-membership status.

- **Cases**: Left column. B is **T**, so is in S. So, $A \rightarrow B$ is in S. $\neg B$ is *not* in S, so by simple $\neg \rightarrow$-Elim, $\neg(A \rightarrow B)$ can't be either.
- **Cases**: Bottom two rows. A is either **F** or **N**, so by hypothesis of induction, A is *not* in S. By Dilemma, $A \vee (A \rightarrow B)$ is in S, so by saturation $(A \rightarrow B) \in S$. In the other direction, simple $\neg \rightarrow$-Elim rule would get from $\neg(A \rightarrow B)$ to A, so $\neg(A \rightarrow B)$ can't be in S.
- **Cases**: Top row: A has value **T**, so $A \in S, \neg A \notin S$
  - **subcase**: B has value **B**, so both $B \in S$ and $\neg B \in S$. Since $B \in S$, $\rightarrow$-Int gives $(A \rightarrow B) \in S$. Since $A \in S$ and $\neg B \in S$, the simple $\neg \rightarrow$-Int rule gives $\neg(A \rightarrow B) \in S$.
  - **subcase**: B has value **N**, so neither $B \in S$ nor $\neg B \in S$. If $(A \rightarrow B)$ were in S, $\rightarrow$-Elim would put B in, so $(A \rightarrow B) \notin S$. If $\neg(A \rightarrow B)$ were in S, simple $\neg \rightarrow$-Elim rule would put $\neg B$ in S, so $\neg(A \rightarrow B) \notin S$.
  - **subcase**: B has value **F**, so $\neg B \in S$ and $B \notin S$. If $(A \rightarrow B)$ were in S, $\rightarrow$-Elim would put B in, so $(A \rightarrow B) \notin S$. On the other hand, since A and $\neg B$ are both available, simple $\neg \rightarrow$-Int gives $\neg(A \rightarrow B) \in S$.
- **Cases**: Second row: A has value **B**, so both $A \in S$ and $\neg A \in S$
  - **subcase**: B has value **B**, so by hypothesis of induction, B and $\neg B$ are both in S. Since B is available, $\rightarrow$-Int gets $(A \rightarrow B) \in S$. Since A and $\neg B$ are both available, simple $\neg \rightarrow$-Int puts $\neg(A \rightarrow B) \in S$.
  - **subcase**: B has value **N**, so neither B nor $\neg B$ is in S. Since A is available, if $(A \rightarrow B)$ were in S, $\rightarrow$-Elim would put B in S. If $\neg(A \rightarrow B)$ were in S, simple $\neg \rightarrow$-Elim would put $\neg B$ in S. So neither $(A \rightarrow B)$ nor $\neg(A \rightarrow B$ is in S.
  - **subcase**: B has value **F**, so $\neg B \in S$ and $B \notin S$. Since A and $\neg B$ are available, $\neg \rightarrow$-Int puts $\neg(A \rightarrow B) \in S$. Since A is available, if $A \rightarrow B$ were in S, B would be also.

□

Normally, in a Henkin-style completeness proof, one defines truth for atoms as membership in the saturated set, and the *verification* stage checks that complex formulas are true if and only if they are members. This requires *two* arguments: one that *if* a formula is true it is a member, and one that *if* a formula belongs to the set, then it is true.

Because the semantic clauses for **T**, **B**, **N**, **F** are more complex, these two parts are now intermingled. If a formula has a certain value, then both the membership and non-membership of the formula and its negation (one for each) in the set have to be checked. Thus **the way up** (verifying that formulas with certain values have the right membership statuses) already in effect includes **the way back down** (verifying that formulas with certain membership statuses have the right values).