A note on a description logic of concept and role typicality

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Abstract

In this work, we propose a meaningful extension of description logics for non-monotonic reasoning. We introduce $\mathcal{ALCH}^\bullet$, a logic allowing for the representation of and reasoning about both typical class-membership and typical instances of a relation. We propose a preferential semantics for $\mathcal{ALCH}^\bullet$ in terms of partially-ordered DL interpretations which intuitively captures the notions of typicality we are interested in. We define a tableau-based algorithm for checking $\mathcal{ALCH}^\bullet$ knowledge-base consistency and show that it is sound and complete w.r.t. our preferential semantics. The general framework we here propose can serve as the foundation for further exploration of non-monotonic reasoning in description logics and similarly structured logics.

Keywords: Description logic; defeasible reasoning; typicality; tableaux

1 Introduction

Description Logics (DLs) [1] are a family of logic-based knowledge representation formalisms with useful computational properties and a variety of applications in artificial intelligence and in databases. In particular, DLs are well-suited for representing and reasoning about terminological knowledge and constitute the formal foundations of semantic-web ontologies. Technically, DLs correspond to decidable fragments of first-order logic and are closely related to modal logics [43].

Notwithstanding their good trade-off between expressive power and computational complexity, DLs remain fundamentally classical formalisms and therefore are not suitable for modelling and reasoning about aspects that are ubiquitous in human quotidian reasoning. Examples of these are exceptions to general rules, incomplete knowledge, and many others, characterising the type of reasoning usually known under the broad term *defeasible reasoning*. In this regard, endowing DLs and their associated reasoning services with the ability to cope with defeasibility is a natural step in their development. Indeed, the past 25 years have witnessed many attempts to introduce defeasible-reasoning capabilities in a DL setting, usually drawing on a well-established body of research on non-monotonic reasoning (NMR). These comprise the so-called preferential approaches [15, 16, 17, 22, 23, 25, 26, 29, 30, 41, 42], circumscription-based ones [7, 8, 44], amongst others [2, 3, 6, 24, 31, 32, 33, 39, 40, 46].

Of particular interest in a non-monotonic context is the ability to express and reason about a notion of typicality (or normality, or expectations). And, as already argued in the propositional case [12], being able to do so *explicitly* in the language brings in many advantages from the standpoint of knowledge representation. In a DL setting, this need is mainly felt when checking whether a given individual is a typical instance of a class or whether a pair of individuals is a typical instance of a given relationship, or some combination involving both. As an example, consider the following scenario, adapted from Giordano et al.'s [25]: Typical students do not pay taxes; employed students typically do; to work for a company typically implies being employed by the company, and John and IBM are in a typical work contract.

It turns out that this issue has only partially been addressed in the literature in that explicit notions of typicality for *concepts* have been introduced [6, 25], but of which the use in logical statements has to adhere to certain syntactic constraints. To the best of our knowledge, a unifying framework for full-fledged typicality in concepts and, important, also in roles has not been developed before. This is precisely the problem that the present paper aims at solving.

The remainder of the paper is organised as follows: In Section 2 we provide the required background on the underlying classical DL we consider in this work and we fix the notation and terminology we shall follow. In Section 3 we introduce $\mathcal{ALCH}^{\bullet}$, a defeasible DL for reasoning about typicality in class- and relation-membership, and we show some of its properties. Section 4 is devoted to the definition of a tableau-based proof procedure for checking satisfiability of $\mathcal{ALCH}^{\bullet}$ knowledge bases. In particular, we show correctness of our tableau algorithm w.r.t. a notion of preferential satisfiability. Finally, after a discussion of, and comparison with, related work (Section 5), we conclude with a summary of our contributions and some directions for further investigation.

2 Logical preliminaries

In this work, we take as point of departure the underlying language of the description logic $\mathcal{ALCH}$, which is the DL $\mathcal{ALC}$ extended with atomic-role hierarchies.[1]

[1]For the reader conversant with modal logics, roughly, $\mathcal{ALCH}$ corresponds to multi-modal logic K allowing for modalities to be dependently axiomatised.

The (concept) language of $\mathcal{ALCH}$ is built upon a finite set of atomic *concept names* C, a finite set of *role names* R (a.k.a. *attributes*) and a finite set of *individual names* I such that C, R and I are pairwise disjoint. In our scenario example, we can have for instance $\mathsf{C} = \{\mathsf{Employee}, \mathsf{Company}, \mathsf{Student}, \mathsf{EmpStudent}, \mathsf{Parent}, \mathsf{Tax}\}$, $\mathsf{R} = \{\mathsf{pays}, \mathsf{employedBy}, \mathsf{worksFor}\}$, and $\mathsf{I} = \{\mathsf{john}, \mathsf{ibm}\}$, with the respective obvious intuitions. With $A, B, \ldots$ we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts are denoted with $C, D, \ldots$ and are built using the constructors $\neg$ (complement), $\sqcap$ (concept conjunction), $\sqcup$ (concept disjunction), $\forall$ (value restriction) and $\exists$ (existential restriction) according to the following grammar rules:

$$C ::= \top \mid \bot \mid \mathsf{C} \mid (\neg C) \mid (C \sqcap C) \mid (C \sqcup C) \mid \forall r.C \mid \exists r.C$$

With $\mathcal{L}$ we denote the *language* of all $\mathcal{ALCH}$ concepts, which is understood as the smallest set of symbol sequences generated according to the rules above. When writing down concepts of $\mathcal{L}$, we shall follow the usual convention and omit parentheses whenever they are not essential for disambiguation. Examples of $\mathcal{ALCH}$ concepts are $\mathsf{Student} \sqcap \mathsf{Employee}$ and $\neg\exists\mathsf{pays}.\mathsf{Tax}$.

The semantics of $\mathcal{ALCH}$ is the standard set-theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping concept names A to subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, role names r to binary relations $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$, and individual names a to elements of the domain $\Delta^{\mathcal{I}}$, i.e., $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

Let $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ be an interpretation and define $r^{\mathcal{I}}(x) := \{y \mid (x, y) \in r^{\mathcal{I}}\}$, for $r \in \mathsf{R}$. We extend the interpretation function $\cdot^{\mathcal{I}}$ to interpret complex concepts of $\mathcal{L}$ as follows:

$$\top^{\mathcal{I}} := \Delta^{\mathcal{I}}, \quad \bot^{\mathcal{I}} := \emptyset, \quad (\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$$
$$(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}, \quad (C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\forall r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \subseteq C^{\mathcal{I}}\}, \quad (\exists r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \cap C^{\mathcal{I}} \neq \emptyset\}$$

Given $C, D \in \mathcal{L}$, $C \sqsubseteq D$ is called a *subsumption statement*, or *general concept inclusion* (GCI), read "C is subsumed by D". A concrete example of GCI is $\mathsf{EmpStudent} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Employee}$. $C \equiv D$ is an abbreviation for both $C \sqsubseteq D$ and $D \sqsubseteq C$. An $\mathcal{ALCH}$ *TBox* $\mathcal{T}$ is a finite set of subsumption statements and formalises the *intensional* knowledge about a given domain of application. Given $r, s \in \mathsf{R}$, a statement of the form $r \sqsubseteq s$ is a *role inclusion axiom* (RIA). An example of RIA is $\mathsf{worksFor} \sqsubseteq \mathsf{employedBy}$. An $\mathcal{ALCH}$ *RBox* $\mathcal{R}$ is a finite set of RIAs. Given $C \in \mathcal{L}$, $r \in \mathsf{R}$ and $a, b \in \mathsf{I}$, an *assertional statement* (*assertion*, for short) is an expression of the form $a : C$ or $(a, b) : r$. Examples of assertions are $\mathsf{john} : \mathsf{EmpStudent}$ and $(\mathsf{john}, \mathsf{ibm}) : \mathsf{worksFor}$. An $\mathcal{ALCH}$ *ABox* $\mathcal{A}$ is a finite set of assertional statements formalising the *extensional* knowledge of the domain. We shall denote statements with $\alpha, \beta, \ldots$. Given $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$, with $\mathcal{KB} := \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ we denote an $\mathcal{ALCH}$ *knowledge base*, a.k.a. an *ontology*.

An interpretation $\mathcal{I}$ *satisfies* a GCI $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. (And then $\mathcal{I} \Vdash C \equiv D$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$.) $\mathcal{I}$ *satisfies* a RIA $r \sqsubseteq s$ (denoted $\mathcal{I} \Vdash r \sqsubseteq s$) if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *satisfies* an assertion $a : C$ (respectively, $(a, b) : r$), denoted $\mathcal{I} \Vdash a : C$ (respectively, $\mathcal{I} \Vdash (a, b) : r$), if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (respectively, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$).

We say that an interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ (respectively, of an RBox $\mathcal{R}$ and of an ABox $\mathcal{A}$), denoted $\mathcal{I} \Vdash \mathcal{T}$ (respectively, $\mathcal{I} \Vdash \mathcal{R}$ and $\mathcal{I} \Vdash \mathcal{A}$) if $\mathcal{I} \Vdash \alpha$ for every α in $\mathcal{T}$ (respectively, in $\mathcal{R}$ and in $\mathcal{A}$). We say that $\mathcal{I}$ is a model of a knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ if $\mathcal{I} \Vdash \mathcal{T}$, $\mathcal{I} \Vdash \mathcal{R}$ and $\mathcal{I} \Vdash \mathcal{A}$. A statement α is (classically) *entailed* by a knowledge base $\mathcal{KB}$, denoted $\mathcal{KB} \models \alpha$, if every model of $\mathcal{KB}$ satisfies α.[2] If $\mathcal{KB} = \emptyset$, then we have that $\mathcal{I} \Vdash \alpha$ for all interpretations $\mathcal{I}$, in which case we say α is a validity and denote with $\models \alpha$.

For more details on Description Logics, the reader is invited to consult the Description Logic handbook [1].

3 The defeasible description logic $\mathcal{ALCH}^{\bullet}$

We now enrich the description logic $\mathcal{ALCH}$ with a typicality operator $\bullet$, applicable to *both* concepts and roles, and of which the intuition is to capture the most typical instances of a class or a relation.

Let C, R and I, as well as the way we denote their respective elements, be as before. The complex roles of $\mathcal{ALCH}^{\bullet}$ are denoted with $R, S, \ldots$ and are defined by the rule:

$$R ::= \mathsf{R} \mid \bullet R$$

Complex $\mathcal{ALCH}^{\bullet}$ concepts are denoted with $C, D, \ldots$ and are built according to the rule:

$$C ::= \top \mid \bot \mid \mathsf{C} \mid (\neg C) \mid (\bullet C) \mid (C \sqcap C) \mid (C \sqcup C) \mid (\forall R.C) \mid (\exists R.C)$$

With $\mathcal{L}^{\bullet}$ we denote the *language* of all $\mathcal{ALCH}^{\bullet}$ concepts (including the $\bullet$-less $\mathcal{ALCH}$ concepts from Section 2), which is understood as the smallest set of symbol sequences generated according to the rules above. When writing down elements of $\mathcal{L}^{\bullet}$, we shall omit parentheses whenever they are not essential for disambiguation. Examples of $\mathcal{ALCH}^{\bullet}$ concepts are $\bullet\mathsf{Student} \sqcap \neg\exists\mathsf{pays.Tax}$ and $\exists\bullet\mathsf{worksFor.Company}$.

The semantics of $\mathcal{ALCH}^{\bullet}$ is in terms of DL interpretations enriched with two partial orders, one on objects and one on pairs of objects:

Definition 1 (Bi-Ordered Interpretation) *An $\mathcal{ALCH}^{\bullet}$ bi-ordered interpretation is a tuple $\mathcal{B} := \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$ such that:*

[2] Hence, DL entailment corresponds to *global consequence* in modal logics [5].

- $\langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}} \rangle$ *is an* $\mathcal{ALCH}$ *interpretation, with* $A^{\mathcal{B}} \subseteq \Delta^{\mathcal{B}}$*, for each* $A \in \mathsf{C}$*,* $r^{\mathcal{B}} \subseteq \Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}$*, for each* $r \in \mathsf{R}$*, and* $a^{\mathcal{B}} \in \Delta^{\mathcal{B}}$*, for each* $a \in \mathsf{I}$*;*
- $<^{\mathcal{B}} \subseteq \Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}$*;*
- $\ll^{\mathcal{B}} \subseteq (\Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}) \times (\Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}})$*, and*
- *Both* $<^{\mathcal{B}}$ *and* $\ll^{\mathcal{B}}$ *are well-founded strict partial orders.*

Given $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$, the intuition of $\Delta^{\mathcal{B}}$ and $\cdot^{\mathcal{B}}$ is the same as in a standard DL interpretation. The intuition underlying the orderings $<^{\mathcal{B}}$ and $\ll^{\mathcal{B}}$ is that they play the role of *preference relations* (or *normality orderings*), in a sense similar to that introduced by Shoham [45] with a preference on worlds in a propositional setting and as investigated by Kraus et al. [34, 35] and others [13, 15, 25]: the objects (respectively, pairs) x (respectively, (x, y)) that are lower down in the ordering $<^{\mathcal{B}}$ (respectively, $\ll^{\mathcal{B}}$) are deemed as the most normal (or typical, or expected, or conventional, depending on the application one is modeling) in the context of a concept (respectively, role) interpretation.

Definition 2 (Semantics of $\mathcal{L}^{\bullet}$) *A bi-ordered interpretation* $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$ *interprets the classical constructors in the usual way, i.e.,* $\top^{\mathcal{B}} := \Delta^{\mathcal{B}}$*,* $\bot^{\mathcal{B}} := \emptyset$*,* $(\neg C)^{\mathcal{B}} := \Delta^{\mathcal{B}} \setminus C^{\mathcal{B}}$*,* $(C \sqcap D)^{\mathcal{B}} := C^{\mathcal{B}} \cap D^{\mathcal{B}}$*,* $(C \sqcup D)^{\mathcal{B}} := C^{\mathcal{B}} \cup D^{\mathcal{B}}$*,* $(\forall R.C)^{\mathcal{B}} := \{x \mid R^{\mathcal{B}}(x) \subseteq C^{\mathcal{B}}\}$ *and* $(\exists R.C)^{\mathcal{B}} := \{x \mid R^{\mathcal{B}}(x) \cap C^{\mathcal{B}} \neq \emptyset\}$*. Typicality-based concepts and roles are interpreted as follows:*

- $(\bullet C)^{\mathcal{B}} := \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$
- $(\bullet r)^{\mathcal{B}} := \min_{\ll^{\mathcal{B}}} r^{\mathcal{B}}$

Hence, under our semantics, to be a typical representative of a class (respectively, relationship) amounts to being amongst the most preferred elements in that class (respectively, relation).

The definitions of GCIs, RIAs, TBox, RBox, ABox and knowledge bases are extended to $\mathcal{ALCH}^{\bullet}$ in the expected way: Given $C, D \in \mathcal{L}^{\bullet}$, $C \sqsubseteq D$ is a GCI; an $\mathcal{ALCH}^{\bullet}$ TBox $\mathcal{T}$ is a finite set of GCIs; given (possibly complex) roles R and S, $R \sqsubseteq S$ is a RIA; an $\mathcal{ALCH}^{\bullet}$ RBox $\mathcal{R}$ is a finite set of RIAs; given $C \in \mathcal{L}^{\bullet}$, R a role and $a, b \in \mathsf{I}$, $a : C$ and $(a, b) : R$ are assertions; moreover, from now on we shall also allow for assertions of the form $(a, b) : \neg R$. An $\mathcal{ALCH}^{\bullet}$ ABox $\mathcal{A}$ is a finite set of assertions. Again, statements are denoted by $\alpha, \beta, \ldots$. With $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ we denote an $\mathcal{ALCH}^{\bullet}$ *knowledge base*, of which the following is an example:

$$\mathcal{T} = \left\{ \begin{array}{c} \mathsf{EmpStudent} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Employee}, \\ \bullet\mathsf{Student} \sqsubseteq \neg\exists \mathsf{pays}.\mathsf{Tax}, \\ \bullet\mathsf{EmpStudent} \sqsubseteq \exists \mathsf{pays}.\mathsf{Tax} \sqcap \neg\bullet\mathsf{Employee}, \\ \mathsf{EmpStudent} \sqcap \mathsf{Parent} \sqsubseteq \bullet\neg\exists \mathsf{pays}.\mathsf{Tax}, \\ \bullet\mathsf{Employee} \sqsubseteq \exists\bullet\mathsf{worksFor}.\mathsf{Company} \end{array} \right\}$$

$$\mathcal{R} = \{\bullet\mathsf{worksFor} \sqsubseteq \mathsf{employedBy}\}$$

$$\mathcal{A} = \{\mathsf{john} : \mathsf{EmpStudent}, \mathsf{john} : \mathsf{Parent}, (\mathsf{john}, \mathsf{ibm}) : \bullet\mathsf{worksFor}\}$$

Definition 3 (Satisfaction) *Let $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$, R a role, $C, D \in \mathcal{L}^{\bullet}$, and $a, b \in \mathsf{I}$. The satisfaction relation $\Vdash$ is defined as follows:*

- $\mathcal{B} \Vdash C \sqsubseteq D$ *if* $C^{\mathcal{B}} \subseteq D^{\mathcal{B}}$;
- $\mathcal{B} \Vdash R \sqsubseteq S$ *if* $R^{\mathcal{B}} \subseteq S^{\mathcal{B}}$;
- $\mathcal{B} \Vdash a : C$ *if* $a^{\mathcal{B}} \in C^{\mathcal{B}}$;
- $\mathcal{B} \Vdash (a, b) : R$ *if* $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in R^{\mathcal{B}}$;
- $\mathcal{B} \Vdash (a, b) : \neg R$ *if* $(a^{\mathcal{B}}, b^{\mathcal{B}}) \notin R^{\mathcal{B}}$.

If $\mathcal{B} \Vdash \alpha$, then we say $\mathcal{B}$ satisfies α. $\mathcal{B}$ satisfies an $\mathcal{ALCH}^{\bullet}$ knowledge base $\mathcal{KB}$, written $\mathcal{B} \Vdash \mathcal{KB}$, if $\mathcal{B} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$, in which case we say $\mathcal{B}$ is a model of $\mathcal{KB}$. We say $C \in \mathcal{L}^{\bullet}$ is satisfiable w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{B}$ of $\mathcal{KB}$ s.t. $C^{\mathcal{B}} \neq \emptyset$.

Given a bi-ordered interpretation $\mathcal{B}$, it is worth observing that (cf. Definition 2):

$$\mathcal{B} \Vdash a : \bullet C \text{ iff } \mathcal{B} \Vdash b : \neg C \text{ for every } b \text{ s.t. } b^{\mathcal{B}} <^{\mathcal{B}} a^{\mathcal{B}} \tag{1}$$

$$\mathcal{B} \Vdash (a, b) : \bullet R \text{ iff } \mathcal{B} \Vdash (c, d) : \neg R \text{ for every } (c, d) \text{ s.t. } (c^{\mathcal{B}}, d^{\mathcal{B}}) \ll^{\mathcal{B}} (a^{\mathcal{B}}, b^{\mathcal{B}}) \tag{2}$$

It is easy to see that the addition of the orderings preserves the truth of all classical (i.e., $\bullet$-less) statements holding in the remaining structure:

Lemma 1 *Let $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$, and define $\mathcal{I}_{\mathcal{B}} := \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}} \rangle$. For every $C, D \in \mathcal{L}$, every $r, s \in \mathsf{R}$ and every $a, b \in \mathsf{I}$:*

- $\mathcal{B} \Vdash C \sqsubseteq D$ *iff* $\mathcal{I}_{\mathcal{B}} \Vdash C \sqsubseteq D$;
- $\mathcal{B} \Vdash r \sqsubseteq s$ *iff* $\mathcal{I}_{\mathcal{B}} \Vdash r \sqsubseteq s$;
- $\mathcal{B} \Vdash a : C$ *iff* $\mathcal{I}_{\mathcal{B}} \Vdash a : C$;
- $\mathcal{B} \Vdash (a, b) : r$ *iff* $\mathcal{I}_{\mathcal{B}} \Vdash (a, b) : r$.

Furthermore, it is not hard to check that our typicality operators are *idempotent*:

Lemma 2 *Let $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$. For every $C \in \mathcal{L}^{\bullet}$ and every role R:*

- $\mathcal{B} \Vdash \bullet\bullet C \equiv \bullet C$;
- $\mathcal{B} \Vdash \bullet\bullet R \equiv \bullet R$.

One of the consequences of Lemma 2 is that we can assume w.l.o.g. that typicality for roles does not occur nested in the knowledge base, a hypothesis that will turn out useful in Section 4. (In principle, we can make the same assumption about concepts, but, besides being unnecessary here, its argument is more intricate [12] and requires the addition of new concept names to the signature.)

Proposition 1 *Let $\mathcal{B}$ be a bi-ordered interpretation and let $C, D \in \mathcal{L}^\bullet$. Then*

1. $\mathcal{B} \Vdash \bullet(\bullet C \sqcap \bullet D) \equiv \bullet C \sqcap \bullet D$;
2. $\mathcal{B} \Vdash \bullet C \sqcap \bullet D \sqsubseteq \bullet(C \sqcap D)$;
3. *If* $\mathcal{B} \not\Vdash \bullet C \sqcap \bullet D \sqsubseteq \bot$, *then* $\mathcal{B} \Vdash \bullet(C \sqcap D) \sqsubseteq \bullet C \sqcap \bullet D$.

Proof:
(1) The left-to-right inclusion follows from $\text{Ref}_\mathcal{T}$ below (cf. Proposition 3). For the right-to-left one, let $x \in (\bullet C \sqcap \bullet D)^\mathcal{B}$. Then $x \in (\bullet C)^\mathcal{B}$ and $x \in (\bullet D)^\mathcal{B}$, i.e., $x \in \min_{<^\mathcal{B}} C^\mathcal{B}$ and $x \in \min_{<^\mathcal{B}} D^\mathcal{B}$. Assume $x \notin (\bullet(\bullet C \sqcap \bullet D))^\mathcal{B}$. In this case, there is $y \in (\bullet C \sqcap \bullet D)^\mathcal{B}$ s.t. $y <^\mathcal{B} x$. Then we have $y \in C^\mathcal{B}$ and $y \in D^\mathcal{B}$, and since $y <^\mathcal{B} x$, we get a contradiction.
(2) Let $x \in (\bullet C \sqcap \bullet D)^\mathcal{B}$. Then $x \in (\bullet C)^\mathcal{B}$ and $x \in (\bullet D)^\mathcal{B}$, i.e., $x \in \min_{<^\mathcal{B}} C^\mathcal{B}$ and $x \in \min_{<^\mathcal{B}} D^\mathcal{B}$. Assume $x \notin (\bullet(C \sqcap D))^\mathcal{B}$. Therefore there is $y \in (C \sqcap D)^\mathcal{B}$ s.t. $y <^\mathcal{B} x$, and this leads to a contradiction.
(3) Let $x \in (\bullet(C \sqcap D))^\mathcal{B}$, i.e., $x \in \min_{<^\mathcal{B}}(C \sqcap D)^\mathcal{B}$, and assume either $x \notin (\bullet C)^\mathcal{B}$ or $x \notin (\bullet D)^\mathcal{B}$. If $x \notin (\bullet C)^\mathcal{B}$, then, since $<^\mathcal{B}$ is well-founded, we know there is $y \in \min_{<^\mathcal{B}} C^\mathcal{B}$ s.t. $y <^\mathcal{B} x$. We claim $y \notin (\bullet D)^\mathcal{B}$; for if it were the case, then we would get $y \in (C \sqcap D)^\mathcal{B}$ and $y <^\mathcal{B} x$, leading us to a contradiction. Hence $(\bullet C)^\mathcal{B} \cap (\bullet D)^\mathcal{B} = \emptyset$, and therefore $\mathcal{B} \Vdash \bullet C \sqcap \bullet D \sqsubseteq \bot$. If $x \notin (\bullet D)^\mathcal{B}$, we reach the same conclusion through an analogous argument. ■

Obviously, the concepts $\neg\bullet C$ and $\bullet\neg C$ do not mean the same, at least not in general. As a result, in the concept $\neg\bullet A$, negation cannot be pushed further inwards. This has as consequence that there can be no negated normal form (NNF) in the usual sense for $\mathcal{L}^\bullet$.

As expected, typicality operators are *non-monotonic*:

Proposition 2 *Let $C, D \in \mathcal{L}^\bullet$ and R, S be roles. It is* ***not*** *the case that, for every bi-ordered interpretation $\mathcal{B}$:*

- *If* $\mathcal{B} \Vdash C \sqsubseteq D$, *then* $\mathcal{B} \Vdash \bullet C \sqsubseteq \bullet D$, *and*
- *If* $\mathcal{B} \Vdash R \sqsubseteq S$, *then* $\mathcal{B} \Vdash \bullet R \sqsubseteq \bullet S$.

Proof:
Let $\mathsf{C} = \{A_1, A_2\}$ and $\mathsf{R} = \{r_1, r_2\}$, and let $\mathcal{B} = \langle \Delta^\mathcal{B}, \cdot^\mathcal{B}, <^\mathcal{B}, \ll^\mathcal{B} \rangle$, with $\Delta^\mathcal{B} = \{x_1, x_2, x_3\}$, $A_1^\mathcal{B} = \{x_1\}$, $A_2^\mathcal{B} = \Delta^\mathcal{B}$, $r_1^\mathcal{B} = \{(x_1, x_2)\}$, $r_2^\mathcal{B} = \{(x_2, x_3)\}$, $<^\mathcal{B} = \{(x_3, x_1)\}$ and $\ll^\mathcal{B} =$

$\{((x_2, x_3), (x_1, x_2))\}$. Then $\mathcal{B} \Vdash A_1 \sqsubseteq A_2$ and $\mathcal{B} \Vdash r_1 \sqsubseteq r_2$, but $\mathcal{B} \not\Vdash \bullet A_1 \sqsubseteq \bullet A_2$ and $\mathcal{B} \not\Vdash \bullet r_1 \sqsubseteq \bullet r_2$ ■

Another consequence of our preferential semantics, but also of the fact we assume a semantic framework as general as possible, is the fact that, as can easily be verified, there are bi-ordered interpretations $\mathcal{B}$ such that:

- $\mathcal{B} \Vdash \bullet C \sqsubseteq D$ but neither $\mathcal{B} \Vdash \bullet\exists R.C \sqsubseteq \exists R.D$ nor $\mathcal{B} \Vdash \exists\bullet R.C \sqsubseteq \exists R.D$;
- $\mathcal{B} \Vdash \bullet R \sqsubseteq S$ but $\mathcal{B} \not\Vdash \bullet\exists R.C \sqsubseteq \exists S.D$;
- Either $\mathcal{B} \not\Vdash \bullet\exists R.C \sqsubseteq \exists\bullet R.C$ or $\mathcal{B} \not\Vdash \exists\bullet R.C \sqsubseteq \bullet\exists R.C$, or both.

Since they are elementary non-monotonic operators, our typicality operators can be used to define further non-monotonic constructs. An interesting example is the notion of defeasible subsumption of the forms $C \precsim D$ [15, 17, 22], for $C, D \in \mathcal{L}^\bullet$, and $R \precsim S$ [18, 20], for R, S roles, and that we can see as abbreviations for, respectively, the $\mathcal{L}^\bullet$-GCI $\bullet C \sqsubseteq D$ and the $\mathcal{L}^\bullet$-RIA $\bullet R \sqsubseteq S$. (Note that both versions of $\precsim$ are defined for full $\mathcal{ALCH}^\bullet$ and that $\bullet$ may also occur on the RHS of such statements.) That this characterisation of defeasible subsumption is appropriate from the NMR point of view is witnessed by the following result:

Proposition 3 *For every bi-ordered interpretation $\mathcal{B}$, every $C, D, E \in \mathcal{L}^\bullet$, and every role R, S, T, the following properties hold:*

$$(Ref_{\mathcal{T}})\ \mathcal{B} \Vdash C \precsim C \qquad\qquad (Ref_{\mathcal{R}})\ \mathcal{B} \Vdash R \precsim R$$

$$(LLE)\ \frac{\mathcal{B} \Vdash C \equiv D,\ \mathcal{B} \Vdash C \precsim E}{\mathcal{B} \Vdash D \precsim E} \qquad (And)\ \frac{\mathcal{B} \Vdash C \precsim D,\ \mathcal{B} \Vdash C \precsim E}{\mathcal{B} \Vdash C \precsim D \sqcap E}$$

$$(Or)\ \frac{\mathcal{B} \Vdash C \precsim E,\ \mathcal{B} \Vdash D \precsim E}{\mathcal{B} \Vdash C \sqcup D \precsim E} \qquad (RW_{\mathcal{T}})\ \frac{\mathcal{B} \Vdash C \precsim D,\ \mathcal{B} \Vdash D \sqsubseteq E}{\mathcal{B} \Vdash C \precsim E}$$

$$(RW_{\mathcal{R}})\ \frac{\mathcal{B} \Vdash R \precsim S,\ \mathcal{B} \Vdash S \sqsubseteq T}{\mathcal{B} \Vdash R \precsim T} \qquad (CM)\ \frac{\mathcal{B} \Vdash C \precsim D,\ \mathcal{B} \Vdash C \precsim E}{\mathcal{B} \Vdash C \sqcap D \precsim E}$$

Proof:
(Ref$_{\mathcal{T}}$): Let $x \in \Delta^{\mathcal{B}}$ be such that $x \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$. Then clearly $x \in C^{\mathcal{B}}$ and therefore $\mathcal{B} \Vdash C \precsim C$.

(Ref$_{\mathcal{R}}$): Analogous to (Ref$_{\mathcal{T}}$) above.

(LLE): Assume that $\mathcal{B} \Vdash C \precsim E$ and $\mathcal{B} \Vdash C \equiv D$. Then $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq E^{\mathcal{B}}$. Since $\mathcal{B} \Vdash C \equiv D$, we have $C^{\mathcal{B}} = D^{\mathcal{B}}$, and therefore $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} = \min_{<^{\mathcal{B}}} D^{\mathcal{B}}$. Hence $\min_{<^{\mathcal{B}}} D^{\mathcal{B}} \subseteq E^{\mathcal{B}}$, and therefore $\mathcal{B} \Vdash D \precsim E$.

(And): Assume we have both $\mathcal{B} \Vdash C \precsim D$ and $\mathcal{B} \Vdash C \precsim E$, i.e., $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq D^{\mathcal{B}}$ and $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq E^{\mathcal{B}}$, and then $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq D^{\mathcal{B}} \cap E^{\mathcal{B}}$, from which follows $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq (D \sqcap E)^{\mathcal{B}}$. Hence $\mathcal{B} \Vdash C \precsim D \sqcap E$.

(Or): Assume we have both $\mathcal{B} \Vdash C \precsim E$ and $\mathcal{B} \Vdash D \precsim E$. Let $x \in \min_{<^{\mathcal{B}}}(C \sqcup D)^{\mathcal{B}}$. Then x is minimal in $C^{\mathcal{B}} \cup D^{\mathcal{B}}$, and therefore either $x \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$ or $x \in \min_{<^{\mathcal{B}}} D^{\mathcal{B}}$. In either case $x \in E^{\mathcal{B}}$. Hence $\mathcal{B} \Vdash C \sqcup D \precsim E$.

($\text{RW}_{\mathcal{T}}$): Assume we have both $\mathcal{B} \Vdash C \precsim D$ and $\mathcal{B} \Vdash D \sqsubseteq E$, i.e., $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq D^{\mathcal{B}}$ and $D^{\mathcal{B}} \subseteq E^{\mathcal{B}}$. Hence $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq E^{\mathcal{B}}$ and therefore $\mathcal{B} \Vdash C \precsim E$.

($\text{RW}_{\mathcal{R}}$): Analogous to ($\text{RW}_{\mathcal{T}}$) above.

(CM): Assume we have both $\mathcal{B} \Vdash C \precsim D$ and $\mathcal{B} \Vdash C \precsim E$. Then $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq D^{\mathcal{B}}$ and $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq E^{\mathcal{B}}$. Let $x \in \min_{<^{\mathcal{B}}}(C \sqcap D)^{\mathcal{B}}$. We show that $x \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$. Suppose this is not the case. Since $<^{\mathcal{B}}$ is well-founded, there must be $x' \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$ s.t. $x' <^{\mathcal{B}} x$. Because $\mathcal{B} \Vdash C \precsim D$, $x' \in D^{\mathcal{B}}$, and then $x' \in C^{\mathcal{B}} \cap D^{\mathcal{B}}$, i.e., $x' \in (C \sqcap D)^{\mathcal{B}}$. From this and $x' <^{\mathcal{B}} x$ it follows that x is not minimal in $(C \sqcap D)^{\mathcal{B}}$, which is a contradiction. Hence $x \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$. From this and $\min_{<^{\mathcal{B}}} C^{\mathcal{B}} \subseteq E^{\mathcal{B}}$, it follows that $x \in E^{\mathcal{B}}$. Hence $\mathcal{B} \Vdash C \sqcap D \precsim E$. ■

That is, defining $\precsim$ for both concepts and roles in terms of $\bullet$, thereby giving it a semantics in terms of our bi-ordered interpretations, delivers a notion of defeasible subsumption satisfying the ($\mathcal{ALCH}^{\bullet}$ versions of the) KLM properties for preferential consequence relations [34]. These properties are usually seen as formalising the minimal requirements that any appropriate notion of defeasible consequence (of which $\precsim$ is an instance) is supposed to satisfy. They have been discussed at length in the literature on non-monotonic reasoning for both the propositional and the DL cases [15, 17, 27, 28, 34, 35] and therefore we shall not repeat so here.

Let $\mathcal{KB}$ be an $\mathcal{ALCH}^{\bullet}$ knowledge base and α a statement. We say $\mathcal{KB}$ *entails* α, denoted $\mathcal{KB} \models \alpha$, if $\mathcal{B} \Vdash \alpha$ for every $\mathcal{B}$ such that $\mathcal{B} \Vdash \mathcal{KB}$. In the case $\mathcal{KB} = \emptyset$, we say α is *preferentially valid* and denote it as $\models \alpha$. Assuming the example knowledge base on Page 5, we have $\mathcal{KB} \models \mathsf{john} : \neg\exists \mathsf{pays}.\mathsf{Tax}$.

The following result will come in handy in the definition of a tableau system in Section 4, as it shows that all reasoning problems for $\mathcal{ALCH}^{\bullet}$ can be reduced to knowledge base satisfiability. Its proof is analogous to that of its classical counterpart in the DL literature and we shall omit it here:

Lemma 3 *Let $\mathcal{KB}$ be an $\mathcal{ALCH}^{\bullet}$ knowledge base and let a be an individual name not occurring in $\mathcal{KB}$. For every $C, D \in \mathcal{L}^{\bullet}$, $\mathcal{KB} \models C \sqsubseteq D$ iff $\mathcal{KB} \models C \sqcap \neg D \sqsubseteq \bot$ iff $\mathcal{KB} \cup \{a : C \sqcap \neg D\} \models \bot$. Moreover, for every $b, c \in \mathsf{I}$, $\mathcal{KB} \models b : C$ iff $\mathcal{KB} \cup \{b : \neg C\} \models \bot$, and $\mathcal{KB} \models (b, c) : R$ iff $\mathcal{KB} \cup \{(b, c) : \neg R\} \models \bot$.*

4 Tableaux for preferential reasoning in $\mathcal{ALCH}^{\bullet}$

In this section, we define a tableau-based algorithm for deciding consistency of an $\mathcal{ALCH}^{\bullet}$ knowledge base. Our main purpose is to show the existence of a proof procedure for $\mathcal{ALCH}^{\bullet}$ that is sound and complete w.r.t. our preferential semantics and therefore we shall not concern ourselves with optimisation matters. (Our terminology and presentation follow those by Baader et al. [4] in the classical case.)

We start by observing that, for every bi-ordered interpretation $\mathcal{B}$ and every $C, D \in \mathcal{L}^{\bullet}$, $\mathcal{B} \Vdash C \sqsubseteq D$ if and only if $\mathcal{B} \Vdash \top \sqsubseteq \neg C \sqcup D$. In that respect, we can assume w.l.o.g. that all GCIs in a TBox are of the form $\top \sqsubseteq E$, for some $E \in \mathcal{L}^{\bullet}$. As we shall see, this assumption will simplify matters when handling the information in a TBox in the tableau rules.

Note also that we can assume w.l.o.g. that the ABox is not empty, for if it is, one can add to it the vacuous assertion $a : \top$, for some new individual name a. It is easy to see that the resulting (non-empty) ABox is preferentially equivalent to the original one.

Next, we define a few notions that will be useful in the remaining of the section.

Definition 4 (Subconcepts) *Let $C \in \mathcal{L}^{\bullet}$. The set of subconcepts of C, denoted* $\mathsf{sub}(C)$ *is inductively defined as follows:*

- *If $C = A \in \mathsf{C} \cup \{\top, \bot\}$, then* $\mathsf{sub}(C) = \{A\}$*;*
- *If $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$, then* $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)$*;*
- *If $C = \neg D$ or $C = \bullet D$ or $C = \exists r.D$ or $C = \forall r.D$, then* $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(D)$.

Given a knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$, the set of subconcepts of $\mathcal{KB}$ is defined as $\mathsf{sub}(\mathcal{KB}) := \mathsf{sub}(\mathcal{T}) \cup \mathsf{sub}(\mathcal{A})$*, where*

$$\mathsf{sub}(\mathcal{T}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} (\mathsf{sub}(C) \cup \mathsf{sub}(D)), \quad \mathsf{sub}(\mathcal{A}) := \bigcup_{a:C \in \mathcal{A}} \mathsf{sub}(C)$$

Definition 5 (a-Concepts and (a, b)-Roles) *Let $\mathcal{A}$ be an ABox and let a, b be individual names appearing in $\mathcal{A}$. With* $\mathsf{con}_{\mathcal{A}}(a) := \{C \mid a : C \in \mathcal{A}\}$ *we denote the set of concepts that a is an instance of w.r.t. $\mathcal{A}$; with* $\mathsf{roles}_{\mathcal{A}}(a, b) := \{R \mid (a, b) : R \in \mathcal{A}\} \cup \{\neg R \mid (a, b) : \neg R \in \mathcal{A}\}$ *we denote the set of roles instantiated by (a, b) w.r.t. $\mathcal{A}$.*[3]

Definition 6 (Ancestor) *Let $\mathcal{A}$ be an ABox. For every $a, b \in \mathsf{I}$ and $r \in \mathsf{R}$, if $(a, b) : r \in \mathcal{A}$, we say b is a (r-) successor of a and a is a predecessor of b. We call ancestor (respectively, descendant) the transitive closure of predecessor (respectively, successor). An individual is called root if it has no ancestor.*

[3]Of course, our language does not have Boolean role constructors [1, Chapter 5] and therefore, strictly speaking, $\neg R$ is not a role. Here we shall abuse notation as it will ease the presentation.

The following definition is used to ensure termination:

Definition 7 (Blocking) *Let $\mathcal{A}$ be an ABox and $a, b \in \mathsf{I}$. We say that a is blocked by b in $\mathcal{A}$ if (*i*) b is an ancestor of a, and (*ii*) $\mathsf{con}_{\mathcal{A}}(a) \subseteq \mathsf{con}_{\mathcal{A}}(b)$. We say a is blocked in $\mathcal{A}$ if itself or some of its ancestors is blocked by some individual name.*

We are now ready for the definition of the expansion rules for $\mathcal{ALCH}^{\bullet}$-concepts. The classical expansion rules are shown in Figure 1, whereas the rules handling typicality-based constructs are shown in Figure 2.

$\neg$-**rule**:	**if**	1. $a : \neg\neg C \in \mathcal{A}$, and 2. $a : C \notin \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : C\}$
$\sqcap^{+}$-**rule**:	**if**	1. $a : C \sqcap D \in \mathcal{A}$, and 2. $\{a : C, a : D\} \not\subseteq \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : C, a : D\}$
$\sqcup^{+}$-**rule**:	**if**	1. $a : C \sqcup D \in \mathcal{A}$, and 2. $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : E\}$, for some $E \in \{C, D\}$
$\sqcap^{-}$-**rule**:	**if**	1. $a : \neg(C \sqcap D) \in \mathcal{A}$, and 2. $\{a : \neg C, a : \neg D\} \cap \mathcal{A} = \emptyset$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : E\}$, for some $E \in \{\neg C, \neg D\}$
$\sqcup^{-}$-**rule**:	**if**	1. $a : \neg(C \sqcup D) \in \mathcal{A}$, and 2. $\{a : \neg C, a : \neg D\} \not\subseteq \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : \neg C, a : \neg D\}$
$\sqsubseteq_{\mathcal{T}}$-**rule**:	**if**	1. $a : C \in \mathcal{A}$, $\top \sqsubseteq D \in \mathcal{T}$, and 2. $a : D \notin \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : D\}$
$\sqsubseteq_{\mathcal{R}}$-**rule**:	**if**	1. $(a, b) : R \in \mathcal{A}$, $R \sqsubseteq S \in \mathcal{R}$, and 2. $(a, b) : S \notin \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{(a, b) : S\}$
$\exists^{+}$-**rule**:	**if**	1. $a : \exists R.C \in \mathcal{A}$, and 2. there is no b s.t. $\{(a, b) : R, b : C\} \subseteq \mathcal{A}$, and 3. a is not blocked
	then	$\mathcal{A} := \mathcal{A} \cup \{(a, c) : R, c : C\}$, for c new in $\mathcal{A}$
$\forall^{+}$-**rule**:	**if**	1. $\{a : \forall R.C, (a, b) : R\} \subseteq \mathcal{A}$, and 2. $b : C \notin \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{b : C\}$
$\exists^{-}$-**rule**:	**if**	1. $\{a : \neg\exists R.C, (a, b) : R\} \subseteq \mathcal{A}$, and 2. $b : \neg C \notin \mathcal{A}$
	then	$\mathcal{A} := \mathcal{A} \cup \{b : \neg C\}$
$\forall^{-}$-**rule**:	**if**	1. $a : \neg\forall R.C \in \mathcal{A}$, and 2. there is no b s.t. $\{(a, b) : R, b : \neg C\} \subseteq \mathcal{A}$, and 3. a is not blocked
	then	$\mathcal{A} := \mathcal{A} \cup \{(a, c) : R, c : \neg C\}$, for c new in $\mathcal{A}$

Figure 1: Classical expansion rules for the $\mathcal{ALCH}^{\bullet}$ tableau.

The rules in Figure 1 are as in the classical case, except for the fact that concepts and roles in the scope of classical operators may contain the typicality operator $\bullet$.

In the $\mathcal{ALCH}^\bullet$ expansion rules we make use of two additional structures, namely $<$ and $\ll$ (see the rules in Figure 2). Their respective purpose is to build the skeleton of a preference relation on individual names and on pairs of individuals appearing in the ABox. In the unravelling of the complete clash-free ABox (see below), if there is any, $<$ and $\ll$ are used to define the preference relations in the constructed bi-ordered interpretation (see proof of Lemma 4 in Appendix A). We shall use $b < \ldots < a$ (respectively, $(c,d) \ll \ldots \ll (a,b)$) to denote the existence of a path from b to a (respectively, from (c,d) to (a,b)) in $<$ (respectively, $\ll$).

Rules $\bullet_C^+$ and $\bullet_r^+$ in Figure 2 take care of positive typical instances of, respectively, concepts and roles. First, they make sure that typical instances of concepts and roles are indeed instances thereof. Second, they ensure Properties (1) and (2) above (cf. paragraph following Definition 3).

Rule $\bullet_C^-$ handles non-typical instances of a concept. There are two possible reasons for an object not to be a typical member of a class C: either it is not in C, or it is, but there is another instance of C that is more preferred than it. This is captured by the or-like branch in the rule.

The $\bullet_r^-$-rule handles the non-typical instantiations of roles and its rationale is analogous to that of the $\bullet_C^-$-rule.

Finally, Rules $\bullet?_C$ and $\bullet?_r$ handle lack of information about an instance's typicality. If we know a is in C, but nothing about a being typical in C or not, then we have to explore two possibilities, namely, if a is a typical instance of C, and if it is not. An analogous reasoning holds for instances of a role.

Definition 8 (Complete and clash-free ABox) *Let $\mathcal{A}$ be an ABox. We say $\mathcal{A}$ contains a clash if there is $a \in \mathsf{I}$ and $C \in \mathcal{L}^\bullet$ such that $\{a : C, a : \neg C\} \subseteq \mathcal{A}$ or there are $a, b \in \mathsf{I}$ and a role R such that $\{(a,b) : R, (a,b) : \neg R\} \subseteq \mathcal{A}$. We say $\mathcal{A}$ is clash-free if it does not contain a clash. $\mathcal{A}$ is complete if contains a clash or if none of the expansion rules in Figures 1 and 2 is applicable to $\mathcal{A}$.*

Let $\mathsf{ndexp}(\cdot)$ denote a function taking as input a clash-free ABox $\mathcal{A}$, a nondeterministic rule ρ from Figures 1 and 2, and an assertion $\alpha \in \mathcal{A}$ such that ρ is applicable to α in $\mathcal{A}$. In our case, the nondeterministic rules are the $\sqcup^+$-, $\sqcap^-$- and $\bullet_C^-$-rules. The function returns a set $\mathsf{ndexp}(\mathcal{A}, \rho, \alpha)$ containing each of the possible ABoxes resulting from the application of ρ to α in $\mathcal{A}$.

The tableau-based procedure for checking consistency of an $\mathcal{ALCH}^\bullet$ knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ is given in Algorithm 1 below. It uses Function Expand to apply the rules in Figures 1 and 2 to $\mathcal{A}$ w.r.t. $\mathcal{T}$ and $\mathcal{R}$.

We can now state the main result of the present section.

Theorem 1 *Algorithm 1 is sound and complete w.r.t. preferential consistency of $\mathcal{ALCH}^\bullet$ knowledge bases.*

$\bullet^+_C$**-rule**:	**if**	1. $a : \bullet C \in \mathcal{A}$, and either
		2.1. $a : C \notin \mathcal{A}$ or
		2.2. $b : \neg C \notin \mathcal{A}$, for some b s.t. $b < \ldots < a$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : C, b : \neg C\}$
$\bullet^-_C$**-rule**:	**if**	1. $a : \neg\bullet C \in \mathcal{A}$, and
		2. $a : \neg C \notin \mathcal{A}$, and
		3. there is no b s.t. $b : C \in \mathcal{A}$ and $b < \ldots < a$
	then	(a) $\mathcal{A} := \mathcal{A} \cup \{a : \neg C\}$, **or**
		(b) $\mathcal{A} := \mathcal{A} \cup \{a : C, c : C\}$ and $< \, := \, < \cup \, \{(c, a)\}$, for c new in $\mathcal{A}$
$\bullet^+_r$**-rule**:	**if**	1. $(a, b) : \bullet r \in \mathcal{A}$, and either
		2.1. $(a, b) : r \notin \mathcal{A}$ or
		2.2. $(c, d) : \neg r \notin \mathcal{A}$, for some (c, d) s.t. $(c, d) \ll \ldots \ll (a, b)$
	then	$\mathcal{A} := \mathcal{A} \cup \{(a, b) : r, (c, d) : \neg r\}$
$\bullet^-_r$**-rule**:	**if**	1. $(a, b) : \neg\bullet r \in \mathcal{A}$, and
		2. $(a, b) : \neg r \notin \mathcal{A}$, and
		3. there are no c, d s.t. $(c, d) : r \in \mathcal{A}$ and $(c, d) \ll \ldots \ll (a, b)$
	then	(a) $\mathcal{A} := \mathcal{A} \cup \{(a, b) : \neg r\}$, **or**
		(b) $\mathcal{A} := \mathcal{A} \cup \{(a, b) : r, (e, f) : r\}$ and $\ll \, := \, \ll \cup \, \{((e, f), (a, b))\}$, for e, f new in $\mathcal{A}$
$\bullet ?_C$**-rule**:	**if**	1. $a : C \in \mathcal{A}$, and
		2. $\{a : \bullet C, a : \neg\bullet C\} \cap \mathcal{A} = \emptyset$
	then	$\mathcal{A} := \mathcal{A} \cup \{a : E\}$, for some $E \in \{\bullet C, \neg\bullet C\}$
$\bullet ?_r$**-rule**:	**if**	1. $(a, b) : r \in \mathcal{A}$, and
		2. $\{(a, b) : \bullet r, (a, b) : \neg\bullet r\} \cap \mathcal{A} = \emptyset$
	then	$\mathcal{A} := \mathcal{A} \cup \{(a, b) : R\}$, for some $R \in \{\bullet r, \neg\bullet r\}$

Figure 2: $\bullet$-based expansion rules for the $\mathcal{ALCH}^\bullet$ tableau.

Proof:
The result follows from Lemmas 4 and 5 in Appendix A. ∎

5 Related work

To the best of our knowledge, the first approach to an explicit notion of typicality in DLs was the one by Giordano et al. [25]. They introduced a typicality operator $\mathbf{T}(\cdot)$, applicable to *concepts* only, and for which they define a preferential semantics that is a special case of ours, in the sense that they place a preference relation only on objects of the domain. In their setting, a concept of the form $\mathbf{T}(C)$, understood as referring to the typical objects falling under C, serves as a macro for the sentence $C \sqcap \Box\neg C$ in a description language extended with a modality capturing the behaviour of a preference relation on objects. Hence, the intuition of $x \in (\mathbf{T}(C))^\mathcal{I} = (C \sqcap \Box\neg C)^\mathcal{I}$ is that x is an instance of C and any other object that is more preferred than x falls under $\neg C$. (This semantic characterisation can be shown to be analogous to the one we have given here if preferences on pair of objects are not taken into account.) It is worth pointing out, though, that in Giordano et al.'s framework, the typicality operator $\mathbf{T}(\cdot)$ is tacitly assumed to occur only in the *left-hand side* of GCIs and not in the scope of other concept constructors. Not having such a syntactic constraint is a feature of our approach that we have put forward in the present work.

Algorithm 1: Consistent($\mathcal{KB}$)

Input: An $\mathcal{ALCH}^\bullet$ knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$

if Expand($\mathcal{KB}$) $\neq \emptyset$ **then**
  return "Consistent"
else
  return "Inconsistent"

Function Expand($\mathcal{KB}$)

Input: An $\mathcal{ALCH}^\bullet$ knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$

while $\mathcal{A}$ *is not complete* **do**
  Select a rule ρ that is applicable to $\mathcal{A}$;
  if ρ *is a nondeterministic rule* **then**
    Select an assertion $\alpha \in \mathcal{A}$ to which ρ is applicable;
    if *there is* $\mathcal{A}' \in \mathsf{ndexp}(\mathcal{A}, \rho, \alpha)$ *with* Expand($\mathcal{T} \cup \mathcal{R} \cup \mathcal{A}'$) $\neq \emptyset$ **then**
      return Expand($\mathcal{T} \cup \mathcal{R} \cup \mathcal{A}'$)
    else
      return $\emptyset$
  else
    Apply ρ to $\mathcal{A}$
if $\mathcal{A}$ *contains a clash* **then**
  return $\emptyset$
else
  return $\langle \mathcal{A}, <, \ll \rangle$

When it comes to reasoning about typicality, Giordano et al. have defined a tableau calculus for their preferential extension of DLs [28]. There are many similarities between their calculus and the one we presented here. Besides having a simpler presentation, our calculus does not have to explicitly handle an extra modality in the way Giordano et al.'s does, and is therefore more elegant.

More recently, Giordano et al. have gone beyond preferential entailment in that they have also explored a definition of non-monotonic entailment for their description logic of typicality [30] corresponding to the well-known notion of *rational closure* as studied by Lehmann and Magidor in the propositional case [35]. Semantically, and roughly, this amounts to a version of a minimal-model semantics, in which some interpretations are preferred over others. This is a promising extension of our work that we may consider. Nevertheless, special care must be taken since Giordano et al.'s approach has a circumscriptive [37, 38] flavour to it (even if not completely) in that it relies on the explicit specification by the knowledge engineer of a set of concepts for which atypical instances must be minimised.

Booth et al. [11, 12] investigated the addition of a typicality operator $\bullet$ to propositional logic, of which the semantics is given in terms of KLM ranked models [35]. The logic thus obtained is more expressive than that of KLM conditional statements, allowing us to move beyond propositional defeasible conditionals. Following up on that, Booth et al. [10] investigated two semantic versions of entailment in the presence of $\bullet$, constructed using two different forms of minimality. Both are based on the notion of rational closure defined by Lehmann and Magidor for KLM-style conditionals. It was shown that (*i*) these notions of entailment can be viewed as generalised definitions of rational closure; (*ii*) that they are equivalent w.r.t. the conditional language originally proposed by Kraus et al., but (*iii*) they are different in the language enriched with $\bullet$. We may consider taking the approach by Booth et al. as a springboard to investigate rationality and different forms of non-monotonic entailment for $\mathcal{ALCH}^{\bullet}$.

Britz et al. [14] have introduced the notion of *defeasible role restrictions*, a variant of *generalised quantifiers* [36]. The idea is to extend the concept language with an additional construct $\forall\!\!\!\!\sim$, the *defeasible value restriction.* The semantics of $\forall\!\!\!\!\sim r.C$ is then given by all objects of the domain such that all of their *minimal* r-related objects are C-instances. This is useful in situations where certain classical concept descriptions may be too strong.

Recently, Britz and Varzinczak have lifted the preferential semantics to also allow for orderings on role-interpretations [18, 20], as we have done here, and multi-orderings on objects of the domain [19, 21]. The latter give us the handle needed to introduce a notion of *context* in defeasible subsumption relations making typicality a relativised construct. The former provides a semantics for defeasible role inclusions of the form $r \sqsubsim s$ and for defeasible role assertions such as "r is usually transitive", "r and s are usually disjoint", as well as others.

Another recent proposal is the approach by Bonatti et al. [6, 9], which introduces a *normality* operator $\mathbf{N}(\cdot)$ on concepts only but that can also be used in the scope of other operators, as in the statement $\mathbf{N}(C)\sqcap\mathbf{N}(D) \sqsubseteq \exists r.\mathbf{N}(E)$. The resulting system, DL^N, is not based on the preferential approach, though, and as a consequence their closure operation does not allow defeasible subsumption to satisfy the preferential properties. Nevertheless, Bonatti et al.'s approach satisfies some interesting properties on the meta-level. It also has the advantage of being computationally tractable for any tractable classical DL.

6 Concluding remarks

We have introduced $\mathcal{ALCH}^{\bullet}$, a description logic allowing for an explicit notion of typicality that can be applied to both concepts and roles and of which the intuition is to capture the most typical instances of, respectively, classes and relations. We have seen that $\mathcal{ALCH}^\bullet$ can be given a simple and intuitive semantics in terms of partially-ordered structures in the spirit of the preferential approach to defeasible reasoning. We have defined a tableau-based proof procedure for $\mathcal{ALCH}^\bullet$ that we have shown to be sound and complete w.r.t. our preferential semantics.

When compared to other approaches to non-monotonicity in DLs, the novelty of $\mathcal{ALCH}^\bullet$ resides in the provision of a unifying framework for typicality of both classes and relations and that can serve as the foundation for extensions of defeasible DLs of increasing expressivity, with non-monotonicity at the level of concepts as well as that of roles.

Space considerations have prevented us from showing termination of our tableau-based algorithm. Nonetheless, the result does hold and therefore we can claim there is a decision procedure for satisfiability of $\mathcal{ALCH}^\bullet$ knowledge bases.

As for the computational complexity of reasoning with general $\mathcal{ALCH}^\bullet$ knowledge bases, we conjecture it is EXPTIME-complete, and therefore in the same complexity class of the problem of reasoning with general (classical) $\mathcal{ALCH}$ knowledge bases. The algorithm we presented is not optimal in that it can be shown to run in time that is doubly exponential in the size of the input knowledge base. An investigation of optimal tableaux for $\mathcal{ALCH}^\bullet$ reasoning is a task we shall for now leave for future work.

The work here presented can be taken further in many ways. Some concrete next steps comprise: (*i*) An extension of the underlying language with further DL constructs such as cardinality restrictions, role operations, nominals and role assertions [1], along with new notions of typicality that those may call for, or even non-monotonic versions of the classical operators [18, 20]; (*ii*) An extension of the preferential semantics to allow for multi-orderings on both objects and role interpretations, each ordering standing for a notion of context [19] and giving rise to a context-based typicality operator for concepts and roles, and (*iii*) An investigation of non-monotonic entailment for $\mathcal{ALCH}^\bullet$, in particular of what the notion of rational closure [35] semantically corresponds to when ordering pairs of objects. (The work by Booth et al. [10] on entailment for propositional typicality may provide us with a starting point for tackling this issue.)

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A Proof of Theorem 1

We remind the reader that we can assume w.l.o.g. that all GCIs in a TBox are of the form $\top \sqsubseteq E$, for $E \in \mathcal{L}^\bullet$, and that the ABox is non-empty (cf. beginning of Section 4).

Lemma 4 *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$. If* Consistent($\mathcal{KB}$) *returns "Consistent", then $\mathcal{KB}$ is preferentially consistent.*

Proof:
Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ and assume Consistent($\mathcal{KB}$) returns "Consistent". Then the result of Expand($\mathcal{KB}$) is non-empty. Let $\langle \mathcal{A}', <_{\mathcal{A}'}, \ll_{\mathcal{A}'} \rangle$ be the result returned by Expand($\mathcal{KB}$). Hence $\mathcal{A}'$ is a complete and clash-free ABox. Moreover, since the expansion rules never delete assertions, we have $\mathcal{A} \subseteq \mathcal{A}'$. In what follows, we will:

1. Define a modification $\langle \mathcal{A}'', <, \ll \rangle$ of $\langle \mathcal{A}', <_{\mathcal{A}'}, \ll_{\mathcal{A}'} \rangle$ to deal with blocked individuals in $\mathcal{A}'$ and such that $\mathcal{A} \subseteq \mathcal{A}''$;
2. Show that $\mathcal{A}''$ is complete and clash-free;
3. Use $\mathcal{A}''$, along with $<$ and $\ll$, to construct a suitable bi-ordered interpretation satisfying $\mathcal{KB}$, which is a witness to the preferential consistency of $\mathcal{KB}$.

Dealing with 1. Let $\mathcal{A}''$, $<$ and $\ll$ be defined as follows:

$$
\begin{aligned}
\mathcal{A}'' \ :=\ & \{a : C \mid a : C \in \mathcal{A}' \text{ and } a \text{ is not blocked}\} \cup \\
& \{(a,b) : R \mid (a,b) : R \in \mathcal{A}' \text{ and } b \text{ is not blocked}\} \cup \\
& \{(a,b') : R \mid (a,b) : R \in \mathcal{A}',\ a \text{ is not blocked and } b \text{ is blocked by } b'\} \cup \\
& \{(a,b) : \neg R \mid (a,b) : \neg R \in \mathcal{A}' \text{ and } a, b \text{ are not blocked}\} \\
< \ :=\ & \{(a,b) \mid (a,b) \in <_{\mathcal{A}'} \text{ and } b \text{ is not blocked}\} \cup \\
& \{(a,b') \mid (a,b) \in <_{\mathcal{A}'},\ a \text{ is not blocked and } b \text{ is blocked by } b'\} \\
\ll \ :=\ & \{((a,b),(c,d)) \mid ((a,b),(c,d)) \in \ll_{\mathcal{A}'} \text{ and } b, d \text{ are not blocked}\} \cup \\
& \{((a,b),(c,d)) \mid ((a,b),(c,d)) \in \ll_{\mathcal{A}'},\ a, c \text{ are not blocked,} \\
& \qquad b \text{ is blocked by } b' \text{ and } d \text{ is blocked by } d'\}
\end{aligned}
$$

It is not hard to see that $\mathcal{A} \subseteq \mathcal{A}''$: first note that $\mathcal{A} \subseteq \mathcal{A}'$; then observe that for all assertions $a : C$, $(a,b) : R$ and $(a,b) : \neg R$ in $\mathcal{A}$, both a and b are root individuals (see Definition 6), and therefore can never be blocked.

An immediate consequence of the definition of $\mathcal{A}''$ is the following property: For every a, b in $\mathcal{A}'', \mathcal{A}'$,

$$\mathsf{con}_{\mathcal{A}''}(a) = \mathsf{con}_{\mathcal{A}'}(a) \text{ and } \mathsf{roles}_{\mathcal{A}''}(a,b) = \mathsf{roles}_{\mathcal{A}'}(a,b) \qquad (*)$$

Moreover, it is not hard to see that, by construction, $<$ and $\ll$ simulate $<_{\mathcal{A}'}$ and $\ll_{\mathcal{A}''}$ for non-blocked (and pairs of non-blocked) individuals.

Dealing with 2. Since $\mathcal{A}'$ is clash-free, $\mathcal{A}''$ is also clash-free, for if $\mathcal{A}''$ contained a clash, Property $(*)$ would imply $\mathcal{A}'$ has a clash, too. It remains to show that $\mathcal{A}''$ is complete, which we do by showing that none of the expansion rules is applicable to $\mathcal{A}''$.

- $\neg$-rule: If $a : \neg\neg C \in \mathcal{A}''$, then by $(*)$ we get $a : \neg\neg C \in \mathcal{A}'$, and since $\mathcal{A}'$ is complete, we have $a : C \in \mathcal{A}'$. By $(*)$ we have $a : C \in \mathcal{A}''$, and then the $\neg$-rule is not applicable to $\mathcal{A}''$.
- $\sqcap^+$-rule: If $a : C \sqcap D \in \mathcal{A}''$, then by $(*)$ we have $a : C \sqcap D \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $\{a : C, a : D\} \subseteq \mathcal{A}'$. By $(*)$ again, $\{a : C, a : D\} \subseteq \mathcal{A}''$ and therefore the $\sqcap^+$-rule is not applicable to $\mathcal{A}''$.
- $\sqcup^+$-rule: If $a : C \sqcup D \in \mathcal{A}''$, then by $(*)$ we have $a : C \sqcup D \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $\{a : C, a : D\} \cap \mathcal{A}' \neq \emptyset$. By $(*)$ again, $\{a : C, a : D\} \cap \mathcal{A}'' \neq \emptyset$ and therefore the $\sqcup^+$-rule is not applicable to $\mathcal{A}''$.
- $\sqcap^-$- and $\sqcup^-$-rules are analogous to the two previous cases.
- $\sqsubseteq_{\mathcal{T}}$-rule: Let $\top \sqsubseteq D \in \mathcal{T}$. If $a : C \in \mathcal{A}''$, then by $(*)$ we have $a : C \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $a : D \in \mathcal{A}'$, too. By $(*)$ again, we get $a : D \in \mathcal{A}''$ and therefore the $\sqsubseteq_{\mathcal{T}}$-rule is not applicable to $\mathcal{A}''$.
- $\sqsubseteq_{\mathcal{R}}$-rule: Let $R \sqsubseteq S \in \mathcal{R}$. If $(a,b) : R \in \mathcal{A}''$, then by $(*)$ we have $(a,b) : R \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $(a,b) : S \in \mathcal{A}'$, and then by $(*)$ we have $(a,b) : S \in \mathcal{A}''$. Hence the $\sqsubseteq_{\mathcal{R}}$-rule is not applicable to $\mathcal{A}''$.
- $\exists^+$-rule: If $a : \exists R.C \in \mathcal{A}''$, then by $(*)$ $a : \exists R.C \in \mathcal{A}'$. This implies a is not blocked in $\mathcal{A}'$, and therefore there is b s.t. $\{(a,b) : R, b : C\} \subseteq \mathcal{A}'$, for $\mathcal{A}'$ is complete. There are two possible cases:
  – b is not blocked: Then $\{(a,b) : R, b : C\} \subseteq \mathcal{A}''$, from the construction of $\mathcal{A}''$;
  – b is blocked: Since a is not blocked and is b's predecessor, we must have that b is blocked by some b' in $\mathcal{A}'$. Hence we have (i) $(a,b') : R \in \mathcal{A}''$, by construction of $\mathcal{A}''$. Clearly, b' is not blocked because it is an ancestor of b which is a successor of an individual that is not blocked. Also, $\mathsf{con}_{\mathcal{A}'}(b) \subseteq \mathsf{con}_{\mathcal{A}'}(b')$, and then $b' : C \in \mathcal{A}'$. This and $(*)$ imply (ii) $b' : C \in \mathcal{A}''$. From (i) and (ii) follows $\{(a,b') : R, b' : C\} \subseteq \mathcal{A}''$.

  In both cases above, the $\exists^+$-rule is not applicable to $\mathcal{A}''$.

- $\forall^+$-rule: If $\{a : \forall R.C, (a, b') : R\} \subseteq \mathcal{A}''$, then $a : \forall R.C \in \mathcal{A}'$, by $(*)$, and neither a nor b' is blocked in $\mathcal{A}'$. There are two possible cases:

  – $(a, b') : R \in \mathcal{A}'$: Then $b' : C \in \mathcal{A}'$, for $\mathcal{A}'$ is complete. From $(*)$ we get $b' : C \in \mathcal{A}''$;

  – $(a, b') : R \notin \mathcal{A}'$: Then there is b s.t. $(a, b) : R \in \mathcal{A}'$, with b blocked by b' in $\mathcal{A}'$, and $b : C \in \mathcal{A}'$, since $\mathcal{A}'$ is complete. Moreover, since $\mathsf{con}_{\mathcal{A}'}(b) \subseteq \mathsf{con}_{\mathcal{A}'}(b')$, we have $b' : C \in \mathcal{A}'$. This and $(*)$ yield $b' : C \in \mathcal{A}''$.

  In both cases above, the $\forall^+$-rule is not applicable to $\mathcal{A}''$.

- $\exists^-$- and $\forall^-$-rules are analogous to the two previous cases.

- $\bullet_C^+$-rule: If $a : \bullet C \in \mathcal{A}''$, then by $(*)$ we have $a : \bullet C \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $a : C \in \mathcal{A}'$ and for all b s.t. $b <_{\mathcal{A}'} \ldots <_{\mathcal{A}'} a$, $b : \neg C \in \mathcal{A}'$. By $(*)$ again and the construction of $<$, we have $a : C \in \mathcal{A}''$ and for all b s.t. $b < \ldots < a$, $b : \neg C \in \mathcal{A}''$. Hence the $\bullet_C^+$-rule is not applicable to $\mathcal{A}''$.

- $\bullet_C^-$-rule: If $a : \neg\bullet C \in \mathcal{A}''$, then by $(*)$ we have $a : \neg\bullet C \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, we have either (i) $a : \neg C \in \mathcal{A}'$, or (ii) $\{a : C, c : C\} \subseteq \mathcal{A}'$ and $(c, a) \in <_{\mathcal{A}'}$. From (i) and $(*)$ follows (iii) $a : \neg C \in \mathcal{A}''$. From (ii), $(*)$ and the construction of $<$ follows (iv) $\{a : C, c : C\} \subseteq \mathcal{A}''$ and $(c, a) \in <$. In either of (ii) and (iv), the $\bullet_C^-$-rule is not applicable to $\mathcal{A}''$.

- $\bullet_r^+$-rule: If $(a, b) : \bullet r \in \mathcal{A}''$, there are two possible cases:

  – $(a, b) : \bullet r \in \mathcal{A}'$: Then, since $\mathcal{A}'$ is complete, $(a, b) : r \in \mathcal{A}'$, and for all (c, d) s.t. $(c, d) \ll_{\mathcal{A}'} \ldots \ll_{\mathcal{A}'} (a, b)$, $(c, d) : \neg r \in \mathcal{A}'$. By $(*)$ and the construction of $\ll$, we get $(a, b) : r \in \mathcal{A}''$ and for all (c, d) s.t. $(c, d) \ll \ldots \ll (a, b)$, $(c, d) : \neg r \in \mathcal{A}''$;

  – $(a, b) : \bullet r \notin \mathcal{A}'$: Then, there is b' s.t. $(a, b') : \bullet r \in \mathcal{A}'$, with b' blocked by b in $\mathcal{A}'$. Since $\mathcal{A}'$ is complete, $(a, b') : r \in \mathcal{A}'$ and for all (c, d) s.t. $(c, d) \ll_{\mathcal{A}'} \ldots \ll_{\mathcal{A}'} (a, b')$, $(c, d) : \neg r \in \mathcal{A}'$. Then, by construction of $\mathcal{A}''$ and $\ll$, we have $(a, b) : r \in \mathcal{A}''$, and for all (c, d) s.t. $(c, d) \ll \ldots \ll (a, b)$, $(c, d) : \neg r \in \mathcal{A}''$.

  In both cases above, the $\bullet_r^+$-rule is not applicable to $\mathcal{A}''$.

- $\bullet_r^-$-rule: If $(a, b) : \neg\bullet r \in \mathcal{A}''$, then by $(*)$ we have $(a, b) : \neg\bullet r \in \mathcal{A}'$. From completeness of $\mathcal{A}'$, we have either (i) $(a, b) : \neg r \in \mathcal{A}'$, or (ii) $\{(a, b) : r, (c, d) : r\} \subseteq \mathcal{A}'$ and $((c, d), (a, b)) \in \ll_{\mathcal{A}'}$. If (i) is the case, $(a, b) : \neg r \in \mathcal{A}''$. If (ii) is the case, since c, d are not blocked (they are root individuals, for they were freshly introduced), we have $\{(a, b) : r, (c, d) : r\} \subseteq \mathcal{A}''$ and $((c, d), (a, b)) \in \ll$. In both (i) and (ii), the $\bullet_r^-$-rule is not applicable to $\mathcal{A}''$.

- $\bullet ?_C$-rule: If $a : C \in \mathcal{A}''$, then by $(*)$ $a : C \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $\{a : \bullet C, a : \neg \bullet C\} \cap \mathcal{A}' \neq \emptyset$. By $(*)$ again, $\{a : \bullet C, a : \neg \bullet C\} \cap \mathcal{A}'' \neq \emptyset$, and therefore the $\bullet ?_C$-rule is not applicable to $\mathcal{A}''$.
- $\bullet ?_r$-rule: If $(a, b) : r \in \mathcal{A}''$, then by $(*)$ $(a, b) : r \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete, $\{(a, b) : \bullet r, (a, b) : \neg \bullet r\} \cap \mathcal{A}' \neq \emptyset$. By $(*)$ again, $\{(a, b) : \bullet r, (a, b) : \neg \bullet r\} \cap \mathcal{A}'' \neq \emptyset$, and therefore the $\bullet ?_r$-rule is not applicable to $\mathcal{A}''$.

Dealing with 3. We use $\mathcal{A}''$ together with $<$ and $\ll$ to construct a suitable model $\mathcal{B}$ for $\mathcal{KB}$ as follows:

- $\Delta^{\mathcal{B}} := \{a \mid a$ is an individual name occurring in $\mathcal{A}''\}$;
- $a^{\mathcal{B}} := a$, for each individual name occurring in $\mathcal{A}''$;
- $A^{\mathcal{B}} := \{a \mid A \in \mathsf{con}_{\mathcal{A}''}(a)\}$, for each concept name occurring in $\mathcal{A}''$;
- $r^{\mathcal{B}} := \{(a, b) \mid (a, b) : r \in \mathcal{A}''\}$, for each role name occurring in $\mathcal{A}''$;
- $<^{\mathcal{B}} := <^+$;
- $\ll^{\mathcal{B}} := \ll^+$.

We show that $\mathcal{B} := \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$ is a bi-ordered interpretation satisfying $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$.

First we show that $\mathcal{B}$ is a bi-ordered interpretation (cf. Definition 1):

- $\Delta^{\mathcal{B}} \neq \emptyset$, as we assumed $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subseteq \mathcal{A}''$;
- By construction, $\cdot^{\mathcal{B}}$ maps every individual name in $\mathcal{A}''$ to an element of $\Delta^{\mathcal{B}}$, every concept name $A \in \mathsf{sub}(\mathcal{A}'')$ to a subset of $\Delta^{\mathcal{B}}$, and every role name r occurring in $\mathcal{A}''$ to a subset of $\Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}$;
- It is easy to see that both $<^{\mathcal{B}}$ and $\ll^{\mathcal{B}}$ are well-founded strict partial orders, for (i) in both $<$ and $\ll$ no reflexive elements are ever introduced, as only pairs containing either a new individual name a or a new pair (a, b) are added at the beginning of the respective chain; (ii) by an analogous argument, no symmetric elements are ever added to $<$ or $\ll$; (iii) taking their transitive closure clearly delivers a transitive relation, and (iv) since both $<$ and $\ll$ are finite, we have that $<^{\mathcal{B}}$ and $\ll^{\mathcal{B}}$ are finite, too, and therefore the orderings are well-founded.

Hence, $\mathcal{B}$ is a bi-ordered interpretation.

Now we show that $\mathcal{B}$ satisfies all concepts and role assertions in $\mathcal{A}$, all GCIs in $\mathcal{T}$, and all RIAs in $\mathcal{R}$.

We start by showing that $\mathcal{B}$ satisfies all concepts and role assertions in $\mathcal{A}''$, and since $\mathcal{A} \subseteq \mathcal{A}''$, we will get $\mathcal{B} \Vdash \mathcal{A}$. First, it is not hard to see that, by its construction, $\mathcal{B}$ satisfies all role assertions in $\mathcal{A}''$. To see that $\mathcal{B}$ satisfies all concept assertions in $\mathcal{A}''$, we show the following property:

$$\text{If } a : C \in \mathcal{A}'', \text{ then } a^{\mathcal{B}} \in C^{\mathcal{B}} \qquad (**)$$

The proof is by induction on the structure of concepts:

Induction basis: Let $C = A \in \mathsf{C}$. By the definition of $\mathcal{B}$, if $a : C \in \mathcal{A}''$, then $a^{\mathcal{B}} \in C^{\mathcal{B}}$.

Induction steps: (Since there is no NNF for $\mathcal{L}^{\bullet}$—cf. paragraph following Proposition 1—we have to analyse more cases than if it had been otherwise. Moreover, note that the case $C = \neg D$, for an arbitrary D, can be reduced to all the others below through De Morgan's laws and therefore we do not address it explicitly here.)

- Let $C = \neg A$, for $A \in \mathsf{C}$. Since $\mathcal{A}''$ is clash-free, $a : \neg A \in \mathcal{A}''$ implies $a : A \notin \mathcal{A}''$, and therefore $A \notin \mathsf{con}_{\mathcal{A}''}(a)$. From this and the construction of $\mathcal{B}$, it follows that $a \notin A^{\mathcal{B}}$.
- Let $C = D \sqcap E$. If $a : D \sqcap E \in \mathcal{A}''$, then, since $\mathcal{A}''$ is complete, $\{a : D, a : E\} \subseteq \mathcal{A}''$, otherwise the $\sqcap^{+}$-rule would be applicable to $\mathcal{A}''$. By the induction hypothesis, $a^{\mathcal{B}} \in D^{\mathcal{B}}$ and $a^{\mathcal{B}} \in E^{\mathcal{B}}$, and therefore $a^{\mathcal{B}} \in D^{\mathcal{B}} \cap E^{\mathcal{B}} = (D \sqcap E)^{\mathcal{B}}$.
- Let $C = D \sqcup E$. If $a : D \sqcup E \in \mathcal{A}''$, then, since $\mathcal{A}''$ is complete, $\{a : D, a : E\} \cap \mathcal{A}'' \neq \emptyset$, otherwise the $\sqcup^{+}$-rule would be applicable to $\mathcal{A}''$. By the induction hypothesis, $a^{\mathcal{B}} \in D^{\mathcal{B}}$ or $a^{\mathcal{B}} \in E^{\mathcal{B}}$, and therefore $a^{\mathcal{B}} \in D^{\mathcal{B}} \cup E^{\mathcal{B}} = (D \sqcup E)^{\mathcal{B}}$.
- Let $C = \forall R.D$. We distinguish two cases, namely $R = r$ and $R = \bullet r$, for $r \in \mathsf{R}$.
  - Case $R = r$: Assume $a : \forall r.D \in \mathcal{A}''$, and let $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$, for an arbitrary b. Then, by construction of $\mathcal{B}$, $(a, b) : r \in \mathcal{A}''$, and since $\mathcal{A}''$ is complete and $a : \forall r.D \in \mathcal{A}''$, we have $b : D \in \mathcal{A}''$, otherwise the $\forall^{+}$-rule would be applicable to $\mathcal{A}''$. By the induction hypothesis, $b^{\mathcal{B}} \in D^{\mathcal{B}}$. Since b is arbitrary, the above holds for all b s.t. $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$ and therefore $a^{\mathcal{B}} \in (\forall r.D)^{\mathcal{B}}$.
  - Case $R = \bullet r$: Assume $a : \forall \bullet r.D \in \mathcal{A}''$, and let $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}}$, for an arbitrary b. Assume $(a, b) : \bullet r \notin \mathcal{A}''$. Then, since $(a, b) : r \in \mathcal{A}''$ (because $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}} \subseteq r^{\mathcal{B}}$), the $\bullet ?_r$-rule is applicable to $\mathcal{A}''$. Hence $(a, b) : \bullet r \in \mathcal{A}''$. Moreover, since $\mathcal{A}''$ is complete and $a : \forall \bullet r.D \in \mathcal{A}''$, we have $b : D \in \mathcal{A}''$, otherwise the $\forall^{+}$-rule would be applicable to $\mathcal{A}''$. By the induction hypothesis, $b^{\mathcal{B}} \in D^{\mathcal{B}}$. Since b is arbitrary, the above holds for all b s.t. $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}}$ and therefore $a^{\mathcal{B}} \in (\forall \bullet r.D)^{\mathcal{B}}$.
- Let $C = \exists R.D$. Again, we distinguish two cases: $R = r$ and $R = \bullet r$, for $r \in \mathsf{R}$.
  - Case $R = r$: Let $a : \exists r.D \in \mathcal{A}''$. Since $\mathcal{A}''$ is complete, $\{(a, b) : r, b : D\} \subseteq \mathcal{A}''$. By the construction of $\mathcal{B}$, $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$. By the induction hypothesis, $b^{\mathcal{B}} \in D^{\mathcal{B}}$. Putting these results together gives us $a^{\mathcal{B}} \in (\exists r.D)^{\mathcal{B}}$.

– Case $R = \bullet r$: Let $a : \exists \bullet r.D \in \mathcal{A}''$. Since $\mathcal{A}''$ is complete, $\{(a, b) : \bullet r, (a, b) : r, b : D\} \subseteq \mathcal{A}''$. By the construction of $\mathcal{B}$, $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$ and there is no c, d s.t. $(c^{\mathcal{B}}, d^{\mathcal{B}}) \ll^{\mathcal{B}} (a^{\mathcal{B}}, b^{\mathcal{B}})$. Hence $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}}$. By the induction hypothesis, $b^{\mathcal{B}} \in D^{\mathcal{B}}$. Therefore, $a^{\mathcal{B}} \in (\exists \bullet r.D)^{\mathcal{B}}$.

- Let $C = \bullet D$. Assume $a : \bullet D \in \mathcal{A}''$. Since $\mathcal{A}''$ is complete, $a : D \in \mathcal{A}''$ (and by the induction hypothesis we have $a^{\mathcal{B}} \in D^{\mathcal{B}}$), and for every b s.t. $b < \ldots < a$, $b : \neg D \in \mathcal{A}''$. As we already know, $b^{\mathcal{B}} \in (\neg D)^{\mathcal{B}}$. Hence, by the construction of $\mathcal{B}$, for every $b^{\mathcal{B}}$ s.t. $b^{\mathcal{B}} <^{\mathcal{B}} a^{\mathcal{B}}$, $b^{\mathcal{B}} \in (\neg D)^{\mathcal{B}}$, and therefore $a^{\mathcal{B}} \in \min_{<^{\mathcal{B}}} D^{\mathcal{B}}$.

- Let $C = \neg \bullet D$. Assume $a : \neg \bullet D \in \mathcal{A}''$. Since $\mathcal{A}''$ is complete, either $a : \neg D \in \mathcal{A}''$ or $\{a : D, c : D\} \subseteq \mathcal{A}''$ and $c < a$. If $a : \neg D \in \mathcal{A}''$, then by the induction hypothesis $a^{\mathcal{B}} \in (\neg D)^{\mathcal{B}}$ and therefore $a^{\mathcal{B}} \in (\neg \bullet D)^{\mathcal{B}}$. If $\{a : D, c : D\} \subseteq \mathcal{A}''$ and $c < a$, then $a^{\mathcal{B}} \in D^{\mathcal{B}}$ and $c^{\mathcal{B}} \in D^{\mathcal{B}}$ (by the induction hypothesis) and $c^{\mathcal{B}} <^{\mathcal{B}} a^{\mathcal{B}}$ (by the construction of $\mathcal{B}$). Hence $a^{\mathcal{B}} \notin (\bullet D)^{\mathcal{B}}$, i.e., $a^{\mathcal{B}} \in (\neg \bullet D)^{\mathcal{B}}$.

This concludes the proof of $(**)$. Hence $\mathcal{B} \Vdash \mathcal{A}''$ and therefore $\mathcal{B} \Vdash \mathcal{A}$.

Now we show that $\mathcal{B}$ is a model of $\mathcal{T}$. Let $\top \sqsubseteq D \in \mathcal{T}$ and let a be an arbitrary individual occurring in $\mathcal{A}''$. Since $\mathcal{A}''$ is complete, $a : D \in \mathcal{A}''$. Hence $a = a^{\mathcal{B}} \in D^{\mathcal{B}}$, since $\mathcal{B} \Vdash \mathcal{A}''$. Given that a is arbitrary (i.e., we assumed any $a \in \Delta^{\mathcal{B}}$, the set of individual names in $\mathcal{A}''$), we have $\Delta^{\mathcal{B}} \subseteq D^{\mathcal{B}}$, as required. Hence $\mathcal{B} \Vdash \mathcal{T}$.

Finally, we show that $\mathcal{B}$ is a model of $\mathcal{R}$. First, recall that the elements of $\mathcal{R}$ have one of four possible forms, namely $r \sqsubseteq s$, $r \sqsubseteq \bullet s$, $\bullet r \sqsubseteq s$ and $\bullet r \sqsubseteq \bullet s$. We analyse each case.

- Assume $r \sqsubseteq s \in \mathcal{R}$. If $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$, then $(a, b) : r \in \mathcal{A}''$, by construction of $\mathcal{B}$. Since $\mathcal{A}''$ is complete, $(a, b) : s \in \mathcal{A}''$, and then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in s^{\mathcal{B}}$.

- Assume $r \sqsubseteq \bullet s \in \mathcal{R}$. If $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$, then $(a, b) : r \in \mathcal{A}''$, by construction of $\mathcal{B}$. Since $\mathcal{A}''$ is complete, $\{(a, b) : \bullet s, (a, b) : s\} \subseteq \mathcal{A}''$ and then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in \min_{\ll^{\mathcal{B}}} s^{\mathcal{B}}$.

- Assume $\bullet r \sqsubseteq s \in \mathcal{R}$. If $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}}$, then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$ and $(a, b) : r \in \mathcal{A}''$. We must have $(a, b) : \bullet r \in \mathcal{A}''$, otherwise the $\bullet ?_r$-rule would be applicable. Hence $(a, b) : s \in \mathcal{A}''$, and then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in s^{\mathcal{B}}$.

- Assume $\bullet r \sqsubseteq \bullet s \in \mathcal{R}$. If $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in (\bullet r)^{\mathcal{B}}$, then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$ and $(a, b) : r \in \mathcal{A}''$. We must have $(a, b) : \bullet r \in \mathcal{A}''$, otherwise the $\bullet ?_r$-rule would be applicable. Hence $(a, b) : \bullet s \in \mathcal{A}''$, and then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in \min_{\ll^{\mathcal{B}}} s^{\mathcal{B}}$.

Hence $\mathcal{B} \Vdash \mathcal{R}$.

Putting all the results together, we have that $\mathcal{B} \Vdash \mathcal{KB}$ and therefore $\mathcal{KB}$ is preferentially satisfiable. ∎

Lemma 5 *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$. If $\mathcal{KB}$ is preferentially consistent, then* Consistent($\mathcal{KB}$) *returns "Consistent".*

Proof:
Assume $\mathcal{KB} = \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ is preferentially consistent, and let $\mathcal{B} = \langle \Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}} \rangle$ be a model of $\mathcal{KB}$. In particular, $\mathcal{B} \Vdash \mathcal{A}$. Since $\mathcal{A}$ is consistent, it does not contain a clash.

If $\mathcal{A}$ is complete, and since it is clash-free, Expand($\mathcal{KB}$) returns $\mathcal{A}$ and Consistent($\mathcal{KB}$) returns "Consistent".

Assume $\mathcal{A}$ is not complete. Then Expand($\mathcal{KB}$) performs iterations of the while loop until $\mathcal{A}$ is complete; each iteration selects a rule and applies it, possibly calling Expand($\cdot$) recursively. We show that this while loop in Expand($\cdot$) preserves consistency. We do so by analysing all possible cases of applicable rules:

- $\neg$-rule: If $a : \neg\neg C \in \mathcal{A}$, then $a^{\mathcal{B}} \in (\neg\neg C)^{\mathcal{B}} = C^{\mathcal{B}}$ and therefore $\mathcal{B}$ is a model of $\mathcal{A} \cup \{a : C\}$. Hence $\mathcal{A}$ is still consistent after the rule is applied.
- $\sqcap^{+}$-rule: If $a : C \sqcap D \in \mathcal{A}$, then $a^{\mathcal{B}} \in (C \sqcap D)^{\mathcal{B}} = C^{\mathcal{B}} \cap D^{\mathcal{B}}$, and then both $a^{\mathcal{B}} \in C^{\mathcal{B}}$ and $a^{\mathcal{B}} \in D^{\mathcal{B}}$. Hence $\mathcal{B}$ is a model of $\mathcal{A} \cup \{a : C, a : D\}$, so $\mathcal{A}$ is still consistent after the application of the rule.
- $\sqcup^{+}$-rule: If $a : C \sqcup D \in \mathcal{A}$, then $a^{\mathcal{B}} \in (C \sqcup D)^{\mathcal{B}} = C^{\mathcal{B}} \cup D^{\mathcal{B}}$, i.e., either $a^{\mathcal{B}} \in C^{\mathcal{B}}$ or $a^{\mathcal{B}} \in D^{\mathcal{B}}$. Hence at least one of the ABoxes $\mathcal{A}' \in \mathsf{ndexp}(\mathcal{A}, \sqcup^{+}, a : C \sqcup D)$ is consistent. Then Expand($\mathcal{T} \cup \mathcal{R} \cup \mathcal{A}'$) is called recursively with $\mathcal{A}'$ being consistent, and we can repeat the same argument.
- $\sqcap^{-}$- and $\sqcup^{-}$-rules are analogous to both cases above.
- $\sqsubseteq_{\mathcal{T}}$-rule: If $a : C \in \mathcal{A}$ and $\top \sqsubseteq D \in \mathcal{T}$, then $a^{\mathcal{B}} \in D^{\mathcal{B}}$ in any model $\mathcal{B}$ of $\mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$, so $\mathcal{B}$ is still a model of $\mathcal{T} \cup \mathcal{R} \cup \mathcal{A} \cup \{a : D\}$.
- $\sqsubseteq_{\mathcal{R}}$-rule: If $(a, b) : R \in \mathcal{A}$ and $R \sqsubseteq S \in \mathcal{R}$, then both $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in R^{\mathcal{B}}$ and $R^{\mathcal{B}} \subseteq S^{\mathcal{B}}$ in any model $\mathcal{B}$ of $\mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$, and therefore $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in S^{\mathcal{B}}$. Hence $\mathcal{B}$ is a model of $\mathcal{A} \cup \{(a, b) : S\}$ and $\mathcal{A}$ is still consistent.
- $\exists^{+}$-rule: If $a : \exists R.C \in \mathcal{A}$, then $a^{\mathcal{B}} \in (\exists R.C)^{\mathcal{B}}$, and then there is some $x \in \Delta^{\mathcal{B}}$ s.t. $(a^{\mathcal{B}}, x) \in R^{\mathcal{B}}$ and $x \in C^{\mathcal{B}}$. It is not hard to see that there is a model $\mathcal{B}'$ of $\mathcal{A}$ that is identical to $\mathcal{B}$, except that for some new individual name d, we have $d^{\mathcal{B}} = x$. Clearly, $\mathcal{B}'$ is a model of $\mathcal{A} \cup \{(a, d) : r, d : C\}$, so $\mathcal{A}$ is still consistent after the application of the rule.
- $\forall^{+}$-rule: If $\{a : \forall R.C, (a, b) : R\} \subseteq \mathcal{A}$, then $a^{\mathcal{B}} \in (\forall R.C)^{\mathcal{B}}$, $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in R^{\mathcal{B}}$, and $b^{\mathcal{B}} \in C^{\mathcal{B}}$. Then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{b : C\}$, and therefore $\mathcal{A}$ is still consistent after the rule is applied.
- $\exists^{-}$- and $\forall^{-}$-rules are analogous to those above.

- $\bullet^+_C$-rule: If $a : \bullet C \in \mathcal{A}$, then $a^{\mathcal{B}} \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$. Let b be s.t. $b < \ldots < a$. If $b : C \in \mathcal{A}$, then b was created by the $\bullet^-_C$-rule (which is the only rule that creates $<$-elements), and then $a : \neg\bullet C \in \mathcal{A}$, which is impossible, as $\mathcal{A}$ is clash-free. Therefore $b : C \notin \mathcal{A}$. It is not hard to see that there is a model $\mathcal{B}'$ of $\mathcal{A}$ that is identical to $\mathcal{B}$, except for the fact that $b^{\mathcal{B}} \in (\neg C)^{\mathcal{B}}$. Hence $\mathcal{B}$ satisfies $\mathcal{A} \cup \{a : C, b : \neg C\}$. Since b is arbitrary, $\mathcal{A}$ is still consistent after the rule is applied.

- $\bullet^-_C$-rule: If $a : \neg\bullet C \in \mathcal{A}$, then $a^{\mathcal{B}} \notin \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$, i.e., either (i) $a^{\mathcal{B}} \notin C^{\mathcal{B}}$ or (ii) $a^{\mathcal{B}} \in C^{\mathcal{B}}$ and there is b s.t. $b^{\mathcal{B}} <^{\mathcal{B}} a^{\mathcal{B}}$ and $b^{\mathcal{B}} \in C^{\mathcal{B}}$. If (i) is the case, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{a : \neg C\}$. If (ii) is the case, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{b : C\}$. In both cases, $\mathcal{A}$ is still consistent after the application of the rule.

- $\bullet^+_r$-rule: If $(a, b) : \bullet r \in \mathcal{A}$, then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in \min_{\ll^{\mathcal{B}}} r^{\mathcal{B}}$. Let c, d be s.t. $(c, d) \ll \ldots \ll (a, b)$. If $(c, d) : r \in \mathcal{A}$, then (c, d) was created by the $\bullet^-_r$-rule (which is the only rule that creates $\ll$-elements), and then $(a, b) : \neg\bullet r \in \mathcal{A}$, which is impossible, since $\mathcal{A}$ is clash-free. Hence $(c, d) : r \notin \mathcal{A}$. It is not hard to see that there is a model $\mathcal{B}'$ of $\mathcal{A}$ that is identical to $\mathcal{B}$, except for the fact that $(a^{\mathcal{B}'}, b^{\mathcal{B}'}) \notin r^{\mathcal{B}'}$. Hence $\mathcal{B}$ satisfies $\mathcal{A} \cup \{(a, b) : r, (c, d) : \neg r\}$. Since c, d are arbitrary, $\mathcal{A}$ is still consistent after the rule is applied.

- $\bullet^-_r$-rule: If $(a, b) : \neg\bullet r \in \mathcal{A}$, then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \notin \min_{\ll^{\mathcal{B}}} r^{\mathcal{B}}$, i.e., either (i) $(a^{\mathcal{B}}, b^{\mathcal{B}}) \notin r^{\mathcal{B}}$ or (ii) $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$ and there is (c, d) s.t. $(c^{\mathcal{B}}, d^{\mathcal{B}}) \ll^{\mathcal{B}} (a^{\mathcal{B}}, b^{\mathcal{B}})$ and $(c^{\mathcal{B}}, d^{\mathcal{B}}) \in r^{\mathcal{B}}$. If (i) is the case, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{(a, b) : \neg r\}$. If (ii) is the case, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{(a, b) : r, (c, d) : r\}$. In both cases, $\mathcal{A}$ is still consistent after the rule is applied.

- $\bullet ?_C$-rule: If $a : C \in \mathcal{A}$, then $a^{\mathcal{B}} \in C^{\mathcal{B}}$. Then either $a^{\mathcal{B}} \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$ or not. If $a^{\mathcal{B}} \in \min_{<^{\mathcal{B}}} C^{\mathcal{B}}$, then $\mathcal{B}$ satisfies $\mathcal{A} \cup \{a : \bullet C\}$. If not, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{a : \neg\bullet C\}$. In both cases, $\mathcal{A}$ is still consistent after the application of the rule.

- $\bullet ?_r$-rule: If $(a, b) : r \in \mathcal{A}$, then $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in r^{\mathcal{B}}$. Hence either $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in \min_{\ll^{\mathcal{B}}} r^{\mathcal{B}}$ or not. If $(a^{\mathcal{B}}, b^{\mathcal{B}}) \in \min_{\ll^{\mathcal{B}}} r^{\mathcal{B}}$, then $\mathcal{B}$ satisfies $\mathcal{A} \cup \{(a, b) : \bullet r\}$. If not, then $\mathcal{B}$ is a model of $\mathcal{A} \cup \{(a, b) : \neg\bullet r\}$. In both cases, $\mathcal{A}$ is still consistent after the rule is applied.

■

The proof of Theorem 1 follows immediately from Lemmas 4 and 5.