Canonical Extensions and Kripke-Galois Semantics for Non-Distributive Propositional Logics

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Abstract

This article presents an approach to the semantics of non-distributive propositional logics that is based on a lattice representation (and duality) theorem that delivers a canonical extension of the lattice. Unlike the framework of generalized Kripke frames (RS-frames), proposed with a similar intension, the semantic approach presented in this article is suitable for modeling applied logics (such as temporal, or dynamic), as it respects the intended interpretation of the logical operators. This is made possible by restricting admissible interpretations.

Keywords: Canonical lattice extensions, logics of lattice expansions, Kripke frames, Kripke-Galois frames, Generalized Kripke frames, Modal operators, Modal lattices, Implicative lattices

1 Introduction

1.1 Motivation and Structure of this Article

In [18] this author and J.M. Dunn published a lattice representation and Stone type duality result. In [9], Proposition 2.6, it was demonstrated by Gehrke and Harding that the representation of [18] is a canonical extension of the represented bounded lattice, which is unique ([9], Proposition 2.7), up to isomorphism. Urquhart's [22] and, subsequently, Hartung's [19] lattice represenations (both predating [18]) also constitute canonical extensions of the represented lattice (see [9], though no proof is presented) and using Urquhart's representation, in particular, it is easily proven (see [22]) that the representation reduces to the Priestley representation [21] when the represented lattice is distributive.

In [8], generalized Kripke frames were introduced, as a relational framework for the semantics of non-distributive propositional logics. The base frames are triples $(X, \perp, Y)$ where $\perp \subseteq X \times Y$ and the further conditions that the frames are separated and reduced (RS-frames, see [8] for definitions) are imposed on the frames. The dual lattice frames of [18] are not RS-frames, but Hartung's dual lattice frames do enjoy both properties. The reader may recall that Hartung [19] defines a dual lattice frame $(X, \perp, Y)$ so that X and Y are the first and second projection, respectively, of the carrier set of Urquhart's doubly-ordered spaces $(U, \leq_1, \leq_2)$, where U is the set of maximally disjoint filter-ideal pairs. By contrast, the frames $(X, \perp, Y)$ of [18] use the set X of all lattice filters and the set Y of all lattice ideals.

The propositions of a generalized Kripke frame are the Galois-stable subsets A of X, $A = \psi\phi A$, where (ψ, ϕ) is the Galois connection generated by the relation $\perp$. Similarly, the co-propositions are the Galois co-stable subsets B of Y, $B = \phi\psi B$. The semantics is thus 2-sorted, with both a relation of satisfaction $\Vdash$ from worlds in X to sentences and a relation $\succ$ of co-satisfaction (or refutation) from co-worlds in Y to sentences. To model additional logical operators the results of [9] on σ and π extensions of maps are used, devising appropriate relations on the frames $(X, \perp, Y)$ to generate the stable set operators obtained from the σ and π extensions. Examples of applications of RS-frames to the semantics of logical calculi can be found in [2, 5, 8]. There are, however, cases of familiar logical operators where the resulting semantics appears to be forced. For example, temporal operators are typically interpreted over linear orders, dynamic operators are interpreted over graphs, both adequately presented as classical Kripke frames and with the standard definition of the interpretation clause, which however fails to be captured in the RS-frames approach, witness the satisfaction condition below, quoted from [5], using an accessibility relation $R_\Diamond \subseteq Y \times X$ from co-worlds to worlds.

$$x \Vdash \Diamond\varphi \;\text{ iff }\; \forall y \in Y\; (\forall z \in X (z \Vdash \varphi \implies yR_\Diamond z) \implies x \leq_R y) \tag{1}$$

This makes it hard to see how the generalized Kripke frames approach can serve as an appropriate semantic framework for applied logics, such a propositional dynamic logic, or temporal logic, on a non distributive propositional basis. Attempts to overcome this difficulty have been made notably in [3, 4, 6] and though some progress has been made, recapturing the standard meaning of familiar operators while working in the context of generalized Kripke frames has proven an evasive task.

On the other hand, this author has recently pursued a line of research aiming precisely at demonstrating that lack of distribution of conjunction over disjunction and vice-versa has no effect on the way we interpret other operators, resulting in establishing the Kripke-Galois frames approach to the semantics of logics over a non-distributive propositional basis [11–16]. The results in [11–16] can be also seen as contributing to the development of the semantics for Dunn's theory of generalized Galois logics (gaggles) [1,7]. To achieve this, the duality for lattice expansions presented in [10] was used, which however does not deliver a canonical lattice extension.

We show in this article that the apparent failure to recapture the standard meaning of familiar operators in the generalized Kripke frames approach can be overcome by restricting the class of admissible interpretations. We demonstrate the approach by targeting a concrete system, as an example, namely the logic of modal implicative (non-distributive) lattices with an intuitionistic type of negation. Our semantic intuitions are based on both the order-dual semantics presented in [16], as well as in the subsequently and recently developed framework of Kripke-Galois semantics [13, 15].

For reader's convenience, we first review in the rest of this section some technical issues underlying the approach. Section 2 defines the algebras of interest, i.e. modal implicative (bounded) lattices, it introduces the syntax and proof system of the logic, it defines frames and models and, finally, it presents a soundness proof. Section 3 proceeds with the construction of the canonical frame and model. σ (and π) extensions of maps are concretely defined, using the filter/ideal operators introduced in [10].

1.2 Preliminaries on Lattice Frames

By a *lattice frame* we mean a triple $(X, \perp, Y)$ where X, Y are nonempty sets (of worlds and co-worlds) and $\perp \subseteq X \times Y$ is a binary relation, to be called the Galois relation of the frame, generating a Galois connection $\wp(X) \underset{{}^{\perp}(\;)=\psi}{\overset{\phi=(\;)^{\perp}}{\rightleftarrows}} \wp(Y)$ defined on $U \subseteq X$ and $V \subseteq Y$ by

$$\begin{aligned} \phi(U) = U^{\perp} &= \{y \in Y \mid \forall u \in U\; u \perp y\} = \{y \in Y \mid U \perp y\} \\ \psi(V) = {}^{\perp}V &= \{x \in X \mid \forall v \in V\; x \perp v\} = \{x \in X \mid x \perp V\} \end{aligned}$$

A subset $A \subseteq X$ is Galois-stable if $A = \psi\phi(A)$ and we let $\mathcal{G}_\psi(X)$ be the complete lattice of Galois-stable subsets of X. Similarly for $\mathcal{G}_\phi(Y)$ and the complete lattice of co-stable subsets of Y, $B = \phi\psi(B)$. In the sequel, we let $\varnothing_\psi, \varnothing_\phi$ be the least elements of $\mathcal{G}_\psi(X)$ and $\mathcal{G}_\phi(Y)$, respectively, i.e. the intersections of all their members, and we note that they need not be empty.

The relations $x \leq z$ iff $\{x\}^{\perp} \subseteq \{z\}^{\perp}$ on X and $y \leq v$ iff ${}^{\perp}\{y\} \subseteq {}^{\perp}\{z\}$ on Y are preorders on X and Y, respectively. We make the further assumptions that

- X, Y are both bounded partial orders under the respective $\leq$-relation
- $\perp$ is increasing in each argument place, i.e. $x \perp y, x \leq z, y \leq v$ imply $z \perp v$

For $x \in X$ (resp. $y \in Y$) we write Γx for the principal upper set over x, as shorthand for the more accurate $\Gamma(\{x\})$: $\Gamma x = \{z \in X \mid x \leq z\}$. Similarly for Γy, with $y \in Y$.

Definition 1.1 (Closed and Open Elements)**.** A stable set A is a *closed* element of $\mathcal{G}_\psi(X)$ iff it is of the form Γx, for some $x \in X$, and similarly for closed elements Γy of $\mathcal{G}_\phi(Y)$, for $y \in Y$. The respective sets of closed elements will be designated by $\mathcal{G}_\kappa(X) \subseteq \mathcal{G}_\psi(X)$ and $\mathcal{G}_\kappa(Y) \subseteq \mathcal{G}_\phi(Y)$.

Dually, a stable set A is an *open* element of $\mathcal{G}_\psi(X)$ iff it is of the form ${}^{\perp}\{y\} = \psi(\Gamma y)$, for some $y \in Y$ and similarly for open elements $\{x\}^{\perp} = \phi(\Gamma x)$ of $\mathcal{G}_\phi(Y)$, for $x \in X$. The respective sets of open elements will be designated by $\mathcal{G}_o(X) \subseteq \mathcal{G}_\psi(X)$ and $\mathcal{G}_o(Y) \subseteq \mathcal{G}_\phi(Y)$.

Finally, a stable set A is a *clopen* element of $\mathcal{G}_\psi(X)$ if it is both closed and open, i.e. $\Gamma x = A = {}^{\perp}\{y\}$, for some $x \in X, y \in Y$, both necessarily unique. Similarly for clopen elements $\{x\}^{\perp} = B = \Gamma y$ of $\mathcal{G}_\phi(Y)$. We let $\mathcal{G}_{\kappa o}$ designate clopen elements.

The *complex algebra* $\mathfrak{F}^+$ of a lattice frame $\mathfrak{F} = (X, \perp, Y)$ is its algebra of clopen elements $\mathcal{G}_{\kappa o}(X) \subseteq \mathcal{G}_\psi(X)$ and its *dual complex algebra* $\mathfrak{F}^{+'}$ is its dual algebra of clopens $\mathcal{G}_{\kappa o}(Y) \subseteq \mathcal{G}_\phi(Y)$. ∎

By the boundedness assumption, each of X, Y is a closed element, too. Finally, we assume that

- $\mathcal{G}_\kappa(X)$ is a sublattice of $\mathcal{G}_\psi(X)$ and similarly for $\mathcal{G}_\kappa(Y)$ and $\mathcal{G}_\phi(Y)$

Lemma 1.2. Let $(X, \perp, Y)$ be a lattice frame. Then the following hold, for any $x \in X, y \in Y$,

1. $(\Gamma x)^{\perp} = \{x\}^{\perp}$ and ${}^{\perp}(\Gamma y) = {}^{\perp}\{y\}$
2. $\Gamma x \in \mathcal{G}_\psi(X)$ and $\Gamma y \in \mathcal{G}_\phi(Y)$

Proof: For 1), left-to-right is immediate and the other direction uses increasingness of $\perp$. For 2), use 1) and the definition of the partial order. ∎

1.3 Normal Lattice Expansions

This section introduces the kind of algebraic structure of interest in the present article, i.e. expansions of bounded lattices by normal operators, typically arising as the Lindenbaum-Tarski algebras of logical calculi.

By a *distribution type* we mean an element δ of the set $\{1, \partial\}^{n+1}$, for some $n \geq 0$, typically to be written as $\delta = (i_1, \ldots, i_n; i_{n+1})$ and where $i_{n+1} \in \{1, \partial\}$ will be referred to as the *output type* of δ. A *similarity type* τ is then defined as a finite sequence of distribution types, $\tau = \langle \delta_1, \ldots, \delta_k \rangle$.

If δ is a distribution type, its dual $\bar{\delta}$ is the distribution type resulting by changing 1 to ∂ and ∂ to 1. For example, if $\delta = (1, \partial; \partial)$, then its dual $\bar{\delta}$ is $(\partial, 1; 1)$. Letting $\bar{i_j} = 1$ if $i_j = \partial$ and $\bar{i_j} = \partial$ if $i_j = 1$, we obtain for a distribution type $\delta = (i_1, \ldots, i_n; i_{n+1})$ that its dual is concisely defined as $\bar{\delta} = (\bar{i_1}, \ldots, \bar{i_n}; \bar{i}_{n+1})$.

Definition 1.3 (Normal Operators)**.** Following [20], an n-ary monotone operator $f : \mathcal{L}^n \longrightarrow \mathcal{L}$ will be called *additive* if it distributes over joins of $\mathcal{L}$ in each argument place. More generally, if $\mathcal{L}_1, \ldots, \mathcal{L}_n, \mathcal{L}$ are bounded lattices, then a monotone function $f : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \longrightarrow \mathcal{L}$ is *additive*, if for each i, f distributes over binary joins of $\mathcal{L}_i$, i.e. $f(a_1, \ldots, a_{i-1}, b \vee d, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \vee f(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n)$.

As a matter of notation, we write $\mathcal{L}$ for $\mathcal{L}^1$ and $\mathcal{L}^\partial$ for its opposite lattice (where order is reversed, usually designated as $\mathcal{L}^{op}$).

An n-ary operator f on a lattice $\mathcal{L}$ is *normal* [10] if it is an additive function $f : \mathcal{L}^{i_1} \times \cdots \times \mathcal{L}^{i_n} \longrightarrow \mathcal{L}^{i_{n+1}}$, where each i_j, for $j = 1, \ldots, n, n+1$, is in the set $\{1, \partial\}$, i.e. $\mathcal{L}^{i_j}$ is either $\mathcal{L}$, or $\mathcal{L}^\partial$. For a normal operator f on $\mathcal{L}$, its *distribution type* is the $(n+1)$-tuple $\delta(f) = (i_1, \ldots, i_n; i_{n+1})$. We call f *completely normal* if it (co)distributes over arbitrary joins, or meets, at each argument place. ∎

Definition 1.4. A *lattice expansion* is a structure $\mathcal{L} = (L, \wedge, \vee, 0, 1, (f_i)_{i\in k})$ where $k > 0$ is a natural number and for each $i \in k$, f_i is a normal operator on $\mathcal{L}$ of some specified arity $\alpha(f_i) \in \mathbb{N}^+$ and distribution type $\delta(i)$. The *similarity type* of $\mathcal{L}$ is the k-tuple $\tau(\mathcal{L}) = \langle\delta(0), \ldots, \delta(k-1)\rangle$. Where τ is a similarity type, $\mathbb{L}_\tau$ is the class of lattice expansions of similarity type τ. ∎

Definition 1.5. A *canonical extension of a lattice expansion* $(L, \wedge, \vee, 0, 1, (f_i)_{i\in k})$ is a canonical lattice extension (α, C) [9] for the underlying bounded lattice together with an n-ary operator F_i, corresponding to the lattice operator f_i such that in each argument place if f_i (co)distributes over finite joins (or meets), then F_i (co)distributes over arbitrary joins (resp. meets). ∎

From [10] recall that if f is an n-ary normal operator of distribution type $\delta = (i_1, \ldots, i_n; i_{n+1})$, then we defined

$$f^\sharp(u_1, \ldots, u_n) = \bigvee\{u_{fa_1\cdots a_n} \mid \bigwedge_j (a_j \in u_j)\} \tag{2}$$

where u_j is $\begin{cases} \text{an ideal} & \text{if } i_j = 1 \\ \text{a filter} & \text{if } i_j = \partial \end{cases}$, $u_{fa_1\cdots a_n}$ is $\begin{cases} \text{a principla ideal} & \text{if } i_{n+1} = 1 \\ \text{a principal filter} & \text{if } i_{n+1} = \partial \end{cases}$

We may similarly define an operator $f^\flat$ corresponding to the dual distribution type $\bar{\delta}$ by setting

$$f^\flat(u_1, \ldots, u_n) = \bigvee\{u_{fa_1\cdots a_n} \mid \bigwedge_j (a_j \in u_j)\} \tag{3}$$

where u_j is $\begin{cases} \text{an ideal} & \text{if } \bar{i_j} = 1 \ (i_j = \partial) \\ \text{a filter} & \text{if } \bar{i_j} = \partial \ (i_j = 1) \end{cases}$, $u_{fa_1\cdots a_n}$ is $\begin{cases} \text{an ideal} & \text{if } \bar{i}_{n+1} = 1 \ (i_{n+1} = \partial) \\ \text{a filter} & \text{if } \bar{i}_{n+1} = \partial \ (i_{n+1} = 1) \end{cases}$

Lemma 1.6. Assume f is an n-ary normal lattice operator of some distribution type $\delta = (i_1, \ldots, i_n; i_{n+1})$. Then each of $f^\sharp$, $f^\flat$ preserves principal filters/ideals. In other words, if $i_j = 1$ and $u_{a_{i_j}}$ is a principal ideal and $i_k = \partial$ and $u_{a_{i_k}}$ is a principal filter, then $f^\sharp(u_{a_{i_1}}, \ldots, u_{a_{i_n}}) = u_{fa_{i_1}\cdots a_{i_n}}$, where $u_{fa_{i_1}\cdots a_{i_n}}$ is a principal ideal if $i_{n+1} = 1$ and a principal filter if $i_{n+1} = \partial$. Similarly for $f^\flat$.

Proof: The proof for $f^\sharp$ was given in [10] and that for $f^\flat$ is completely similar. ∎

The following theorem was proven in [10], Theorem 6.6.

Theorem 1.7 (Extensions of Normal Operators)**.** Every normal lattice operator, of some distribution type $\delta(f) = (i_1, \ldots, i_n; i_{n+1})$, i.e. $f : L^{i_1} \times \cdots \times L^{i_n} \longrightarrow L^{i_{n+1}}$ extends to a completely normal operator $\hat{f} : I_{i_1} \times \cdots \times I_{i_n} \longrightarrow I_{i_{n+1}}$, where I_{i_j} is the lattice of ideals of L^{i_j}. ∎

To prove that every lattice expansion has a (unique up to isomorphism) canonical extension, [9] introduces a notion of σ and π-extension of maps. σ and π-extensions, reviewed in Section 3.2, are explicitly defined for unary monotone maps. It is then pointed out that if $\mathcal{L}, \mathcal{M}$ are bounded lattices and $\mathcal{L}_\sigma, \mathcal{M}_\sigma$ their respective (unique, up to isomorphism) canonical extensions, then up to isomorphism we have $(\mathcal{L}^\partial)_\sigma = (\mathcal{L}_\sigma)^\partial$, which we therefore denote simply by $\mathcal{L}^\partial_\sigma$, and, again up to isomorphism, $(\mathcal{L}\times\mathcal{M})_\sigma = \mathcal{L}_\sigma \times \mathcal{M}_\sigma$. This allows for extending n-ary maps with various monotonicity properties in each argument place.

We shall demonstrate in the sequel that σ and π-extensions are in fact directly definable from the point operators of equations (2, 3) introduced in [10] and extensively used in [11–16] (see Section 3.2).

2 The Logic of Modal Implicative Lattices

2.1 Syntax, Proof System and Algebraic Semantics

If $\tau = \langle(\partial;\partial), (1;1), (1,\partial;\partial)\rangle$, then a τ-algebra is a bounded lattice expansion $\mathcal{L} = (L, \wedge, \vee, 0, 1, \Box, \Diamond, \rightarrow)$ with a Box, a Diamond and an Implication operator, i.e. a modal algebra on an underlying non-distributive lattice. Its axiomatization is given by the familiar bounded lattice axioms, extended with monotonicity and the respective distribution axioms for $\Box$ (over meets) and $\Diamond$ (over joins) and for the implication operator which, as indicated by its distribution type, co-distributes over joins in the first place and distributes over meets in the second place. An intuitionistic type of negation is introduced by setting $\neg a = a \to 0$.

The language of τ-algebras is the extension of the language of Positive Lattice Logic (**PLL**) obtained by adding modal operators and implication, built on a non-empty, countable set P of propositional variables.

$$\mathcal{L}(P) \ni \varphi \quad = \quad p\ (p \in P) \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \varphi \to \varphi$$

Since we have no interest in proof-theoretic matters in this article, we present a symmetric consequence proof system, in groups of axioms and rules, beginning with the pure **PLL** system.

1. Axioms and rules for Positive Lattice Logic

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi \qquad \frac{\varphi \vdash \psi \qquad \varphi \vdash \vartheta}{\varphi \vdash \psi \wedge \vartheta} \qquad \frac{\varphi \vdash \psi \qquad \psi \vdash \vartheta}{\varphi \vdash \vartheta}$$

$$\bot \vdash \varphi \qquad \varphi \vdash \top$$

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \qquad \frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta}$$

2. Axioms and rules for the box operator

$$\top \vdash \Box\top \qquad \frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \qquad \Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$$

3. Axioms and rules for the diamond operators

$$\Diamond\bot \vdash \bot \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi} \qquad \Diamond(\varphi \vee \psi) \vdash \Diamond\varphi \vee \Diamond\psi$$

4. Axioms and Rules for Implication

$$\frac{\chi \vdash \varphi \qquad \psi \vdash \vartheta}{\varphi \to \psi \vdash \chi \to \vartheta} \qquad (\varphi \to \vartheta) \wedge (\psi \to \vartheta) \vdash (\varphi \vee \psi) \to \vartheta \qquad (\varphi \to \chi) \wedge (\varphi \to \vartheta) \vdash \varphi \to (\chi \wedge \vartheta)$$

Negation is introduced into the system by definition, setting $\neg\varphi \equiv \varphi \to \bot$. We leave it to the interested reader to verify that the Lindenbaum-Tarski algebra of the logic is a τ-algebra in the sense introduced at the beginning of this section.

Algebraic soundness and completeness is immediate, by the way we presented the proof system of the logic.

2.2 Frames, Models and Soundness

Let τ be the similarity type $\tau = \langle(\partial;\partial), (1;1), (1,\partial;\partial)\rangle$.

Definition 2.1. *A refined τ-frame* is a tuple $\mathfrak{F}_\tau = ((X, \perp, Y), (R_{\Box}, R^\partial_{\Box}), (R_{\Diamond}, R^\partial_{\Diamond}), (R_{>}, R^\partial_{>}))$ where

1. $(X, \perp, Y)$ is a lattice frame in the sense of Section 1.2.
2. $R_{\Diamond}, R_{\Box} \subseteq X \times X$ are binary accessibility relations on X and $R^\partial_{\Diamond}, R^\partial_{\Box} \subseteq Y \times Y$ are dual accessibility relations on Y, such that
   (a) for any $A \in \mathcal{G}_\kappa(X)$, $\diamondminus A = \psi \diamondminus^\partial (\phi A)$ and $\boxminus A = \psi \boxminus^\partial (\phi A)$, where $\diamondminus, \boxminus, \diamondminus^\partial, \boxminus^\partial$ are the induced image operators on $\wp(X)$ defined as follows
      - $\diamondminus U = \{x \in X \mid \exists z \in X\ (xR_{\Diamond}z \text{ and } z \in U)\}$
      - $\boxminus U = \{x \in X \mid \forall z \in U\ (xR_{\Box}z \implies z \in U)\}$
      - $\diamondminus^\partial V = \{y \in Y \mid \forall v \in Y\ (yR^\partial_{\Diamond}v \implies v \in V)\}$
      - $\boxminus^\partial V = \{y \in Y \mid \exists v \in V\ (yR^\partial_{\Box}v \text{ and } v \in V)\}$

(b) For all $x, z \in X$, if $xR_\diamond z$ and $z \in \varnothing_\psi$, then $x \in \varnothing_\psi$

3. $R_>, R^\partial_>$ are ternary relations $R_> \subseteq X^2 \times Y$ and $R^\partial_> \subseteq X \times Y^2$ generating operators $\Rightarrow, \Rightarrow^\partial$ on $\wp(X)$

   - $U \Rightarrow U' = \{x \in X \mid \forall z \in X \forall y \in Y\ (z \in U \text{ and } xzR_> y \implies y \notin \phi U')\}$
   - $V \Rightarrow^\partial V' = \{y \in Y \mid \forall v \in Y \forall x \in X\ (xR^\partial_> vy \text{ and } v \in V \implies x \notin \psi V')\}$

   such that $\Rightarrow$ is an operator of distribution type $(1, \partial; \partial)$ on $\mathcal{G}_\kappa(X)$ and then $\Rightarrow^\partial$ is one of type $(\partial, 1; 1)$ on $\mathcal{G}_\kappa(Y)$. ■

Remark 2.2. The reason we wish to associate to a distribution type a pair of relations $R_\delta, R^\partial_\delta$, rather than a single relation, is grounded on the essential feature of every lattice representation theorem, where two dually isomorphic concrete meet-semilattices $\mathcal{S} \simeq \mathcal{K}^\partial$ are shown to be isomorphic and dually isomorphic to the original lattice $\mathcal{L}$, see [10,18,19,22]. Thereby, a normal n-ary lattice operator f is also both represented as an operator $\odot_f$ and dually represented as an operator $\odot^\partial_f$ in each of $\mathcal{S}$ and $\mathcal{K}$, respectively, and so that the two operators are interdefinable using the dual isomorphism. Each of the relations $R_\delta, R^\partial_\delta$ is used to generate its respective operator $\odot_f$ (on $\mathcal{S}$) and $\odot^\partial_f$ (on $\mathcal{K}$). Refined frames are equipped with a pair $(R_\delta, R^\partial_\delta)$ of accessibility relations for each distribution type. A frame is *not refined* (called *reduced* in [16]) if for some (perhaps all) distribution type only one of R_δ or R^∂_δ is part of the model structure. In this case one of the operators is defined using the corresponding accessibility relation and the definition of the other can be then derived, using the Galois connection. ■

Definition 2.3 (Models and Validity)**.** Let $\mathfrak{F}$ be a refined τ-frame (Definition 2.1). A *τ-model* $\mathfrak{M} = (\mathfrak{F}, V)$ on $\mathfrak{F}$ is additionally equipped with a pair $V = (V_1, V_2)$ of admissible interpretation and co-interpretation maps $V_1 : P \longrightarrow \mathcal{G}_\kappa(X)$ and $V_2 : P \longrightarrow \mathcal{G}_\kappa(Y)$ such that for any propositional variable p, $V_2(p) = V_1(p)^{\perp\!\!\!\!-}$. The relations of satisfaction $\Vdash \subseteq X \times \mathcal{L}(P)$ and co-satisfaction (dual satisfaction, refutation) $\succ \subseteq Y \times \mathcal{L}(P)$ are generated by mutual recursion as shown in Table 1, where $x, z \in X$ and $y, v \in Y$, subject to the requirement that for any sentence φ, $x \Vdash \varphi$ iff $\forall y\ (y \succ \varphi \implies x \perp\!\!\!\!- y)$ and, dually, $y \succ \varphi$ iff $\forall x\ (x \Vdash \varphi \implies x \perp\!\!\!\!- y)$.

Table 1: Interpretation and Dual Interpretation
($x, z \in X,\ y, v \in Y$ and $R = \not\perp\!\!\!\!-$))

$x \Vdash p$	iff	$x \in V_1(p)$	$y \succ p$	iff	$y \in V_2(p)$
$x \Vdash \top$		always	$y \succ \bot$		always
$x \Vdash \bot$	iff	$x \in \varnothing_\psi$	$y \succ \top$	iff	$y \in \varnothing_\phi$
$x \Vdash \varphi \wedge \psi$	iff	$x \Vdash \varphi$ and $x \Vdash \psi$	$y \succ \varphi \vee \psi$	iff	$y \succ \varphi$ and $y \succ \psi$
$x \Vdash \varphi \vee \psi$	iff	$\forall y\ (xRy \implies y \not\succ \varphi$ or $y \not\succ \psi)$	$y \succ \varphi \wedge \psi$	iff	$\forall x\ (xRy \implies x \nVdash \varphi$ or $x \nVdash \psi)$
$x \Vdash \Box\varphi$	iff	$\forall z\ (xR_\Box z \implies z \Vdash \varphi)$	$y \succ \Box\varphi$	iff	$\exists v\ (yR^\partial_\Box v$ and $v \succ \varphi)$
$x \Vdash \Diamond\varphi$	iff	$\exists z\ (xR_\Diamond z$ and $z \Vdash \varphi)$	$y \succ \Diamond\varphi$	iff	$\forall v\ (yR^\partial_\Diamond v \implies v \succ \varphi)$
$x \Vdash \varphi \to \psi$	iff	$\forall z \forall y\ (z \Vdash \varphi\ \wedge\ xzR_> y \implies y \not\succ \psi)$	$y \succ \varphi \to \psi$	iff	$\forall z \forall v\ (v \succ \varphi\ \wedge\ zR^\partial_> yv \implies z \nVdash \psi)$

A sentence φ is *(dually) satisfied* in a model $\mathfrak{M} = (\mathfrak{F}, V)$ if there is a world $x \in X$ such that $x \Vdash \varphi$ (respectively, $y \succ \varphi$, for some $y \in Y$). It is *(dually) valid* in $\mathfrak{M}$ iff it is satisfied (respectively, dually satisfied) at all worlds $x \in X$ (respectively, at all $y \in Y$).

A sequent $\varphi \vdash \psi$ is valid in a model $\mathfrak{M}$ iff for every world x of $\mathfrak{M}$, if $x \Vdash \varphi$, then $x \Vdash \psi$. Equivalently, the sequent is valid in the model $\mathfrak{M}$ iff for every co-world $y \in Y$, if $y \succ \psi$, then $y \succ \varphi$. The sequent is valid in a frame $\mathfrak{F}$ if it is valid in every model $\mathfrak{M}$ based on the frame $\mathfrak{F}$. Finally, we say that the sequent is valid in a class $\mathbb{F}$ of frames iff it holds in every frame in $\mathbb{F}$. ■

Lemma 2.4. For every sentence φ, $[\![\varphi]\!] = \{x \in X \mid x \Vdash \varphi\} \in \mathcal{G}_\kappa(X) \subseteq \mathcal{G}_\psi(X)$ and $(\!|\varphi|\!) = \{y \in Y \mid y \succ \varphi\} \in \mathcal{G}_\kappa(Y) \subseteq \mathcal{G}_\phi(Y)$. In particular, $[\![\varphi]\!]^{\perp\!\!\!\!-} = (\!|\varphi|\!)$ and ${}^{\perp\!\!\!\!-}(\!|\varphi|\!) = [\![\varphi]\!]$.

Proof: Immediate, by the requirements set on models. ■

Theorem 2.5 (Soundness)**.** The logic of modal implicative lattices is sound in the class of τ-models (Definition 2.3).

Proof: By the semantic clauses, we have in particular $\llbracket \top \rrbracket = X \in \mathcal{G}_\psi(X)$, $\llbracket \bot \rrbracket = \bigcap \mathcal{G}_\psi(X) \in \mathcal{G}_\psi(X)$ and $(\!| \bot |\!) = Y \in \mathcal{G}_\phi(Y)$, $(\!| \top |\!) = \bigcap \mathcal{G}_\phi(Y) \in \mathcal{G}_\phi(Y)$, where we used the boundedness assumption for X, Y. Thereby, the **PLL** axioms for top and bottom are valid. The clause for disjunction can be equivalently written in the form $x \Vdash \vartheta \vee \chi$ iff $\forall y \in Y\ (y \succ \vartheta \vee \chi \implies x \perp y)$, hence $\llbracket \vartheta \vee \chi \rrbracket = {}^{\perp}((\!|\vartheta|\!) \cap (\!|\chi|\!)) = {}^{\perp}(\llbracket \vartheta \rrbracket^{\perp} \cap \llbracket \chi \rrbracket^{\perp}) = {}^{\perp}((\llbracket \vartheta \rrbracket \cup \llbracket \chi \rrbracket)^{\perp})$, i.e. disjunction is interpreted as the closure of a union and co-interpreted as intersection. By the above observations, the axioms and rules for **PLL** are sound, given the assumption that the set of closed elements isa sublattice of $\mathcal{G}_\psi(X)$.

For necessity, by the definition of the $\boxminus$ set-operator from the relation $R_\Box$, both the monotonicity rule and the axiom of distribution over intersections are immediately seen to be valid. For validity of the axiom $\top \vdash \Box\top$, observe that $\llbracket \Box\top \rrbracket = \boxminus X = \{x \in X \mid \forall z \in X\ (x R_\Box z \implies z \in X)\} = X = \llbracket \top \rrbracket$.

For possibility, the monotonicity rule is valid by monotonicity of the set-operator $\diamondminus$. Validity of the axiom $\Diamond\bot \vdash \bot$ is ensured by condition 2(b) on frames. Validity of the distribution axiom over disjunction follows from the frame condition 2(a) and the fact that $\diamondminus^\partial$ distributes over intersections, since $\diamondminus(A \vee B) = \diamondminus A \vee \diamondminus B$ iff $\phi \diamondminus (A \vee B) = \phi(\diamondminus A \vee \diamondminus B)$ iff $\diamondminus^\partial \phi(A \vee B) = \diamondminus^\partial(\phi A) \cap \diamondminus^\partial(\phi B)$ iff $\diamondminus^\partial(\phi(A) \cap \phi(B)) = \diamondminus^\partial(\phi A) \cap \diamondminus^\partial(\phi B)$. Just like for the set operator $\boxminus$, distribution of $\diamondminus^\partial$ over intersections follows from the definition.

For implication, the monotonicity rule holds on $\wp(X)$ in general, as the reader can easily check, given the definition of $U \Rightarrow V$ using the relation $R_{>}$. Validity of the axioms for the (co)distribution properties of implication have been forced by the definition of frames. ∎

3 Completeness

Completeness is proved by the standard technique of representation of the Lindenbaum-Tarski algebra of the logic, a τ-algebra, where τ is the similarity type $\tau = \langle(\partial;\partial), (1;1), (1,\partial;\partial)\rangle$. We show that $\varphi \vdash \psi$ iff $[\varphi] \leq [\psi]$ iff $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ (iff $(\!|\psi|\!) \subseteq (\!|\varphi|\!)$), where the first equivalence is the algebraic completeness part discussed in Section 2. For the underlying lattice representation the reader is referred to [18], though the brief review presented in Section 3.1 should suffice for the purposes of this article.

3.1 Canonical Lattice Extensions

In [18], a follow up paper to the report [17], the following result was proven.

Theorem 3.1 (Lattice Representation, [17, 18])**.** Every bounded lattice $\mathcal{L}$ can be represented and co-represented in the lattice frame $(X, \perp, Y)$ where X is the set of lattice filters of $\mathcal{L}$ and Y its set of ideals, while $\perp \subseteq X \times Y$ is defined by $x \perp y$ iff $x \cap y \neq \varnothing$. The representation and co-representation maps α, β are defined by $\alpha(a) = \{x \in X \mid a \in x\}$ and $\beta(a) = \{y \in Y \mid a \in y\}$. Endowing X, Y with the natural Stone topologies generated by the subbasis elements $\{\alpha(a) \mid a \in \mathcal{L}\} \cup \{-\alpha(a) \mid a \in \mathcal{L}\}$ and similarly for Y, $\{\beta(a) \mid a \in \mathcal{L}\} \cup \{-\beta(a) \mid a \in \mathcal{L}\}$, the image $\alpha[\mathcal{L}]$ of the representation map is characterized as the family of compact-open stable subsets of X and similarly for $\beta[\mathcal{L}]$ and Y. ∎

In other words, there is an isomorphism $\mathcal{L} \simeq (\mathcal{L}_+)^+$ of a lattice with the complex algebra of its dual frame and an anti-isomorphism $\mathcal{L}^\partial \simeq (\mathcal{L}_+)^{+'}$ of $\mathcal{L}$ with the dual complex algebra of its dual frame.

In [9] Gehrke and Harding introduced a notion of canonical extension of bounded lattices, generalizing the corresponding notion for distributive lattices and Boolean algebras and which characterizes the dual objects of lattices in purely lattice-theoretic terms, without resorting to topological properties. They define a *canonical extension* of a bounded lattice $\mathcal{L}$ as a pair (α, C), where C is a complete lattice and $\alpha : \mathcal{L} \hookrightarrow C$ is a lattice embedding and where

• (density) $\alpha[\mathcal{L}]$ is *dense* in C, where the latter means that every element of C can be expressed both as a meet of joins and as a join of meets of elements in $\alpha[\mathcal{L}]$
• (compactness) for any set A of closed elements and any set B of open elements of C, $\bigwedge A \leq \bigvee B$ iff there exist finite subcollections $A' \subseteq A, B' \subseteq B$ such that $\bigwedge A' \leq \bigvee B'$

where the *closed elements* of C are defined in [9] as the elements in the meet-closure of the representation map α and the *open elements* of C are defined dually as the join-closure of the image of α.

In [9], Proposition 2.6, Gehrke and Harding, prove existence of canonical extensions for bounded lattices by showing that the completion of a bounded lattice $\mathcal{L}$ obtained in the lattice representation theorem of [17, 18] is a canonical extension of $\mathcal{L}$. Furthermore, canonical extensions are proven to be unique, up to isomorphism ([9], Proposition 2.7). Urquhart's [22] and, subsequently, Hartung's [19] lattice represenations (both predating [18]) also constitute canonical extensions of the represented lattice (see [9], though no proof is presented) and using Urquhart's representation, in particular, it is easily proven (see [22]) that the representation reduces to the Priestley representation [21] when the represented lattice is distributive.

In the sequel we present some technical clarifications that will be useful in the rest of this article.

Let $\mathfrak{L}$ be a bounded lattice and $\mathfrak{L}_+ = \mathfrak{F} = (X, \perp, Y)$, following [18], be the canonical dual frame of the lattice $\mathfrak{L}$, where X is the set of lattice filters, Y is the set of lattice ideals and where $\perp \subseteq X \times Y$ is defined by $x \perp y$ iff $x \cap y \neq \emptyset$.

Designate the canonical (co)representation maps by $\alpha_X(a) = \alpha(a) = \{x \in X \mid a \in x\}$ and $\alpha_Y(a) = \beta(a) = \{y \in Y \mid a \in y\}$. Letting $x_a = a\uparrow$ be the principal filter generated by the lattice element a and writing Γx for the principal cone over the filter x, $\Gamma x = \{x' \in X \mid x \leq x'\}$ (where we use $\leq$ for filter-inclusion), it is straightforward to see, by join-density of principal filters, that the closed elements of $\mathcal{G}_\psi(X)$ in the sense of [9] are precisely the elements of the form Γx, with $x \in X$, since $\bigwedge_{a \in A \subseteq \mathcal{L}} \Gamma x_a = \Gamma(\bigvee_{a \in A \subseteq \mathcal{L}} x_a) = \Gamma x_A$. Similarly, the closed elements of $\mathcal{G}_\phi(Y)$, defined as the meet-closure of the image of the co-representation map β in [9], are the principal cones over ideals $\Gamma y = \{y' \in Y \mid y \leq y'\}$, this time using join-density of principal ideals.

The following is an immediate consequence of definitions and its proof is left to the reader.

Lemma 3.2. Let $(X, \perp, Y)$ be the canonical dual frame of a bounded lattice. For any $x \in X, y \in Y$

- $x \perp y_a$ iff $a \in x$ and $x_a \perp y$ iff $a \in y$
- ${}^{\perp}\{y_a\} = \{x \in X \mid a \in x\} = \Gamma x_a$ and $\{x_a\}^{\perp} = \{y \in Y \mid a \in y\} = \Gamma y_a$. ∎

Recall that the *open elements* of $\mathcal{G}_\psi(X)$ are defined dually in [9] as the join-closure of the image of the representation map α. Given the previous Lemma and the fact that $\psi\beta = \alpha$ we obtain the following.

Corollary 3.3. The open elements of $\mathcal{G}_\psi(X)$ are the elements of the form ${}^{\perp}\{y\}$, for $y \in Y$. Similarly, the open elements of $\mathcal{G}_\phi(Y)$ are the elements of the form $\{x\}^{\perp}$, for $x \in X$. ∎

It follows from the above that our definition of closed and open elements of the canonical lattice frame (Definition 1.1) coincides with that introduced in [9].

Corollary 3.4. The clopen elements in $\mathcal{G}_\psi(X)$ are the elements $\Gamma x_a = {}^{\perp}\{y_a\} = \alpha(a)$ and the clopen elements of $\mathcal{G}_\phi(Y)$ are the elements $\Gamma y_a = \{x_a\}^{\perp} = \beta(a)$. ∎

It was observed in [9] that if $\mathcal{L}$ is a bounded lattice and $M \subseteq \mathcal{L}$, then

1. $\bigcap \alpha[M] = \{x \in X \mid M \subseteq x\}$
2. $\bigcap \beta[M] = \{y \in Y \mid M \subseteq y\}$
3. $\bigvee \alpha[M] = \psi\phi(\bigcup \alpha[M]) = \{x \in X \mid \forall y \in Y\ (M \subseteq y \Longrightarrow x \perp y)\}$
4. $\bigvee \beta[M] = \phi\psi(\bigcup \beta[M]) = \{y \in Y \mid \forall x \in X\ (M \subseteq x \Longrightarrow x \perp y)\}$

Therefore, we obtain:

Corollary 3.5. If $z \in X$ is a filter, then

1. $\bigcap \alpha[z] = \{x \in X \mid z \leq x\} = \Gamma z$
2. $\bigvee \beta[z] = \phi\psi(\bigcup \beta[z]) = \{y \in Y \mid \forall x \in X\ (z \leq x \Longrightarrow x \perp y)\} = \{y \in Y \mid \Gamma z \perp y\} = \{z\}^{\perp} = \phi(\{z\})$

and if $v \in Y$ is an ideal, then

1. $\bigcap \beta[v] = \{y \in Y \mid v \leq y\} = \Gamma v$
2. $\bigvee \alpha[v] = \psi\phi(\bigcup \alpha[v]) = \{x \in X \mid \forall y \in Y\ (v \leq y \Longrightarrow x \perp y)\} = {}^{\perp}\{v\} = \psi(\{v\})$. ■

By the results of [9], as specialized to the case of the canonical extension of a bounded lattice given in [17,18], for any $U \subseteq X$ and $V \subseteq Y$ we have:

- $\psi\phi U = \bigvee\{\bigwedge \alpha[x] \mid x \in U\} = \bigvee\{\Gamma x \mid x \in U\}$
- $\phi\psi V = \bigvee\{\bigwedge \beta[v] \mid v \in V\} = \bigvee\{\Gamma v \mid v \in V\}$.

Given the above together with the fact that ψ, ϕ are dual isomorphisms (hence they switch joins to meets and meets to joins), then the following is the case for a stable set $A \in \mathcal{G}_\psi(X) \subseteq \wp(X)$ and a co-stable set $B \in \mathcal{G}_\phi(Y) \subseteq \wp(Y)$:

$$\bigvee_{x \in A} \Gamma x \;=\; A \;=\; \bigwedge_{A \perp y} ({}^{\perp}\{y\}) \tag{4}$$

$$\bigvee_{y \in B} \Gamma y \;=\; B \;=\; \bigwedge_{x \perp B} (\{x\}^{\perp}) \tag{5}$$

3.2 σ, π-Extensions of Lattice Maps

Recall from [9] that if (α, C) is a canonical extension of a bounded lattice L, and $\mathtt{K}, \mathtt{O}$ are its sets of closed and open elements, the σ and π-extensions $f_\sigma, f_\pi : \mathcal{L}_\sigma \longrightarrow \mathcal{L}_\sigma$ (where, following the notation of [9], $\mathcal{L}_\sigma = \mathcal{G}_\psi(X)$ designates the canonical extension of $\mathcal{L}$) of a unary monotone map $f : L \longrightarrow L$ are defined in [9], taking also into consideration Lemma 4.3 of [9], by setting, for $k \in \mathtt{K}$, $o \in \mathtt{O}$ and $u \in C$

$$f_\sigma(k) = \bigwedge\{f(a) \mid k \leq a \in L\} \qquad f_\sigma(u) = \bigvee\{f_\sigma(k) \mid \mathtt{K} \ni k \leq u\} \tag{6}$$

$$f_\pi(o) = \bigvee\{f(a) \mid L \ni a \leq o\} \qquad f_\pi(u) = \bigwedge\{f_\pi(o) \mid u \leq o \in \mathtt{O}\} \tag{7}$$

where in these definitions $\mathcal{L}$ is identified with its isomorphic image in C and $a \in \mathcal{L}$ is then identified with its representation image.

Working concretely with the canonical extension of [18], the σ extension $f_\sigma : \mathcal{G}_\psi(X) \longrightarrow \mathcal{G}_\psi(X)$ of a monotone map f as in equation (6) and the dual σ-extension $f_\sigma^\partial : \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$ (not used in [9]) are defined by instantiating equation (6) in the concrete canonical extension of [18] considered here by setting, for $x \in X$ and $y \in Y$ and where x_e is a principal filter and y_e a principal ideal.

$$f_\sigma(\Gamma x) = \bigwedge\{\alpha_X(fa) \mid a \in \mathcal{L}, \Gamma x \leq \alpha_X(a)\} = \bigwedge\{\Gamma x_{fa} \mid \Gamma x \subseteq \Gamma x_a\} = \bigwedge\{\Gamma x_{fa} \mid a \in x\} = \Gamma(\bigvee\{x_{fa} \mid a \in x\}) \tag{8}$$

$$f_\sigma^\partial(\Gamma y) = \bigwedge\{\alpha_Y(fa) \mid a \in \mathcal{L}, \Gamma y \leq \alpha_Y(a)\} = \bigwedge\{\Gamma y_{fa} \mid \Gamma y \subseteq \Gamma y_a\} = \bigwedge\{\Gamma y_{fa} \mid a \in y\} = \Gamma(\bigvee\{y_{fa} \mid a \in y\}) \tag{9}$$

We can now state the following Proposition, linking the results of [9] to those proven in [10].

Proposition 3.6. Let f be a unary monotone map.

1. If $\delta(f) = (1;1)$, then $f_\sigma(\Gamma x) = \Gamma(f^\flat x)$
2. If $\delta(f) = (\partial;\partial)$, then $f_\sigma(\Gamma x) = \Gamma(f^\sharp x)$

Proof: Immediate, by comparing the definition of f_σ in equations (8, 9) and the definitions of $f^\sharp, f^\flat$ in equations (2, 3), repeated from [10]. ∎

Note that Proposition 3.6 readily extends to the case of any distribution type δ, but we shall restrict ourselves here to the operators of our target system.

Lemma 3.7. The π-extension of a monotone map f is obtained from the dual σ-extension by setting $f_\pi({}^{\perp}\{y\}) = \psi(f^\partial_\sigma(\phi({}^{\perp}\{y\}))) = \psi(f^\partial_\sigma(\Gamma y))$.

Proof: We have

$$\begin{aligned}
\psi(f^\partial_\sigma(\Gamma y)) &= \psi(\bigwedge\{\Gamma y_{fa} \mid a \in y\}) &&= \bigvee\{\psi\Gamma y_{fa} \mid a \in y\} &&= \bigvee\{\Gamma x_{fa} \mid a \in y\} \\
&= \bigvee\{\Gamma x_{fa} \mid y_a \leq y\} &&= \bigvee\{\Gamma x_{fa} \mid \Gamma y \subseteq \Gamma y_a\} &&= \bigvee\{\Gamma x_{fa} \mid \psi\Gamma y_a \subseteq \psi\Gamma y\} \\
&= \bigvee\{\Gamma x_{fa} \mid {}^{\perp}\{y_a\} \subseteq {}^{\perp}\{y\}\} &&= \bigvee\{\Gamma x_{fa} \mid \alpha_Y(a) \subseteq {}^{\perp}\{y\}\} &&= f_\pi({}^{\perp}\{y\})
\end{aligned}$$

which proves the claim. In particular, we obtained $f_\pi(\psi\Gamma y) = \psi f^\partial_\sigma(\Gamma y)$. ∎

For antitone maps, since filters of L^∂ are the ideals of L, i.e. $F(L^\partial) = I(L)$, and conversely $I(L^\partial) = F(L)$, the canonical frame for L^∂ is the frame $(X', \perp', Y') = (Y, \perp^{-1}, X)$, where X is the set of filters of L (hence the ideals of L^∂), Y is its set of ideals (the filters of L^∂) and where $\perp^{-1} \subseteq Y \times X$, $y \perp^{-1} x$ iff $x \perp y$ iff $x \cap y \neq \varnothing$. Let ϕ', ψ' be the generated Galois connection $\wp(Y) \underset{\psi'}{\overset{\phi'}{\rightleftarrows}} \wp(X)^\partial$, where for $V \subseteq Y$, $U \subseteq X$ we have $\phi'(V) = \{x \in X \mid V \perp^{-1} x\} = \{x \in X \mid x \perp V\} = \psi(V) = {}^{\perp}V$ and $\psi'(U) = \{y \in Y \mid y \perp^{-1} U\} = \{y \in Y \mid U \perp y\} = \phi(U) = U^{\perp}$.

Therefore, $\mathcal{G}_{\psi'}(X') = \mathcal{G}_\phi(Y) \subseteq \wp(Y)$ and $\mathcal{G}_{\phi'}(Y') = \mathcal{G}_\psi(X) \subseteq \wp(X)$. Since $\mathcal{G}_\psi(X) \simeq \mathcal{G}_\phi(Y)^\partial$, we have $\mathcal{G}_{\psi'}(X') \simeq \mathcal{G}_\psi(X)^\partial$. In other words, $(L^\partial)_\sigma \simeq (L_\sigma)^\partial$. As already pointed out when defining closed and open elements, we have $\mathcal{G}_\kappa(\mathcal{L}^\partial_\sigma) \simeq \mathcal{G}_o(\mathcal{L}_\sigma)$ and, dually, $\mathcal{G}_o(\mathcal{L}^\partial_\sigma) \simeq \mathcal{G}_\kappa(\mathcal{L}_\sigma)$ where the bijections involved are given by $\Gamma y \mapsto {}^{\perp}\{y\} = \psi(\Gamma y)$ and $\Gamma x \mapsto \{x\}^{\perp} = \phi(\Gamma x)$.

For n-ary maps and product lattices a similar analysis shows that $(\mathcal{L} \times \mathcal{M})_\sigma \simeq \mathcal{L}_\sigma \times \mathcal{M}_\sigma$. We will provide more details when we discuss the representation of implication.

For later use we list the following results from [9]

Proposition 3.8 ([9], Lemmas 4.3, 4.4, 4.6)**.** The following hold for a unary monotone map f

1. The σ and π extensions f^σ, f^π of f agree on closed or open elements
2. If either f^σ preserves all joins, or f^π preserves all meets, then $f^\sigma = f^\pi$
3. If f preserves binary joins then f^σ preserves all joins anf if f preserves binary meets, then f^π preserves all meets. ∎

3.3 Representing Implication and the Modal Operators

This section contains all the technical representation results needed for our completeness theorem. Its main objective is twofold. First, for each of the operators of interest in this article, we demonstrate that the corresponding σ, π extensions of [9] are definable directly from the filter/ideal operators defined in [10] as in equations (2, 3). Defining σ, π-extensions using the filter/ideal operators allows us to devise appropriate accessibility relations that generate the respective operators on stable sets. Furthermore, it is precisely this connection between σ, π-extensions and filter/ideal operators that bridges the gap between the semantics based on the canonical extensions theory of [8] and the Kripke-Galois semantics recently proposed by this author [11–16].

3.3.1 Possibility

Instantiating equations (2, 3) of Section 1.3 to the case of the unary diamond operator, of distribution type (1;1), we obtain filter and ideal operators $\Diamond^\flat : X \longrightarrow X$ and $\Diamond^\sharp : Y \longrightarrow Y$.

$$\Diamond^\flat\, x = \bigvee\{x_{\Diamond a} \mid a \in x \in X\} \qquad\qquad \Diamond^\sharp\, y = \bigvee\{y_{\Diamond a} \mid a \in y \in Y\} \tag{10}$$

By Lemma 1.6, $\Diamond^\flat x_a = x_{\Diamond a}$ and $\Diamond^\sharp y_a = y_{\Diamond a}$. Next, following our approach in [10], we define an operator $\diamondminus$ on the closed elements $\mathcal{G}_\kappa(X) = \{\Gamma x \mid x \in X\}$ of $\mathcal{G}_\psi(X)$ and $\diamondminus^\partial$ on the closed elements $\mathcal{G}_\kappa(Y) = \{\Gamma y \mid y \in Y\}$ of $\mathcal{G}_\phi(Y)$ by setting

$$\diamondminus(\Gamma x) = \Gamma(\Diamond^\flat x) \qquad\qquad \diamondminus^\partial\,(\Gamma y) = \Gamma(\Diamond^\sharp y)$$

By the results of [9], the set of closed elements of $\mathcal{G}_\psi(X)$ is join-dense in $\mathcal{G}_\psi(X)$, hence the operator extends immediately to the whole of $\mathcal{G}_\psi(X)$, $\diamondminus A = \bigvee_{x \in A} \diamondminus(\Gamma x)$. Similarly for the dual operator $\diamondminus^\partial$ on $\mathcal{G}_\phi(Y)$.

Lemma 3.9. The following hold:

1. $\diamondminus(\Gamma x) = \Diamond_\sigma(\Gamma x) = \bigwedge\{\Gamma x_{\Diamond a} \mid a \in x \in X\}$
2. $\diamondminus^\partial(\Gamma y) = \Diamond^\partial_\sigma(\Gamma y)$
3. $\Diamond_\pi({}^\perp\{y\}) = \psi(\diamondminus^\partial(\phi({}^\perp\{y\}))) = \psi(\diamondminus^\partial(\Gamma y)) = \bigvee\{\Gamma x_{\Diamond a} \mid a \in y \in Y\}$.
4. $\Diamond^\partial_\pi(\{x\}^\perp) = \phi(\Diamond_\sigma \psi(\{x\}^\perp)) = \phi(\Diamond_\sigma \Gamma x) = \phi(\Gamma(\Diamond^\flat x)) = \{\Diamond^\flat x\}^\perp$
5. $\diamondminus$ is the σ-extension, in the terminology of [9], of the lattice $\Diamond$ operator and, similarly, the π-extension of $\Diamond$, in the terminology of [9] is the image under the Galois map ψ of the map $\diamondminus^\partial$

Proof: For 1) and 2) Proposition 3.6 provides the proof for the general case, but the calculation is simply $\Diamond_\sigma(\Gamma x) = \bigwedge\{\Gamma x_{\Diamond a} \mid a \in x \in X\} = \Gamma(\bigvee\{x_{\Diamond a} \mid a \in x \in X\}) = \Gamma(\Diamond^\flat x) = \diamondminus(\Gamma x)$.

3) and 4) follow from Lemma 3.7, but we give details for the sake of the reader.

$$\begin{aligned}
\psi(\diamondminus^\partial(\Gamma y)) &= \psi\Gamma(\Diamond^\sharp y) = \psi\Gamma(\bigvee\{y_{\Diamond a} \mid a \in y \in Y\}) \\
&= \psi(\bigwedge\{\Gamma y_{\Diamond a} \mid a \in y \in Y\}) \\
&= \bigvee\{\psi\Gamma y_{\Diamond a} \mid a \in y \in Y\}) \\
&= \bigvee\{{}^\perp\{y_\Diamond a\} \mid a \in y \in Y\}) \\
&= \bigvee\{\Gamma x_{\Diamond a} \mid a \in y \in Y\}) \text{ (because principal filters/ideals are clopen)} \\
&= \Diamond_\pi({}^\perp\{y\})
\end{aligned}$$

Hence, $\diamondminus$ is the σ-extension, in the terminology of [9], of the lattice $\Diamond$ operator and, similarly, the π-extension of $\Diamond$, in the terminology of [9] is the image under the Galois map ψ of the map $\diamondminus^\partial$. Extensions of σ,π-maps to the whole of $\mathcal{G}_\psi(X)$ are obtained in [9] using join-density of closed and meet-density of open elements. ■

We may conclude from the above that the σ,π extensions are really obtained from the point operators introduced in [10]. This allows us to define now the appropriate accessibility relation.

Definition 3.10. Define $R_\diamond \subseteq X \times X$ by setting $xR_\diamond z$ iff $\Diamond^\flat z \leq x$. The co-accessibility relation $R^\partial_\diamond$ is the binary relation on ideals defined by $yR^\partial_\diamond v$ iff $\forall a\,(\Diamond a \in y \implies a \in v)$. ■

The definition of the accessibility relation as above was first introduced in [14]. Note that, by the first claim of the next lemma, the definition of $R_\diamond$ coincides with the classical definition of the accessibility relation for possibility.

Lemma 3.11. The following hold:

1. $xR_\diamond z$ iff $\forall a\,(a \in z \implies \Diamond a \in x)$

2. For any filter u, $\Diamond_\sigma(\Gamma u) = \{x \in X \mid \exists z \in X\ (xR_\diamond z \text{ and } u \leq z)\}$
3. For any lattice element a and filter x, $\Diamond a \in x$ iff $\exists z\ (xR_\diamond z$ and $a \in z)$
4. For any ideal y, $\diamondminus^\partial(\Gamma y) = \{y' \mid \forall v\ (y' R^\partial_\diamond v \implies y \leq v)\}$
5. For any lattice element a and ideal y, $\Diamond a \in y$ iff $\forall v \in Y\ (yR^\partial_\diamond v \implies a \in v)$.

Proof: For 1), the proof follows merely from the definition of $\Diamond^\flat$.

For 2), by the previous Lemma and by definition, $\Diamond_\sigma(\Gamma u) = \diamondminus(\Gamma u) = \Gamma(\Diamond^\flat u)$, hence the left-to-right inclusion follows by taking $z = u$. For the other direction, notice first that the filter operator $\Diamond^\flat$ is monotone. Therefore, if z is such that $\Diamond^\flat z \leq x$ and $u \leq z$, then it follows that $\Diamond^\flat u \leq x$, hence $x \in \Gamma(\Diamond^\flat u) = \diamondminus(\Gamma u) = \Diamond_\sigma(\Gamma u)$.

For 3), we have $\Diamond a \in x$ iff $x_{\Diamond a} \leq x$ iff $x \in \Gamma x_{\Diamond a}$ iff $x \in \Gamma(\Diamond^\flat x_a)$ iff $x \in \Diamond_\sigma(\Gamma x_a)$ iff $\exists z\ (xR_\diamond z$ and $x_a \leq z)$ iff $\exists z\ (xR_\diamond z$ and $a \in z)$.

For 4), given that $\diamondminus^\partial(\Gamma y) = \Gamma(\Diamond^\sharp y)$ it suffices to show that $\Diamond^\sharp y \leq y'$ iff $\forall v\ (y' R^\partial_\diamond v \implies y \leq v)$.

Note that from the definition of $\Diamond^\sharp$ it follows that $\Diamond^\sharp y \leq y'$ iff $\forall a\ (a \in y \implies \Diamond a \in y')$. Assuming $\Diamond^\sharp y \leq y'$ and $y' R^\partial_\diamond v$, for an arbitrary ideal v it follows from definitions that if $a \in y$, then $\Diamond a \in y'$ and then $a \in v$, hence $y \leq v$.

Conversely, assume $\forall v\ (y' R^\partial_\diamond v \implies y \leq v)$ and choose $v = \{a \mid \Diamond\, a \in y'\}$, an ideal, since y' is one. Then $y' R^\partial_\diamond v$ holds and thereby $y \leq v = \{a \mid \Diamond\, a \in y'\}$. In other words, for any a, if $a \in y$, then $\Diamond a \in y'$, which is equivalent to $\Diamond^\sharp y \leq y'$.

For 5), the direction left to right is enforced by the definition of the dual accessibility relation $R^\partial_\diamond$. For the other direction, assume $\forall v\ (yR^\partial_\diamond v \implies a \in v)$. Let $y^\diamond = \{a \mid \Diamond\, a \in y\}$, an ideal. By definition of $R^\partial_\diamond$ and of $y^\diamond$ it follows that $yR^\partial_\diamond y^\diamond$ holds. The hypothesis then entails that $a \in y^\diamond$. But this yields $\Diamond a \in y$, by definition of $y^\diamond$. ■

Lemma 3.12. For any stable set A we have $\Diamond_\sigma A = \psi \Diamond^\partial_\sigma (\phi A)$ and if $A = \Gamma x \in \mathcal{G}_\kappa(X)$, then also $\diamondminus A = \psi \diamondminus^\partial \phi A$.

Proof:

$$
\begin{array}{rcll}
\Diamond_\sigma A & = & \psi\phi \Diamond_\sigma A & \\
& = & \psi\phi \Diamond_\sigma \bigvee_{x \in A} \Gamma x & \text{(by join density of closed elements)} \\
& = & \psi\phi \bigvee_{x \in A} \Diamond_\sigma(\Gamma x) & \text{(by Lemma 3.8)} \\
& = & \psi\phi \bigvee_{x \in A} \Gamma(\Diamond^\flat x) & \text{(by definition of } \diamondminus \text{ and by } \diamondminus = \Diamond_\sigma) \\
& = & \psi \bigwedge_{x \in A} \phi\Gamma(\Diamond^\flat x) & \text{(because } \psi, \phi \text{ is a dual isomorphism)} \\
& = & \psi \bigwedge_{x \in A} \{\Diamond^\flat x\}^{\perp} & \text{(by definition of } \diamondminus \text{ and by Lemma 3.9)} \\
& = & \psi \bigwedge_{x \in A} \Diamond^\partial_\pi(\{x\}^{\perp}) & \text{(by Lemma 3.9)} \\
& = & \psi \bigwedge_{x \in A} \Diamond^\partial_\sigma(\{x\}^{\perp}) & \text{(by Lemma 3.8)} \\
& = & \psi \Diamond^\partial_\sigma \bigwedge_{x \in A} \{x\}^{\perp} & \text{(by Lemma 3.8)} \\
& = & \psi \Diamond^\partial_\sigma \phi \bigvee_{x \in A} \Gamma x & \text{(by Lemma 1.2)} \\
& = & \psi \Diamond^\partial_\sigma \phi A &
\end{array}
$$

The second part follows from Lemma 3.9, given the above calculation. ■

3.3.2 Necessity

Instantiating equations (2, 3) of Section 1.3 to the case of the box operator, of distribution type $(\partial;\partial)$, we obtain filter and ideal operators $\Box^\sharp : X \longrightarrow X$ and $\Box^\flat : Y \longrightarrow Y$

$$\Box^\sharp x = \bigvee \{x_{\Box a} \mid a \in x \in X\} \qquad\qquad \Box^\flat\, y = \bigvee \{y_{\Box a} \mid a \in y \in Y\}$$

By Lemma 1.6, the operators preserve principal filters and ideals, respectively.

Next, define an operator $\boxminus$ on the closed elements $\mathcal{G}_\kappa(X)$ of $\mathcal{G}_\psi(X)$ and, dually, an operator $\boxminus^\partial$ on the closed elements $\mathcal{G}_\kappa(Y)$ of $\mathcal{G}_\phi(Y)$ by setting

$$\boxminus(\Gamma x) = \Gamma(\Box^\sharp x) \qquad\qquad \boxminus^\partial(\Gamma y) = \Gamma(\Box^\flat y)$$

Lemma 3.13. The following hold

1. For any filter $u \in X$, $\boxminus(\Gamma u) = \Box_\sigma(\Gamma u)$
2. For any ideal $y \in Y$, $\boxminus^\partial \Gamma y = \Box_\sigma^\partial \Gamma y$
3. For any ideal $y \in Y$, $\Box_\pi({}^\perp\{y\}) = \psi(\boxminus^\partial \Gamma y) = \{x \in X \mid x \perp \Box^\flat y\} = {}^\perp\{\Box^\flat y\}$

Proof: For 1) and 2), we refer the reader to Proposition 3.6.
For 3), Lemma 3.7 may be consulted. We leave details to the interested reader. ∎

Definition 3.14. The accessibility and co-accessibility relations $R_{\scriptscriptstyle\Box} \subseteq X \times X, R_{\scriptscriptstyle\Box}^\partial \subseteq Y \times Y$ are defined by setting

- $xR_{\boxminus}z$ iff $\forall a\,(\Box a \in x \implies a \in z)$
- $yR_{\scriptscriptstyle\Box}^\partial v$ iff $\Box^\flat v \leq y$. ∎

Lemma 3.15. The following hold

1. For any $u \in X$, $\boxminus\Gamma u = \{x \in X \mid \forall z \in X\,(xR_{\scriptscriptstyle\Box}z \implies u \leq x)\}$
2. For any $u \in Y$, $\Box_\sigma^\partial(\Gamma u) = \{y \in Y \mid \exists v \in Y\,(yR_{\scriptscriptstyle\Box}^\partial v \text{ and } u \leq v)\}$
3. For any lattice element a and ideal y we have $\Box a \in y$ iff $\exists v \in Y\,(yR_{\scriptscriptstyle\Box}^\partial v \text{ and } a \in v)$.

Proof: For 1), we first prove the particular case $\boxminus\Gamma x_a = \Gamma x_{\Box a}$ for clopen elements Γx_a of $\mathcal{G}_\psi(X)$. This follows by a simple calculation: $x \in \Gamma x_{\Box a}$ iff $\Box a \in x$ iff $\forall z \in X\,(xR_{\boxminus}z \implies a \in z)$ iff $\forall z \in X\,(xR_{\boxminus}z \implies z \in \Gamma x_a)$ iff $x \in \boxminus(\Gamma x_a)$.

Now assume $x \in \boxminus\Gamma u$, for some filter $u \in X$. It suffices to show that $\Box^\sharp u \leq x$. Given the definition of the filter operator $\Box^\sharp$, the desired conclusion follows if we can show that for any a, if $a \in u$, then $\Box a \in x$. The hypothesis is equivalent to $\Gamma u \subseteq \Gamma x_a$ from which we obtain, by monotonicity of $\boxminus$, that $\boxminus\Gamma u \subseteq \boxminus\Gamma x_a = \Gamma x_{\Box a}$, by the special case we first proved above. Hence $\Box a \in x$, given the hypotheses $x \in \boxminus\Gamma u$ and $a \in u$. This shows $\Box^\sharp u \leq x$, i.e. $x \in \Gamma(\Box^\sharp u) = \Box_\sigma(\Gamma u)$.

For the converse, assume $\Box^\sharp u \leq x$ and let z be any filter such that $xR_{\boxminus}z$. If $a \in u$, which is equivalent to $x_a \leq u$, we get $\Box^\sharp x_a \leq \Box^\sharp u \leq x$, which shows that $\Box a \in x$. But we are assuming $xR_{\boxminus}z$, hence $a \in z$. This shows that $x \in \boxminus(\Gamma u)$, q.e.d.

For 2), if $y \in \Box_\sigma^\partial(\Gamma u) = \Gamma(\Box^\flat u)$, then $\Box^\flat u \leq y$, hence taking $v = u$ we obtain $yR_{\scriptscriptstyle\Box}^\partial v$ and $u \leq v$. Conversely, let v be such that $yR_{\scriptscriptstyle\Box}^\partial v$ and $u \leq v$, i.e. $\Box^\flat v \leq y$ and $v \in \Gamma u$. By monotonicity of $\Box^\flat$ we obtain $\Box^\flat u \leq \Box^\flat v \leq y$ and since $\Box_\sigma^\partial(\Gamma u) = \Gamma(\Box^\flat u)$ we obtain $y \in \Box_\sigma^\partial(\Gamma u)$.

Finally, by the following straightforward calculation

$$\begin{array}{ll}\Box a \in y & \text{iff } y_{\Box a} \leq y \text{ iff } y \in \Gamma y_{\Box a} \\ & \text{iff } y \in \Gamma(\Box^\flat y_a) \text{ iff } y \in \Box_\sigma^\partial(\Gamma y_a) \text{ iff } \exists v\,(yR_{\scriptscriptstyle\Box}^\partial v \text{ and } y_a \leq v) \\ & \text{iff } \exists v \in Y\,(yR_{\scriptscriptstyle\Box}^\partial v \text{ and } a \in v)\end{array}$$

the proof of 3) is complete. ∎

Lemma 3.16. For any stable set A we have $\Box_\sigma A = \psi\Box_\sigma^\partial A$ and if $A = \Gamma x \in \mathcal{G}_\kappa(X)$, then also $\boxminus A = \psi\boxminus^\partial(\phi A)$.

Proof: The proof is completely analogous to the proof of the same fact for $\Diamond_\sigma, \diamondminus$ (Lemma 3.12) and it can be safely left to the interested reader. ∎

3.3.3 Implication

The distribution type of implication is $\delta(\rightarrow) = (1, \partial; \partial)$, i.e. it is a map $\rightarrow\ : \mathcal{L} \times \mathcal{L}^\partial \longrightarrow \mathcal{L}^\partial$. Hence its extension in the sense of Section 1.3, following [10], is a map $\xrightarrow{\sharp} : Y \times X \longrightarrow X$. Similarly, the extension for a map of the dual distribution type $(\partial, 1; 1)$ is a map $\xrightarrow{\flat} : X \times Y \longrightarrow Y$, both defined below, after [10]:

$$y \xrightarrow{\sharp} x \;=\; \bigvee_X \{x_{a \to b} \mid a \in y,\; b \in x\} \qquad\qquad x \xrightarrow{\flat} y \;=\; \bigvee_Y \{y_{a \to b} \mid a \in x, b \in y\}$$

σ-Extension: Define an operator $\Rightarrow_0 : \mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$ on the closed elements of $\mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y)$, by setting $(\Gamma x) \Rightarrow_0 (\Gamma y) = \Gamma(x \xrightarrow{\flat} y)$, for $x \in X, y \in Y$, and extend it on the whole of $\mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y)$ using join-density of the set of closed elements together with the fact that $\Rightarrow_0$ is monotone in both argument places: $A \Rightarrow_0 B = \bigvee_{x \in A} \bigvee_{y \in B} (\Gamma x \Rightarrow_0 \Gamma y)$.

A dual extension map $\Rightarrow_0^\partial : \mathcal{G}_\phi(Y) \times \mathcal{G}_\psi(X) \longrightarrow \mathcal{G}_\psi(X)$ can be also defined in an analogous manner by setting $(\Gamma y) \Rightarrow_0^\partial (\Gamma x) = \Gamma(y \xrightarrow{\sharp} x)$.

Lemma 3.17. $\Rightarrow_0$ is the σ-extension $\Rightarrow_\sigma$ of the lattice implication operator and $\Rightarrow_0^\partial$ is its dual σ-extension $\Rightarrow_\sigma^\partial$.

Proof: The σ-extension of $\rightarrow\ : \mathcal{L} \times \mathcal{L}^\partial \longrightarrow \mathcal{L}^\partial$ is a map $\Rightarrow_\sigma : \mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$ and it is defined after [9] by instantiating equation (6) $(\Gamma x) \Rightarrow_\sigma (\Gamma y) = \bigwedge\{\Gamma y_{a \to b} \mid a \in x,\; b \in y\}$ and we obtain

$$(\Gamma x) \Rightarrow_\sigma (\Gamma y) = \bigwedge\{\Gamma y_{a \to b} \mid a \in x,\; b \in y\} = \Gamma(\bigvee_Y \{y_{a \to b} \mid a \in x,\; b \in y\}) = \Gamma(x \xrightarrow{\flat} y) = (\Gamma x) \Rightarrow_0 (\Gamma y) \tag{11}$$

Since $\Rightarrow_0, \Rightarrow_\sigma$ agree on closed elements and they are both extended using join-density of the set of closed elements we may conclude that $\Rightarrow_0 = \Rightarrow_\sigma$.

For the dual extension the argument is completely similar. Indeed, we obtain from definitions and with small calculations that $(\Gamma y) \Rightarrow_\sigma^\partial (\Gamma x) = \bigwedge\{\Gamma x_{a \to b} \mid a \in y,\; b \in x\} = \Gamma(\bigvee\{x_{a \to b} \mid a \in y,\; b \in x\}) = \Gamma(y \xrightarrow{\sharp} x) = (\Gamma y) \Rightarrow_0^\partial (\Gamma x)$. ∎

π-Extension: Applying Lemma 3.7 we obtain a definition on open elements of the π-extension of implication $\Rightarrow_\pi : \mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$ by conjugating the dual σ-extension with the dual isomorphism ψ, ϕ. We set

$${}^\perp\{y\} \Rightarrow_\pi \{x\}^\perp = \phi(\phi({}^\perp\{y\}) \Rightarrow_\sigma^\partial \psi(\{x\}^\perp)) = \phi(\Gamma y \Rightarrow_\sigma^\partial \Gamma x) = \phi(\Gamma(y \xrightarrow{\sharp} x)) = \{y \xrightarrow{\sharp} x\}^\perp$$

Similarly, the dual π-extension $\Rightarrow_\pi^\partial : \mathcal{G}_\phi(Y) \times \mathcal{G}_\psi(X) \longrightarrow \mathcal{G}_\psi(X)$ is obtained by conjugating with the σ-extension

$$\{x\}^\perp \Rightarrow_\pi^\partial {}^\perp\{y\} = \psi(\psi(\{x\}^\perp) \Rightarrow_\sigma \phi({}^\perp\{y\})) = \psi(\Gamma x \Rightarrow_\sigma \Gamma y) = \psi(\Gamma(x \xrightarrow{\flat} y)) = {}^\perp\{x \xrightarrow{\flat} y\}$$

Representing Implication: Gehrke and Harding [9] work only up to equivalence and do not differentiate between $\mathcal{G}_\phi(Y)$ and $\mathcal{G}_\psi(X)^\partial$, given that $\mathcal{G}_\phi(Y) \simeq \mathcal{G}_\psi(X)^\partial$, but drawing the distinction is of significance for our semantic purposes.

Using the dual equivalence of $\mathcal{G}_\psi(X), \mathcal{G}_\phi(Y)$ we obtain from $\Rightarrow_0$ a map $\Rightarrow_1 : \mathcal{G}_\psi(X) \times \mathcal{G}_\psi(X)^\partial \longrightarrow \mathcal{G}_\psi(X)^\partial$ defined on the closed elements of $\mathcal{G}_\psi(X) \times \mathcal{G}_\psi(X)^\partial$ by setting

$$(\Gamma x) \Rightarrow_1 ({}^\perp\{y\}) \;=\; \psi((\Gamma x) \Rightarrow_0 (\phi({}^\perp\{y\}))) = \psi(\Gamma x \Rightarrow_0 \Gamma y) = \psi\Gamma(x \xrightarrow{\flat} y) = {}^\perp\{x \xrightarrow{\flat} y\} \tag{12}$$

as illustrated in the diagram below.

$$
\begin{array}{cccccc}
 & (\Gamma x) & \Rightarrow_0 & (\Gamma y) & = & \Gamma(x \xrightarrow{\flat} y) \\
 & & & & & \\
\Rightarrow_0: & \mathcal{G}_\psi(X) & \times & \mathcal{G}_\phi(Y) & \longrightarrow & \mathcal{G}_\phi(Y) \\
 & \Big\| & & {\scriptstyle\phi}\big\uparrow\big\downarrow{\scriptstyle\psi} & & {\scriptstyle\phi}\big\uparrow\big\downarrow{\scriptstyle\psi} \\
\Rightarrow_1: & \mathcal{G}_\psi(X) & \times & \mathcal{G}_\psi(X)^\partial & \longrightarrow & \mathcal{G}_\psi(X)^\partial \\
 & & & & & \\
 & (\Gamma x) & \Rightarrow_1 & ({}^{\perp}\{y\}) & = & {}^{\perp}\{x \xrightarrow{\flat} y\}
\end{array}
$$

Similarly, for the dual extension $\Rightarrow_0^\partial$ and considering again the dual equivalence of $\mathcal{G}_\phi(Y)$ and $\mathcal{G}_\psi(X)^\partial$ we define a map $\Rightarrow_1^\partial: \mathcal{G}_\phi(Y) \times \mathcal{G}_\phi(Y)^\partial \longrightarrow \mathcal{G}_\phi(Y)^\partial$ as displayed in the diagram below

$$
\begin{array}{cccccc}
 & (\Gamma y) & \Rightarrow_0^\partial & (\Gamma x) & = & \Gamma(y \xrightarrow{\sharp} x) \\
 & & & & & \\
\Rightarrow_0^\partial: & \mathcal{G}_\phi(Y) & \times & \mathcal{G}_\psi(X) & \longrightarrow & \mathcal{G}_\psi(X) \\
 & \Big\| & & {\scriptstyle\psi}\big\uparrow\big\downarrow{\scriptstyle\phi} & & {\scriptstyle\psi}\big\uparrow\big\downarrow{\scriptstyle\phi} \\
\Rightarrow_1^\partial: & \mathcal{G}_\phi(Y) & \times & \mathcal{G}_\phi(Y)^\partial & \longrightarrow & \mathcal{G}_\phi(Y)^\partial \\
 & & & & & \\
 & (\Gamma y) & \Rightarrow_1^\partial & (\{x\}^{\perp}) & = & \{y \xrightarrow{\sharp} x\}^{\perp}
\end{array}
$$

In conclusion, we represent lattice implication $\rightarrow\ : \mathcal{L} \times \mathcal{L}^\partial \longrightarrow \mathcal{L}^\partial$ by $\Rightarrow_1 : \mathcal{G}_\psi(X) \times \mathcal{G}_\psi(X)^\partial \longrightarrow \mathcal{G}_\psi(X)^\partial$ (extended to all stable sets using join-density of closed elements) and we co-represent it by $\Rightarrow_1^\partial : \mathcal{G}_\phi(Y) \times \mathcal{G}_\phi(Y)^\partial \longrightarrow \mathcal{G}_\phi(Y)^\partial$, defined as above.

Note that $\Rightarrow_1^\partial$ can be equivalently regarded, given the dual isomorphism ϕ, ψ, as a map from $\mathcal{G}_\psi(X)^\partial \times \mathcal{G}_\psi(X)$ and into $\mathcal{G}_\psi(X)$, which better reflects the fact that the dual distribution type of implication is $(\partial, 1; 1)$. However, this representation does not deliver an operator on co-stable sets, i.e. on $\mathcal{G}_\phi(Y)$, which is desirable, as this map underlies the way we define the canonical co-interpretation.

Compositionality of the Representation: To illustrate the compositionality of the representation, let $a{\to}b \in \mathcal{L}$ and $\alpha(a{\to}b) = [\![a{\to}b]\!] = \Gamma x_{a\to b}$, $\beta(a{\to}b) = (\!|a{\to}b|\!) = \Gamma y_{a\to b}$. By the above constructions we have

$$[\![a{\to}b]\!] = \Gamma x_{a\to b} = {}^{\perp}\{y_{a\to b}\} = {}^{\perp}\{x_a \xrightarrow{\flat} y_b\} = \Gamma x_a \Rightarrow_1 {}^{\perp}\{y_b\} = \Gamma x_a \Rightarrow_1 \Gamma x_b = [\![a]\!] \Rightarrow_1 [\![b]\!]$$

where we used the fact that $\Gamma x_b = {}^{\perp}\{y_b\}$ because principal filters/ideals are clopen elements. Similarly,

$$(\!|a{\to}b|\!) = \Gamma y_{a\to b} = \{x_{a\to b}\}^{\perp} = \{y_a \xrightarrow{\sharp} x_b\}^{\perp} = \Gamma y_a \Rightarrow_1^\partial \{x_b\}^{\perp} = \Gamma y_a \Rightarrow_1^\partial \Gamma y_b = (\!|a|\!) \Rightarrow_1^\partial (\!|b|\!)$$

where we again used the same facts about clopen elements.

Normality of the Representation: By Proposition 3.8 and given that $\rightarrow : \mathcal{L} \times \mathcal{L}^\partial \longrightarrow \mathcal{L}^\partial$ distributes over joins in each argument place, where joins in the second argument place are joins in $\mathcal{L}^\partial$, hence meets in $\mathcal{L}$, delivering joins in $\mathcal{L}^\partial$, i.e. meets in $\mathcal{L}$, its sigma extension $\Rightarrow_\sigma = \Rightarrow_0 : \mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$ is completely normal, distributing over joins in each argument place and delivering a join in $\mathcal{G}_\phi(Y)$.

Considering $\Rightarrow_1$, for distribution over meets in the second argument place and given meet-density of open elements it suffices to show that $A \Rightarrow_1 \bigwedge_{i\in I} {}^{\underline{\perp}}\{y_i\} = \bigwedge_{i\in I}(A \Rightarrow_1 {}^{\underline{\perp}}\{y_i\})$. Similarly for co-distribution over joins in the first argument place.

$$
\begin{array}{lcl}
A \Rightarrow_1 \bigwedge_{i\in I} {}^{\underline{\perp}}\{y_i\} & = & \psi(A \Rightarrow_0 \phi(\bigwedge_{i\in I} {}^{\underline{\perp}}\{y_i\})) = \psi(A \Rightarrow_0 \bigvee_{i\in I} \Gamma y_i) = \psi \bigvee_{i\in I}(A \Rightarrow_0 \Gamma y_i) \\
 & = & \bigwedge_{i\in I} \psi(A \Rightarrow_0 \Gamma y_i) = \bigwedge_{i\in I}(A \Rightarrow_1 {}^{\underline{\perp}}\{y_i\}) \\
(\bigvee_{i\in I} \Gamma x_i) \Rightarrow_1 {}^{\underline{\perp}}\{y\} & = & \psi((\bigvee_{i\in I} \Gamma x_i) \Rightarrow_0 \psi({}^{\underline{\perp}}\{y\})) = \psi(\bigvee_{i\in I} \Gamma x_i \Rightarrow_0 \Gamma y) = \psi \bigvee_{i\in I}(\Gamma x_i \Rightarrow_0 \Gamma y) \\
 & = & \bigwedge_{i\in I} \psi(\Gamma x_i \Rightarrow_0 \Gamma y) = \bigwedge_{i\in I}(\Gamma x_i \Rightarrow_1 {}^{\underline{\perp}}\{y\})
\end{array}
$$

Relational Representation: For the relational representation of $\Rightarrow_1, \Rightarrow_1^\partial$, define accessibility relations $R_> \subseteq X^2 \times Y$ by setting $xzR_>y$ iff $x \not\underline{\perp} (z \xrightarrow{\flat} y)$ and $R_>^\partial \subseteq X \times Y^2$ by $zR_>^\partial wy$ iff $(w \xrightarrow{\sharp} z) \not\underline{\perp}\, y$. We let $\overline{R}_>, \overline{R}_>^\partial$ be the complements of $R_>, R_>^\partial$.

Lemma 3.18. The following hold:

1. $\Gamma z \Rightarrow_1 {}^{\underline{\perp}}\{v\} = \{x \in X \mid \forall z' \in X\ \forall v' \in Y\ (z \leq z' \text{ and } v \leq v' \implies xz'\overline{R}_>v')\}$
   $= \{x \in X \mid \forall z' \in X\ \forall v' \in Y\ (z' \in \Gamma z \text{ and } xz'R_>v' \implies v \not\underline{\perp}\, v')\}$

2. $a \rightarrow b \in x \in X$ iff $\forall z \in X \forall y \in Y\ (a \in z \text{ and } b \in y \implies xz\overline{R}_>y)\}$
   iff $\forall z \in X \forall y \in Y\ (xzR_>y \text{ and } a \in z \implies b \notin y)$

3. $\Gamma y \Rightarrow_1^\partial \{x\}^{\underline{\perp}} = \{v \in Y \mid \forall y' \in Y \forall x' \in X (y \leq y' \ \wedge\ x \leq x' \implies x'\overline{R}_>^\partial y'v)\}$

4. $a \rightarrow b \in v \in Y$ iff $\forall y' \in Y \forall x' \in X\ (x'R_>^\partial y'v \ \wedge\ a \in y' \implies b \notin x')$

Proof: The case for 1) is immediate from definitions, given monotonicity of $\xrightarrow{\flat}$.
For 2), we have the following calculation

$$
\begin{array}{ll}
a \rightarrow b \in x & \text{iff } x \,\underline{\perp}\, y_{a\rightarrow b} \text{ iff } x \,\underline{\perp}\, (x_a \xrightarrow{\flat} y_b) \\
 & \text{iff } \forall z \in X \forall y \in Y\ (x_a \leq z \text{ and } y_b \leq y \implies x \,\underline{\perp}\, (z \xrightarrow{\flat} y)) \\
 & \text{iff } \forall z \in X \forall y \in Y\ (a \in z \text{ and } b \in y \implies xz\overline{R}_>y) \\
 & \text{iff } \forall z \in X \forall y \in Y\ (xzR_>y \text{ and } a \in z \implies b \notin y)
\end{array}
$$

which establishes its truth. 3 and 4 are proven by similar argument. ■

Lemma 3.19. For any $A, B \in \mathcal{G}_\psi(X)$ we have $A \Rightarrow_1 B = \psi(\phi A \Rightarrow_1^\partial \phi B)$.

Proof: Recall that $\Rightarrow_1 : \mathcal{G}_\psi(X) \times \mathcal{G}_\psi(X)^\partial \longrightarrow \mathcal{G}_\psi(X)^\partial$ was obtained from the σ-extension of implication, which is the map $\Rightarrow_0 : \mathcal{G}_\psi(X) \times \mathcal{G}_\phi(Y) \longrightarrow \mathcal{G}_\phi(Y)$, by setting $A \Rightarrow_1 B = \psi(A \Rightarrow_0 \phi B) = \psi(A \Rightarrow_\sigma \phi B)$. The π-extension, as we have shown, is obtained from the dual σ-extension, i.e. $A' \Rightarrow_\pi B' = \phi(\phi A' \Rightarrow_\sigma^\partial \psi B')$. By Lemma 3.8 (which reviews results obtained in [9]), given the (co)distribution properties of implication, its σ and π-extensions coincide. Hence, we obtain

$$A \Rightarrow_1 B = \psi(A \Rightarrow_\sigma \phi B) = \psi(A \Rightarrow_\pi \phi B) = \psi(\phi(\phi A \Rightarrow_\sigma^\partial \psi(\phi B))) = \phi A \Rightarrow_\sigma^\partial B = \psi(\phi A \Rightarrow_1^\partial \phi B)$$

which establishes the claim. ■

3.4 Completeness Proof

Based on the representation results of the previous sections we introduce the definition of the canonical frame (Definition 3.20), we prove it to be a τ-frame (Lemma 3.21) in the sense of Definition 2.1, we then verify (Lemma 3.22) that the canonical interpretation and co-interpretation satisfy the model constraints of Definition 2.3 and we conclude with the statement of our completeness theorem (Theorem 3.23).

Definition 3.20 (Canonical Frame and Model)**.**
The canonical frame is the structure $\mathfrak{F}_\tau = ((X, \perp, Y), (R_\Box, R^\partial_\Box), (R_\Diamond, R^\partial_\Diamond), (R_>, R^\partial_>))$ where

1. X is the set of filters of the Lindenbaum-Tarski algebra of the logic, Y is its set of ideals and $x \perp y$ iff $x \cap y \neq \emptyset$

2. The accessibility and co-accessibility relations are defined as follows

   $(R_\Diamond \subseteq X \times X)$ $xR_\Diamond z$ iff $\Diamond^\flat z \leq x$, where for $z \in X$, $\Diamond^\flat z = \bigvee\{x_{\Diamond a} \mid a \in z\}$
   $(R^\partial_\Diamond \subseteq Y \times Y)$ $yR^\partial_\Diamond v$ iff $\forall a\,(\Diamond a \in y \implies a \in v)$
   $(R_\Box \subseteq X \times X)$ $xR_\boxminus z$ iff $\forall a\,(\Box a \in x \implies a \in z)$
   $(R^\partial_\Box \subseteq Y \times Y)$ $yR^\partial_\Box v$ iff $\Box^\flat v \leq y$ where for an ideal v, $\Box^\flat v = \bigvee\{y_{\Box a} \mid a \in v\}$
   $(R_> \subseteq X^2 \times Y)$ $xzR_> y$ iff $x \not\leq (z \xrightarrow{\flat} y)$, where $z \xrightarrow{\flat} y \;=\; \bigvee\{y_{a\to b} \mid a \in z, b \in y\}$
   $(R^\partial_> \subseteq X \times Y^2)$ $zR^\partial_> wy$ iff $(w \xrightarrow{\sharp} z) \not\leq y$, where $w \xrightarrow{\sharp} z \;=\; \bigvee\{x_{a\to b} \mid a \in w,\ b \in z\}$

   where for a lattice element e we write x_e for the principal filter and y_e for the principal ideal generated by e and $\leq$ designates filter, respectively ideal, inclusion.

   The canonical model is the structure $\mathfrak{M}_\tau = (\mathfrak{F}_\tau, [\![\;]\!], (\!|\;|\!))$, where for a sentence φ and where $[\varphi] = a$ is its equivalence class under provability we set $[\![\varphi]\!] = \{x \in X \mid a \in x\} = \{x \in X \mid x_a \leq x\} = \Gamma x_a$ and $(\!|\varphi|\!) = \{y \in Y \mid a \in y\} = \{y \in Y \mid y_a \leq y\} = \Gamma y_a$. ∎

Lemma 3.21 (Canonical Frame Lemma)**.** The canonical frame is a τ-frame in the sense of Definition 2.1.

Proof: By definition, the canonical Galois relation $\perp$ of the frame is increasing in each argument place. This implies that if $x \leq z$, then $\{x\}^{\perp} \subseteq \{z\}^{\perp}$. For the converse, let $a \in x$, so that $x \perp y_a$, i.e. $y_a \in \{x\}^{\perp}$ and then by hypothesis $y_a \in \{z\}^{\perp}$, i.e. $z \perp y_a$ and so $a \in z$, using Lemma 3.2, hence $x \leq z$. Thus condition 1 of Definition 2.1 holds for the canonical frame.

For condition 2(a), closure of $\mathcal{G}_\kappa(X)$ under the induced image operators $\diamondminus, \boxminus$ and of $\mathcal{G}_\kappa(Y)$, respectively, under $\diamondminus^\partial, \boxminus^\partial$ has been verified in Sections 3.3.1 and 3.3.2, while duality of the operators $\diamondminus, \diamondminus^\partial$ (and similarly for $\boxminus, \boxminus^\partial$) has been verified in Lemma 3.12 (and similarly for box and Lemma 3.16). In what regards $\boxminus$ it is obvious from its definition that it distributes over arbitrary intersections. For $\diamondminus$, we may derive complete distribution over arbitrary joins in $\mathcal{G}_\psi(X)$ from the fact that its dual, namely $\diamondminus^\partial$, being really a box-operator on $\wp(Y)$, as its definition reveals, distributes over arbitrary intersections. Then use interdefinability of $\diamondminus$ and $\diamondminus^\partial$ using the dual equivalence ψ, ϕ. Alternatively, the distribution facts of σ-extensions obtained in [9] can be directly used.

For condition 2(b), notice that $\emptyset_\psi = \{\omega\}$, where ω designates the improper filter (the whole lattice). Details are left to the reader.

For condition 3, closure of $\mathcal{G}_\kappa(X)$ under the image operator $\Rightarrow$ induced by $R_>$ and of $\mathcal{G}_\kappa(Y)$ under $\Rightarrow^\partial$, induced by $R^\partial_>$, was demonstrated in Section 3.3.3, where the interdefinability requirement was verified in Lemma 3.19. For the implication operator, we verified the complete (co)distribution properties in Section 3.3.3, in the paragraph on normality of the representation.

By the above, the canonical frame is a τ-frame in the sense of Definition 2.1. ∎

Lemma 3.22 (Canonical Interpretation Lemma)**.** Let $V_1(p) = [\![p]\!]$ and $V_2(p) = (\!|p|\!) = \phi([\![p]\!])$, for a propositional variable p. Define $x \Vdash \varphi$ iff $x \in [\![\varphi]\!]$ and $y \succ \varphi$ iff $y \in (\!|\varphi|\!)$, for any sentence φ. Then $\Vdash, \succ$ satisfy the recursive clauses of Table 1 and, in particular, $(\!|\varphi|\!) = \phi([\![\varphi]\!])$, for any φ.

Proof: Clearly $[\![\top]\!] = X$, since $[\top]$ is the upper bound of the Lindenbaum-Tarski algebra of the logic and $[\top] = 1 \in x$ for any filter x. Similarly $(\!|\bot|\!) = Y$. Letting ω be the improper filter (the whole Lindenbaum-Tarski algebra), for any filter x we have $x \leq \omega$, hence $\omega \in \Gamma x$ and since a stable set A is the join $A = \bigvee\{\Gamma x \mid x \in A\}$ it follows that $\omega \in A$ for an $A \in \mathcal{G}_\psi(X)$, hence $\{\omega\} \subseteq \varnothing_\psi = \bigcap\{A \mid A \in \mathcal{G}_\psi(X)\}$. If a filter $z \in \varnothing_\psi$, then in particular $z \in \Gamma x_0 = \{\omega\}$ (where $0 = [\bot]$ is the lower bound of the lattice), hence $z = \omega$ and thus $\varnothing_\psi = \{\omega\}$. Hence $[\![\bot]\!] = \varnothing_\psi = \{\omega\}$ and $\omega \Vdash \bot$ since $[\bot] \in \omega$. Similarly, $(\!|\top|\!) = \{y_1\}$ is the improper ideal generated by the top element $1 = [\top]$ (i.e. the whole lattice) and then $y \succ \top$ iff $y = y_1$ holds.

The cases for conjunction and disjunction are immediate, given that conjunction is interpreted as intersection, disjunction is co-interpreted as intersection, hence the cases $x \Vdash \varphi \wedge \psi$ and $y \succ \varphi \vee \psi$ hold in the canonical interpretation. The remaining two cases are obtained using the Galois connection ψ, ϕ and thereby the clauses $x \Vdash \varphi \vee \psi$ and $y \succ \varphi \wedge \psi$ obtain in the canonical model.

The cases $x \Vdash \Diamond\varphi$ and $y \succ \Diamond\varphi$ hold by Lemma 3.11. Similarly, the cases for $x \Vdash \Box\varphi$ and $y \succ \Box\varphi$ hold by Lemma 3.15.

Finally, the cases for implication, $x \Vdash \varphi \rightarrow \psi$ and $y \succ \varphi \rightarrow \psi$ follow from Lemma 3.18.

The fact that $(\!|\vartheta|\!) = \phi([\![\vartheta]\!])$ follows from the interdefinability, using the duality ψ, ϕ of operators and their duals (Lemmas 3.12, 3.16, 3.19). ∎

Therefore, by the above results completeness is established.

Theorem 3.23 (Completeness)**.** The logic of modal implicative lattices (Section 2) is sound and complete in the class of frames and models of Definition 2.1. ∎

4 Conclusions

In a series of recent articles [11–16] we have shown that the semantics of non-distributive propositional logics need not abandon the standard interpretation of familiar operators and we developed to this effect the framework of Kripke-Galois (or order-dual) semantics, which is based on the representation and duality of lattice expansions presented in [10]. In this article, we have demonstrated that this is also the case, in particular, when the semantics is based on a lattice representation and duality theorem [18] that delivers a canonical extension [9], provided a restriction on the admissible interpretations is made. The frames and models presented in this article can be seen as a 2-sorted version of the Kripke-Galois frames and models we have recently developed. The canonical frame construction is based, first, on the lattice representation presented in [18], as well as on the representation of normal lattice operators given in [10].

What made it possible to bridge the gap between a semantics based on canonical extensions and the Kripke-Galois frames approach is that (a) the σ and π-extensions of maps of [9] are canonically obtained from the representation of normal operators of [10], (b) that the set of closed elements is closed under σ-extensions and that (c) the canonical frame of [10] consists of filters alone, hence its set of Galois stable sets is restricted to the set of closed elements of $\mathcal{G}_\psi(X)$ (which join-generate $\mathcal{G}_\psi(X)$).

The approach of [8], followed also in [5], constructs the canonical frame by taking the canonical extension $\mathcal{A}_\sigma$ of the Lindenbaum-Tarski algebra $\mathcal{A}$ of the logic and then letting $(X, \perp, Y) = (J^\infty(\mathcal{A}_\sigma), \perp, M^\infty(\mathcal{A}_\sigma))$, where $J^\infty(\mathcal{A}_\sigma) \subseteq \mathcal{G}_\kappa(X)$ is the set of completely join-irreducibles of $\mathcal{A}_\sigma$ and $M^\infty(\mathcal{A}_\sigma) \subseteq \mathcal{G}_o(X)$ is its set of completely meet-irreducibles. In the distributive case these are the completely prime elements of the canonical extension. Defining semantics in a uniform way for the distributive and non-distributive case, however, stumbles on the fact that in the latter case completely join-irreducibles need not be completely prime. This is precisely what is responsible for the failure to re-capture the standard semantic clauses for familiar operators as this is very clearly illustrated for the case of diamond in [5].

In this article we have obtained a relational representation of $\Diamond, \Box$ and $\rightarrow$ which allows for interpreting the corresponding logical operators as in the distributive case. The cost of this has been the restriction of admissible interpretations to those assigning a closed element of the complete lattice of stable sets. Indeed, examining the case of $\Diamond$ as en example, its extension on all stable sets is defined using join-density of closed elements, $\Diamond_\sigma A = \bigvee_{x \in A} \Diamond_\sigma \Gamma x$. Given that, as we have shown in Lemma 3.9, $\Diamond_\sigma \Gamma x = \diamondminus \Gamma x = \Gamma(\Diamond^\flat x)$, a straightforward calculation shows that

$$x \in \Diamond_\sigma A \text{ iff } \forall y \in Y\ (\forall z \in X(z \in A \longrightarrow yR_\diamond z) \longrightarrow x \perp y)$$

where we defined $yR_\diamond z$ iff $\Diamond^\flat z \perp y$. This is precisely condition (1) that we mentioned in the Introduction, quoting it from [5] (where $x \leq y$ is defined by $x \perp y$). Similar observations can be made for the other operators.

There is then a choice to be made, between pursuing a uniform algebraic approach based on canonical extensions and then abandoning the standard interpretation of e.g. boxes and diamonds, or taking a more applied stance and preferring to abandon uniformity of approach when it comes to semantic issues. The choice boils down to either (a) considering all interpretations assigning just any stable set to propositional variables as admissible, but then the received interpretation of familiar operators must be abandoned, or (b) we may opt for recapturing the familiar meaning of operators despite the absence of distribution, but then interpretations must be restricted to the closed ones, assigning a closed element to a propositional variable.

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