Dualities for Płonka sums

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Abstract. Płonka sums consist of an algebraic construction similar, in some sense, to direct limits, which allows to represent classes of algebras defined by means of regular identities (namely those equations where the same set of variables appears on both sides). Recently, Płonka sums have been connected to logic, as they provide algebraic semantics to logics obtained by imposing a syntactic filter to given logics. In this paper, I present a very general topological duality for classes of algebras admitting a Płonka sum representation in terms of dualisable algebras.

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1. Introduction

A formal identity $\varphi \approx \psi$ is said to be *regular* provided that exactly the same variables occur in the terms φ and ψ. A variety $\mathcal{V}$ is called *regular* whenever it satisfies identities which are regular only. The aim of this paper is showing a very simple way to construct topological dualities for regular varieties via the use of Płonka sums.

On the other hand, a variety satisfying at least one identity which is not regular, is called *irregular*. A relevant subclass of irregular variety is formed by the strongly irregular ones. A variety $\mathcal{V}$ is called *strongly irregular* if it satisfies an identity of the kind $f(x, y) \approx x$, where $f(x, y)$ is any term of the language in which x and y really occur. Examples of strongly irregular varieties abound in logic, since every variety with a lattice reduct is irregular as witnessed by the term $f(x, y) \coloneqq x \wedge (x \vee y)$.

The algebraic study of regular varieties traces back to the pioneering work of Płonka [22], who introduced a new class-operator $\mathcal{P}_l(\cdot)$ nowadays called *Płonka sum*, and used it to prove that any regular variety $\mathcal{V}$ can be represented as Płonka sum of a suitable strongly irregular variety $\mathcal{V}'$, in symbols $\mathcal{P}_l(\mathcal{V}') = \mathcal{V}$. In this case $\mathcal{V}$ is called the *regularization* of $\mathcal{V}'$, in a sense that we will be made precise.

Although the whole theory of Płonka sum is purely algebraic, regular varieties have found applications in computer science, in particular in the theory of program semantics (see [18, 17, 28]). Recently, Płonka sums have been surprisingly connected to logic. Indeed, the algebraic semantics of one among the logics within the so-called Kleene family [16], namely paraconsistent Weak Kleene logic, PWK for short, coincide with the regularization of the variety of Boolean algebras, firstly axiomatised in [24, 25].

PWK has been essentially introduced by Halldén [15] and defended by Prior [27] as a logic for handling reasonings that involve meaningless expressions and references to non-existing objects, respectively. The relation between PWK and classical logic has been recently investigated in [6], while proofs systems can be found in [9, 21]. The details of the connection between the logic PWK and the regularization of Boolean algebras, also referred to as *involutive bisemilattices*, are extensively studied in [1, 20].

The link between logics and Płonka sums can indeed be pushed further: the construction of Płonka sums, originally devised for algebras only, can be extended also to logical matrices [4], in such a way to provide algebraic semantics to the logics of variables inclusion. In detail, given a logic $\vdash$, two new consequence relations, denoted[1] by $\vdash^l$ and $\vdash^r$, respectively, can defined as follows:

$$\Gamma \vdash^l \varphi \iff \text{ there is } \Delta \subseteq \Gamma \text{ s.t. } Var(\Delta) \subseteq Var(\varphi) \text{ and } \Delta \vdash \varphi.$$

and

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) & \text{or} \\ \Sigma \subseteq \Gamma. \end{cases}$$

The models of the logics $\vdash^l$ and $\vdash^r$ are obtained out of matrix models of $\vdash$ via the appropriate construction of the Płonka sum (see [4] and [5] for details). As a consequence, logics of variables inclusion embrace the class of logics often referred as *infectious* (see [32, 12]), as they are semantically defined by a matrix containing a value that infects every operation in which it takes part (the logic PWK is a prototypical example): Płonka sums is indeed to most appropriate algebraic tool to express, algebraically, the notion of *contamination*. Examples of logics of variables inclusion are also introduced in [8, 7]. In particular, they are applied for both modeling computer-programs affected by errors [11] and in recent developments in the theory of truth [33].

On a different stream of research, the study of topological dualities for regular varieties traces back to the work of Gierz and Romanowska for distributive bisemilattices [14], the regularization of distributive lattices. The

[1]The notation aims at stressing that the variable inclusion constraint goes from premises to conclusion (roughly speaking, from left to right), in one case, and from conclusion to premises (from right to left) in the other.

technique used there has been generalized a few years later to regular varieties in [31, 29] (a different approach can be found in [10]).

We recently stated a slightly different duality, still based on Płonka sums, for involutive bisemilattices, see [3] (differences will be briefly explained in Section 3).

At the light of the above mentioned connection between logics (of variables inclusion) and Płonka sums of (system of) algebras, the aim of this paper is provide a very general method for constructing topological dualities for algebras admitting a Płonka sum representation in terms of dualisable algebraic structures (see Corollary 4.6).

The paper is structured as follows: Section 2 recalls the main results concerning the construction of Płonka sums and their connection with regular varieties which will be used to implement our duality. Section 3 is devoted to introduce the categories used to build the duality, namely semilattice direct and inverse systems of an arbitrary category. Finally, Section 4 presents the main result.

2. Preliminaries

We start by providing all the necessary notions to construct Płonka sums; then we will recall the connection with regular varieties.

For standard information on Płonka sums we refer the reader to [23, 22, 26, 30]. A *semilattice* is an algebra $\mathbf{A} = \langle A, \vee \rangle$, where $\vee$ is a binary commutative, associative and idempotent operation. Given a semilattice $\mathbf{A}$ and $a, b \in A$, we set

$$a \leq b \Longleftrightarrow a \vee b = b.$$

It is easy to see that $\leq$ is a partial order on A.

Definition 2.1. A *semilattice direct system of algebras* consists in

(i) a semilattice $I = \langle I, \vee \rangle$;
(ii) a family of algebras $\{\mathbf{A}_i : i \in I\}$ with disjoint universes;
(iii) a homomorphism $f_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$;

moreover, f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Let X be a semilattice direct system of algebras as above. The *Płonka sum* over X, in symbols $\mathcal{P}_l(X)$ or $\mathcal{P}_l(\mathbf{A}_i)_{i\in I}$, is the algebra defined as follows. The universe of $\mathcal{P}_l(\mathbf{A}_i)_{i\in I}$ is the union $\bigcup_{i\in I} A_i$. Moreover, for every n-ary basic operation f (with $n \geqslant 1$) [2], and $a_1, \ldots, a_n \in \bigcup_{i\in I} A_i$, we set

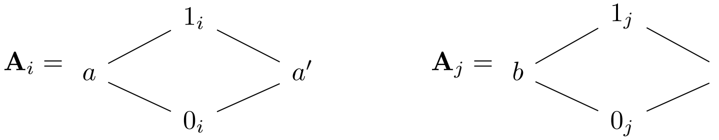
$$f^{\mathcal{P}_l(\mathbf{A}_i)_{i\in I}}(a_1, \ldots, a_n) := f^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n))$$

where $a_1 \in A_{i_1}, \ldots, a_1 \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

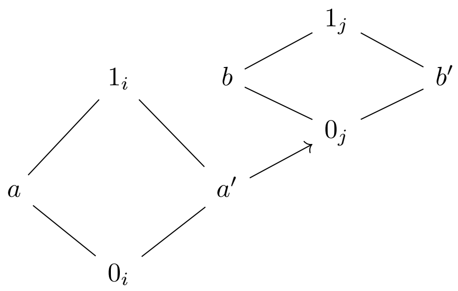
A simple example can be helpful to clarify the above definition.

[2] In presence of nullary operations it is necessary to assume that I also has a least element. See [25] for details.

Example 2.2. Let $\mathbf{A}_i$ and $\mathbf{A}_j$ be isomorphic copies of the 4"-element Boolean algebra, with elements labelled as follows:



Let $\mathbf{I}$ be the linear order with two elements $i < j$, and $\mathbb{A}$ the semilattice direct system over $\mathbf{I}$ in which the homomorphism $p_{ij}\colon \mathbf{A}_j \to \mathbf{A}_j$ is given by $p_{ij}(a) = 1_j$ (and therefore $p_{ij}(a') = 0_j$). Hence the Płonka sum $\mathcal{P}_l(\mathbb{A})$ over this system is drawn in the following diagram (the arrow indicates the homomorphism p_{ij}):



We briefly sketch the way binary operations work in $\mathcal{P}_l(\mathbb{A})$. For instance,

$$a \wedge^{\mathcal{P}_l} a' = a \wedge^{\mathbf{A}_i} a' = 0_i$$

More precisely, any operation involving two elements belonging to the same algebra is performed via the operations in such an algebra. On the other hand,

$$a' \wedge^{\mathcal{P}_l} b = p_{ij}(a') \wedge^{\mathbf{A}_j} b = 0_j \wedge^{\mathbf{A}_j} b = 0_j.$$

□

The theory of Płonka sums is strictly related with a special kind of operation:

Definition 2.3. Let $\mathbf{A}$ be an algebra of type ν. A function $\cdot\colon A^2 \to A$ is a *partition function* in $\mathbf{A}$ if the following conditions are satisfied for all $a, b, c \in A$, $a_1, ..., a_n \in A^n$ and for any operation $g \in \nu$ of arity $n \geqslant 1$.

1. $a \cdot a = a$
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. $a \cdot (b \cdot c) = a \cdot (c \cdot b)$
4. $g(a_1, \dots, a_n) \cdot b = g(a_1 \cdot b, \dots, a_n \cdot b)$
5. $b \cdot g(a_1, \dots, a_n) = b \cdot a_1 \cdot_{\dots} \cdot a_n$

The next result makes explicit the relation between Płonka sums and partition functions:

Theorem 2.4 (**[22], Thm. II**). *Let* $\mathbf{A}$ *be an algebra of type* ν *with a partition funtion* $\cdot$. *The following conditions hold:*

1. *A can be partitioned into $\{A_i : i \in I\}$ where any two elements $a, b \in A$ belong to the same component A_i exactly when*
$$a = a \cdot b \text{ and } b = b \cdot a.$$
Moreover, every A_i is the universe of a subalgebra $\mathbf{A}_i$ of $\mathbf{A}$.
2. *The relation $\leq$ on I given by the rule*
$$i \leq j \iff \text{there exist } a \in A_i, b \in A_j \text{ s.t. } b \cdot a = b$$
is a partial order and $\langle I, \leq \rangle$ is a semilattice.
3. *For all $i, j \in I$ such that $i \leq j$ and $b \in A_j$, the map $f_{ij} \colon A_i \to A_j$, defined by the rule $f_{ij}(x) = x \cdot b$ is a homomorphism. The definition of f_{ij} is independent from the choice of b, since $a \cdot b = a \cdot c$, for all $a \in A_i$ and $c \in A_j$.*
4. *$Y = \langle \langle I, \leq \rangle, \{\mathbf{A}_i\}_{i \in I}, \{f_{ij} : i \leq j\} \rangle$ is a direct system of algebras such that $\mathcal{P}_l(Y) = \mathbf{A}$.*

The above result states that every algebra possessing a partition function can be associated to a semilattice system $\mathbb{A}$ and, most importantly, the Płonka sum over $\mathbb{A}$ is a representation of $\mathbf{A}$.

The construction of Płonka sums preserves the validity of the so-called *regular identities* (see [22, Theorem III]), i.e. identities of the form $\varphi \approx \psi$ such that $Var(\varphi) = Var(\psi)$. In particular:

Theorem 2.5 (**[22], Thm. I**). *If $\mathbb{A}$ is a semilattice direct system of algebras containing at least two algebras, then in the algebra $\mathcal{P}_l(\mathbb{A})$ all regular equations satisfied in all algebras of $\mathbb{A}$ are satisfied, whereas every other equations is false in $\mathcal{P}_l(\mathbb{A})$.*

A variety of algebras is called *regular* if it does satisfies regular identities only. It is called irregular if it is not regular. In particular, an irregular variety $\mathcal{V}$ which possesses a term-definable operation $f(x, y)$ such that $\mathcal{V} \models f(x, y) \approx x$ is said to be *strongly irregular*. Strongly irregular varieties are actually very common in mathematics: indeed, examples include the variety of groups and rings (as witnessed by the terms $f(x, y) := x + (y - y)$, in additive notation) and any variety which has a lattice reduct (this includes, for instance, any variety of residuated lattices [13]), as witnessed by the term $f(x, y) := x \wedge (x \vee y)$.

Whenever $\mathcal{V}$ is an irregular variety, then we indicate by $R(\mathcal{V})$, the *regularization* of $\mathcal{V}$, namely the variety satisfying only the regular identities holding in $\mathcal{V}$.

The importance of regular and strongly irregular varieties, in the context of Płonka sums, is resumed in the following:

Theorem 2.6 (**[26], Thm. 7.1**). *Let $\mathcal{V}$ be a strongly irregular variety. Then any element $\mathbf{A} \in R(\mathcal{V})$ is isomorphic to the Płonka sum over a direct system of algebras in $\mathcal{V}$.*

3. The categories of semilattice systems

The present section is meant to introduce the categories of direct and inverse semilattice systems which will be used to establish the main results (see Section 4).

We briefly recall the categories so as they are introduced in our previous work [3]. Semilattice direct (and inverse) systems are, roughly speaking, obvious generalizations of direct (and inverse) systems in a given category, obtained by assuming the index set to be a semilattice instead of a (directed) pre-ordered set. These concepts find applications in several fields of mathematics (see for example [19]).

Definition 3.1. Let $\mathfrak{C}$ be an arbitrary category. A *semilattice direct system* in $\mathfrak{C}$ is a triple $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$ such that

(i) I is a semilattice.
(ii) X_i is an object in $\mathfrak{C}$, for each $i \in I$;
(iii) $p_{ii'} : X_i \to X_{i'}$ is a morphism of $\mathfrak{C}$, for each pair $i \leqslant i'$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{i'i''} \circ p_{ii'} = p_{ii''}$.

As a matter of convention, we indicate by $\vee$ the semilattice operation on I.

Given two strongly direct systems $\mathbb{X}$ and $\mathbb{Y}$, a morphism is a pair (φ, f_i) : $\mathbb{X} \to \mathbb{Y}$ such that:

i) $\varphi\colon I \to J$ is a semilattice homomorphism
ii) $f_i\colon X_i \to Y_{\varphi(i)}$ is a morphism of $\mathfrak{C}$, making the diagram in Figure 1 commutative for each $i, i' \in I$, $i \leq i'$:

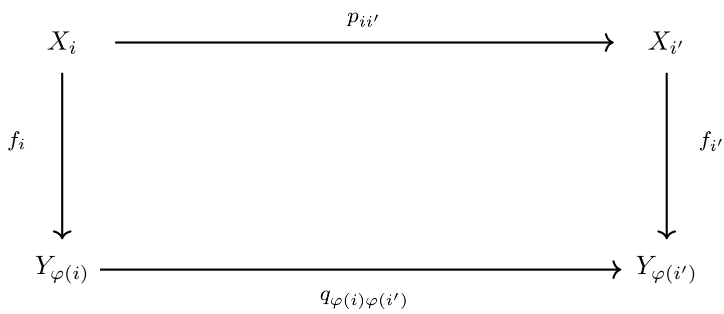


Figure 1. The commuting diagram defining morphisms of semilattice direct systems

Semilattice inverse systems, for an arbitrary category, are defined in an analogous, dual way.

Definition 3.2. Let $\mathfrak{C}$ be an arbitrary category, a *semilattice inverse system* in the category $\mathfrak{C}$ is a tern $\mathcal{X} = \langle X_i, p_{ii'}, I\rangle$ such that

(i) I is a semilattice;
(ii) for each $i \in I$, X_i is an object in $\mathfrak{C}$;
(iii) $p_{ii'} : X_{i'} \to X_i$ is a morphism of $\mathfrak{C}$, for each pair $i \leqslant i'$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{ii'} \circ p_{i'i''} = p_{ii''}$.

As already mentioned, the only difference making an inverse system a semilattice inverse system is the requirement on the index set to be a semilattice instead of a directed preorder.

Definition 3.3. Given two *semilattice inverse systems* $\mathcal{X} = \langle X_i, p_{ii'}, I\rangle$ and $\mathcal{Y} = \langle Y_j, q_{jj'}, J\rangle$, a *morphism* between $\mathcal{X}$ and $\mathcal{Y}$ is a pair (φ, f_j) such that

i) $\varphi : J \to I$ is a semilattice homomorphism;
ii) for each $j \in J$, $f_j : X_{\varphi(j)} \to Y_j$ is a morphism in $\mathfrak{C}$, such that whenever $j \leq j'$, then the diagram in Figure 2 commutes.

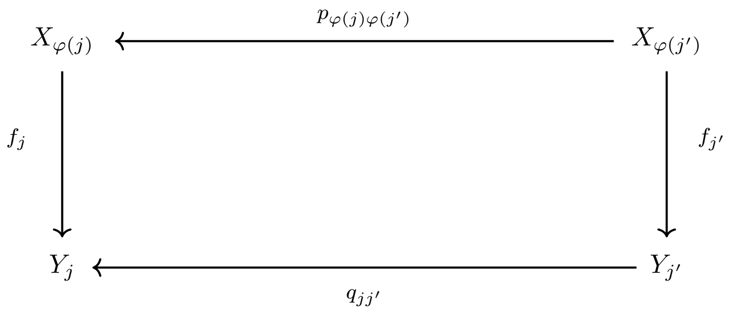


FIGURE 2. The commuting diagram defining morphisms of semilattice inverse systems.

Notice that, the assumption that $\varphi \colon J \to I$ is a (semilattice) homomorphism implies that whenever $j \leq j'$ then $\varphi(j) \leq \varphi(j')$.

It is easily checked that, whenever $\mathfrak{C}$ an arbitrary category, then semilattice direct and inverse systems form two categories which we will refer to sem-dir-$\mathfrak{C}$ and sem-inv-$\mathfrak{C}$, respectively.

Dualities of arbitrary categories can be lifted to dualities of semilattice systems constructed via such duals categories. The following result will be used in the next section:

Theorem 3.4 ([3])**.** *Let $\mathfrak{C}$ and $\mathfrak{D}$ be dually equivalent categories. Then* sem-dir-$\mathfrak{C}$ *and* sem-inv-$\mathfrak{D}$ *are dually equivalent categories.*

A similar idea of lifting a duality has been ideated by Romanowska and Smith [31, 29]. In contrast with their approach, our duality is obtained constructing the categories sem-dir-$\mathfrak{C}$ and sem-inv-$\mathfrak{D}$ using the very same index set. On the other hand, they consider, on the algebraic side, the semilattice sum of an algebraic category and, on the topological, the semilattice representation of the dual spaces: the duality is then obtained by *dualising* the semilattice of the index sets (the proof involves sophisticated categorical machinery).

4. The Duality

The notions of strongly irregular variety and regularization of a variety can be clearly defined as categories. We will say that $\mathfrak{C}$ is a strongly irregular algebraic category provided that its objects are strongly irregular varieties. In such case, $R(\mathfrak{C})$ is the algebraic category whose objects are regularizations of the objects in $\mathfrak{C}$. Moreover, we say that an algebraic category $\mathfrak{C}$ is *dualisable* whenever it admits a dually equivalent topological category.

Theorem 2.6 states that, whenever $\mathfrak{C}$ is a strongly irregular category, the objects in $R(\mathfrak{C})$ are isomorphic to the objects of the category sem-dir-$\mathfrak{C}$. We will show that they are also equivalent as categories.

Definition 4.1. Given two semilattice direct systems $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ and $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ from an arbitrary category $\mathfrak{C}$ and an homomorphism $h\colon \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$, we say that h *preserves* the Płonka fibres if, for every $i \in I$ there exists an index $j \in J$ such that $h(A_i) \subseteq B_j$.

We are interested in the following question: for which classes $\mathcal{K}$ of algebras, any homomorphism (between Płonka sums of elements in $\mathcal{K}$) preserves the fibres?

For the purpose of this paper, we confine our analysis to the case where $\mathcal{K}$ is a strongly irregular variety.

Theorem 4.2. *Let $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ and $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ be semilattice direct systems of algebras, with $\{\mathbf{A}_i\}_{i\in I}$ and $\{\mathbf{B}_j\}_{j\in J}$ belonging to a variety $\mathcal{V}$, for each $i \in I$, $j \in J$. Then any homomorphism $h : \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$ preserves the fibres if and only if $\mathcal{V}$ is a strongly irregular variety.*

Proof. To simplify notation, set $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$ and $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$.

($\Leftarrow$) Suppose $\mathcal{K}$ is a strongly irregular variety, i.e. it possesses a binary term definable operation $\circ$ such that $\mathcal{V} \models x \circ y \approx x$. Notice that $\circ$ defines a partition function on $\mathbf{A}$ and $\mathbf{B}$. Let a_1, $a_2 \in A_i$ for some $i \in I$ and suppose, towards a contradiction, that $h(a_1) = b_1 \in B_j$ and $h(a_2) = b_2 \in B_k$ with $j \neq k \in J$. It follows that:

$$b_1 = h(a_1) = h(a_1 \circ a_2) = h(a_1) \circ h(a_2) = b_1 \circ b_2$$

and

$$b_2 = h(a_2) = h(a_2 \circ a_1) = h(a_2) \circ h(a_1) = b_2 \circ b_1,$$

which implies that the elements b_1, b_2 belong to the same algebra in $\mathbb{B}$, i.e. $j = k$, a contradiction.

($\Rightarrow$) Suppose h preserves the fibres of the Płonka sum, i.e. for each $i \in I$, there exists a $j \in J$ such that $h(A_i) \subseteq B_j$ and suppose, towards a contradiction that $\mathcal{V}$ is not strongly irregular. Since $\mathbf{A}$ and $\mathbf{B}$ are Płonka sums of algebras in $\mathcal{V}$, there exists a binary term definable operation $f(x, y)$

which is a partition function on both $\mathbf{A}$ and $\mathbf{B}$. By Theorem 2.4, two elements $a, b \in \mathcal{P}_l(\mathbb{A})$ belong to the same component $\mathbf{A}_i$ if and only if $f(a, b) = a$ and $f(b, a) = b$. Therefore, for each $i \in I$, $\mathbf{A}_i \models f(x, y) \approx x$. Moreover, for each $i \in I$, $\mathbf{B}_j \in \mathbb{H}(A_i)$, hence (since $\mathcal{V}$ is a variety) $\mathbf{B}_j \models f(x, y) \approx x$, for each $j \in J$. Then, since $\mathcal{V}$ is not strongly irregular, it follows that $\mathbf{A}_i$, $\mathbf{B}_j \notin \mathcal{V}$, a contradiction. □

Lemma 4.3. *Let* $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ *and* $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ *be semilattice direct systems of an arbitrary algebraic category* $\mathfrak{C}$ *and* (φ, f_i) *a morphism from* $\mathbb{A}$ *to* $\mathbb{B}$. *Then* $h\colon \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$, *defined as*

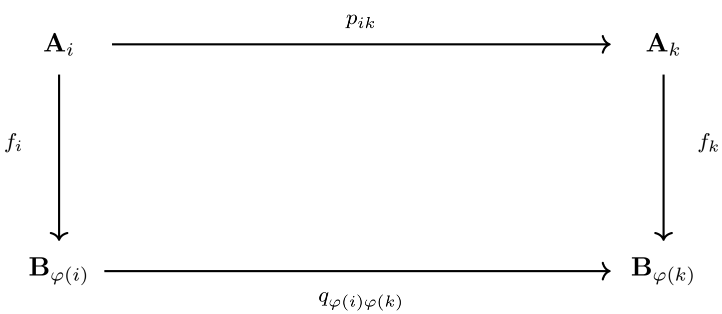
$$h(a) := f_i(a),$$

where $i \in I$ *is the index such that* $a \in A_i$, *is a morphism in* $\mathfrak{C}$.

Proof. The map h is well defined for every $i \in I$, as by assumption f_i is morphism in $\mathfrak{C}$. Since $\mathfrak{C}$ is an algebraic category (where morphisms are homomorphisms of algebras), we only have to check that h is compatible with all the operations of the Płonka sum. To simplify notation, we set $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$, $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$, $a_1, ..., a_n \in \mathbf{A}$ with $i_1, ..., i_n$ indexing the algebras to which they belong, g a generic n-ary operation in the type of the considered algebras and, finally, $k = i_1 \vee ... \vee i_n$. Then,

$$\begin{aligned}
h(g^{\mathbf{A}}(a_1, ..., a_n)) &= h(g^{\mathbf{A}_k}(p_{i_1 k}(a_1), ..., p_{i_n k}(a_n))) \\
&= f_k(g^{\mathbf{A}_k}(p_{i_1 k}(a_1), ..., p_{i_n k}(a_n))) \\
&= g^{\mathbf{B}_{\varphi(k)}}(f_k(p_{i_1 k}(a_1)), ..., f_k(p_{i_n k}(a_n))) \\
&= g^{\mathbf{B}_{\varphi(k)}}(q_{\varphi(i_1)\varphi(k)}(f_{i_1}(a_1)), ..., q_{\varphi(i_n)\varphi(k)}(f_{i_n}(a_n)) \\
&= g^{\mathbf{B}}(f_{i_1}(a_1), ..., f_{i_n}(a_n)) \\
&= g^{\mathbf{B}}(h(a_1), ..., h(a_n)),
\end{aligned}$$

where the fourth equality is justified by the commutativity of the following diagram (which holds as, by assumption, (φ, f_i) is morphism in sem-dir-$\mathfrak{C}$), for every $i \in \{i_1, ..., i_n\}$:



□

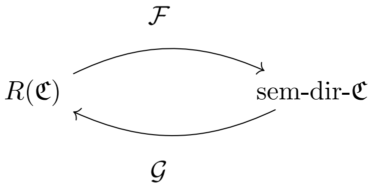
Lemma 4.4. *Let $\mathcal{V}$ be a variety, $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$, $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ be semilattice direct systems of algebras in $\mathcal{V}$ and $h\colon \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$ a homomorphism. Let $\varphi_h : I \to J$ be a map such that $h(A_i) \subseteq B_{\varphi_h(i)}$. Then φ_h is a semilattice homomorphism.*

Proof. Let $a_1, \dots, a_n \in \bigcup_{i\in I} A_i$, with $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$ $(i_1, \dots i_n \in I)$ and set $k = i_1 \vee \dots \vee i_n$. We want to show that $\varphi_h(k) = \varphi_h(i_1) \vee \dots \vee \varphi_h(i_n)$. To simplify notation, set $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$ and $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$. Consider an arbitrary operation in the type of $\mathcal{V}$. Clearly, $h(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(h(a_1), \dots, h(a_n))$. By hypothesis, $h(a_1) \in B_{\varphi_h(i_1)}, \dots, h(a_n) \in B_{\varphi_h(i_n)}$, therefore $f^{\mathbf{B}}(h(a_1), \dots, h(a_n)) \in B_j$ with $j = \varphi(i_1) \vee \dots \vee \varphi(i_n)$. On the other hand, $f^{\mathbf{A}}(a_1, \dots a_n) \in A_k$, hence $h(f^{\mathbf{A}}(a_1, \dots a_n)) \in B_{\varphi(k)}$. This shows that $\varphi_h(k) = \varphi_h(i_1) \vee \dots \vee \varphi_h(i_n)$, i.e. is a semilattice homorphism.

□

Theorem 4.5. *Let $\mathfrak{C}$ be a strongly irregular algebraic category. Then the categories $R(\mathfrak{C})$ and* sem-dir-$\mathfrak{C}$ *are equivalent.*

Proof. The equivalence is proved via the following functors:



Let $\mathbf{A}$ be an object in the category $R(\mathfrak{C})$. Since $\mathfrak{C}$ is strongly irregular, by Theorem 2.6, we know that $\mathbf{A} \cong \mathcal{P}_l(\mathbb{A})$, with $\mathbb{A}$ a semilattice direct system of algebras in $\mathfrak{C}$. $\mathcal{F}$ associates to $\mathbf{A}$ the semilattice direct system $\mathbb{A}$.

Consider a morphism in $R(\mathfrak{C})$, $h\colon \mathbf{A} \to \mathbf{B}$ and set $\mathbf{A} \cong \mathcal{P}_l(\mathbb{A})$, $\mathbf{B} \cong \mathcal{P}_l(\mathbb{B})$, with with $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ and $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ semilattice direct systems of algebras in $\mathfrak{C}$. Since $\mathfrak{C}$ is a strongly irregular variety, we know, by Theorem 4.2, that h preserves the Płonka fibres of the direct system $\mathbb{A}$ (arising from the Płonka sum representation of $\mathbf{A}$), i.e. $h(A_i) \subseteq B_j$, for some $j \in J$. Hence, we can define a map $\varphi_h : I \to J$ satisfying the assumptions of Lemma 4.4, which assures that φ_h is a semilattice homomorphism. Moreover, for each $i \in I$, the restriction of h over $\mathbf{A}_i$, $h_{|A_i}$ is a homomorphism of algebras (objects) in $\mathfrak{C}$. $\mathcal{F}$ associate to the morphism h, the pair $(\varphi_h, h_{|A_i})$.

Moreover, it is easily checked that the following diagram is commutative for each $i \leq i'$ (indeed $i \leq i'$ implies $\varphi_h(i) \leq \varphi_h(i')$)

$$
\begin{array}{ccc}
\mathbf{A}_i & \xrightarrow{\quad p_{ii'} \quad} & \mathbf{A}_{i'} \\
\Big\downarrow{\scriptstyle h_{|A_i}} & & \Big\downarrow{\scriptstyle h_{|A_{i'}}} \\
\mathbf{B}_{\varphi_h(i)} & \xrightarrow[\quad q_{\varphi_h(i)\varphi_h(i')} \quad]{} & \mathbf{B}_{\varphi_h(i')}
\end{array}
$$

Therefore $\mathcal{F}(h)$ is a morphism from $\mathbb{A}$ to $\mathbb{B}$, showing that $\mathcal{F}$ is a covariant functor.

On the other hand, $\mathcal{G}$ associates to an object $\mathbb{A}$ in the category sem-dir-$\mathfrak{C}$, the Płonka sum $\mathcal{P}_l(\mathbb{A})$ over $\mathbf{A}$, which is an object in $R(\mathfrak{C})$ (as $\mathfrak{C}$ is strongly irregular). Moreover, to each morphism (φ, f_i), $\mathcal{G}$ associates the map $h\colon \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$, defined as $h(a) := f_i(a)$, for each $a \in A_i$ and $i \in I$. Lemma 4.3 assures that h is indeed a morphism in $R(\mathfrak{C})$.

It is easy to check that the compositions of the two functors are naturally isomorphic with the identities (in both categories). □

As already mentioned, many algebraic structures arising in the study of logics are strongly irregular, since they possess a lattice reduct. Considering those ones admitting topological duals, the combination of Theorem 4.5 with Theorem 3.4 allows to construct the topological dual of the regularization of a variety.

Corollary 4.6. *Let $\mathfrak{C}$ be a dualisable strongly irregular algebraic category with $\mathfrak{C}^*$ as topological dual. Then the categories $R(\mathfrak{C})$ and* sem-inv-$\mathfrak{C}^*$ *are dually equivalent.*

It is worthless to say that, to our's best knowledge, the construction of Płonka sum has no analogous on the side of the topological representation spaces, so the class *sem-inv*-$\mathfrak{C}^*$ remains basically a collection of spaces organized into a semilattice inverse system. A partial attempt to fill this gap is [2].

A related question concerns the possibility of describing semilattice inverse systems of topological spaces as a unique space. This is done in some known special cases, as distributive bisemilattices [14], the Płonka sum of distributive lattices and involutive bisemilattices [3], the Płonka sum of Boolean algebras.

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