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Proof Theory for Genuine Paraconsistent Logics L3A and L3B (A preliminary draft)

Abstract: In this paper, we present cut-free sequent calculi and natural deduction systems for Béziau and Franceschetto's three-valued genuine paraconsistent logics **L3A** and **L3B**. Besides, we consider logics $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$ which are extensions of **L3A** and **L3B**, respectively, by Heyting's (Gödel's) implication.

Keywords: paraconsistent logic, three-valued logic, sequent calculus, natural deduction

1 Introduction

To begin with, let us specify two propositional languages: $\mathscr{L}$ and $\mathscr{L}_{\rightarrow}$. The first one has the following alphabet: $\langle \mathscr{P}, \wedge, \vee, \neg, (,) \rangle$, where $\mathscr{P} = \{p, q, r, p_n, q_n, r_n \mid n \in \mathbb{N}\}$ is the set of propositional variables. The second one is $\mathscr{L}$'s extension by an implication $\rightarrow$. The sets $\mathscr{F}$ and $\mathscr{F}_{\rightarrow}$, respectively, of all $\mathscr{L}$'s and $\mathscr{L}_{\rightarrow}$'s formulas are defined in a standard way. Let **L** be a logic which is built either in $\mathscr{L}$ or $\mathscr{L}_{\rightarrow}$.

In logical literature, there are several definitions of paraconsistent logic. The following list contains the most popular ones and does not pretend to be complete:[1]

- Priest's definition [16]. A logic **L** is paraconsistent iff for *some* $A, B \in \mathscr{F}'$ it holds that $A, \neg A \not\models_{\mathbf{L}} B$, where $\mathscr{F}' \in \{\mathscr{F}, \mathscr{F}_{\rightarrow}\}$.
- Da Costa's definition [7]. A logic **L** is paraconsistent iff *there is* $A \in \mathscr{F}'$ such that $\not\models_{\mathbf{L}} \neg(A \wedge \neg A)$, where $\mathscr{F}' \in \{\mathscr{F}, \mathscr{F}_{\rightarrow}\}$.
- Jaśkowski's definition [11]. A logic **L** is paraconsistent iff for *some* $A, B \in \mathscr{F}_{\rightarrow}$ it holds that $\not\models_{\mathbf{L}} A \rightarrow (\neg A \rightarrow B)$.[2] .

In general case, these definitions are not equivalent. First of all, they have different requirements to languages in which the considered logics are built. Priest's definition seems to be the most uniform one: it requires the presence of negation only in the concerned logic. Da Costa's definition requires the presence of negation as well as conjunction and Jaśkowski's one — the presence of negation as well as implication. However, even if two different logics are built in the same language, these definitions still can be different. Let us present some examples, using the framework of many-valued logics.

[1] In order to be more precise, we consider logics which are built in $\mathscr{L}$ and $\mathscr{L}_{\rightarrow}$ only. However, one can deal with the other languages.

[2] There is also a variation on the theme of Jaśkowski's definition such that $(A \wedge \neg A) \rightarrow B$ is used instead of $A \rightarrow (\neg A \rightarrow B)$.

- Asenjo-Priest's three-valued logic of paradox **LP** [1, 15, 16]. It is built in the language $\mathscr{L}$ and has the following set of truth values: $\mathscr{V} = \{1, 1/2, 0\}$. Its connectives are defined as follows, for each $A, B \in \mathscr{F}$: $v(A \wedge B) = min(v(A), v(B))$, $v(A \vee B) = max(v(A), v(B))$, and $v(\neg A) = 1 - v(A)$, where v is a function, called *valuation*, such that $v(P) \in \mathscr{V}$, for each $P \in \mathscr{P}$. The set $\mathscr{D}_2$ of *designated* values is $\{1, 1/2\}$ and the entailment relation is defined as follows, for each $\Gamma \subseteq \mathscr{F}$ and $A \in \mathscr{F}$: $\Gamma \models_{\mathbf{LP}} A$ iff for each valuation v, $v(G) \neq 0$ (for each $G \in \Gamma$) implies $v(A) \neq 0$. In the case of **LP**, it holds that $p, \neg p \not\models_{\mathbf{LP}} q$ while $\models_{\mathbf{LP}} \neg(A \wedge \neg A)$, for each $A \in \mathscr{F}$. Thus, **LP** is paraconsitent according to Priest's definition, but at the same time it is *not* paraconsitent according to da Costa's one.

- An implicative extension of **LP** (which we will call $\mathbf{LP}_{\rightarrow}$) such that $v(A \rightarrow B) = v(\neg A \vee B)$. In the case of $\mathbf{LP}_{\rightarrow}$, it holds that $p, \neg p \not\models_{\mathbf{LP}} q$, $\models_{\mathbf{LP}} \neg(A \wedge \neg A)$, and $\models_{\mathbf{LP}} (A \rightarrow \neg A) \rightarrow B$, for each $A, B \in \mathscr{F}_{\rightarrow}$. Thus, $\mathbf{LP}_{\rightarrow}$ is *not* paraconsitent in Jaśkowski's sense as well as in Da Costa's one, but it is paraconsitent in Priest's sense.

- Yet another implicative extension of **LP** — Thomas' $\mathbf{LP}_{\Rightarrow}$ [20]. In the case of $\mathbf{LP}_{\Rightarrow}$, $v(A \rightarrow B) = 1$, if $v(A) \leqslant v(B)$; $v(A \rightarrow B) = 0$, otherwise. This implication is Rescher's one [17]. In this logic, it holds that $p, \neg p \not\models_{\mathbf{LP}} q$, $\models_{\mathbf{LP}} \neg(A \wedge \neg A)$, for each $A \in \mathscr{F}_{\rightarrow}$, and $\not\models_{\mathbf{LP}} (p \rightarrow \neg p) \rightarrow q$. So, $\mathbf{LP}_{\Rightarrow}$ is paraconsistent both in Priest's sense and Jaśkowski's one, but not in Da Costa's one.

- Kleene's strong logic $\mathbf{K_3}$ [13] which differs from **LP** with respect to the set of designated values: $\mathscr{D}_1 = \{1\}$ instead of $\mathscr{D}_2$. As a consequence, the entailment relation is defined as follows: $\Gamma \models_{\mathbf{K_3}} A$ iff for each valuation v, $v(G) = 1$ (for each $G \in \Gamma$) implies $v(A) = 1$, for each $\Gamma \subseteq \mathscr{F}$ and $A \in \mathscr{F}$. In the case of $\mathbf{K_3}$, $\not\models_{\mathbf{K_3}} \neg(p \wedge \neg p)$ while $A, \neg A \models_{\mathbf{K_3}} B$, for each $A, B \in \mathscr{F}_{\rightarrow}$. Thus, $\mathbf{K_3}$ is paraconsistent in Da Costa's sense, but not in Priest's one.[3]

One can find many other examples which show the difference between various definitions of paraconsistency. We are interested in a combination of these definitions, especially Priest's and Da Costa's ones. Béziau and Franceschetto [3] (see also Béziau's paper [2]) presented two logics, **L3A** and **L3B**, which and are paraconsistent according to both Priest's and Da Costa's approaches. In [3], they call these logics strong paraconsistent, but in [2] this notion is changed to *genuine paraconsistent* logics.

L3A and **L3B** are built in the language $\mathscr{L}$, $\mathscr{V}$ is the set of their truth values, and $\mathscr{D}_2$ is the set of their designated values. The connectives of **L3A** are defined as follows:

A	$\neg$
1	0
1/2	1
0	1

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\wedge$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	0	0	0

[3] Usually $\mathbf{K_3}$ is considered as *paracomplete* logic. According to Sette and Carnielli [19], a logic **L** is paracomplete iff *there is* $A \in \mathscr{F}'$ such that $\not\models_{\mathbf{L}} A \vee \neg A$, where $\mathscr{F}' \in \{\mathscr{F}, \mathscr{F}_{\rightarrow}\}$.

In **L3B**, the definition of disjunction is the same as for **L3A** while the other connectives are defined as follows:

A	$\neg$
1	0
$1/2$	$1/2$
0	1

$\wedge$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	1	0
0	0	0	0

The entailment relation in both **L3A** and **L3B** is defined in the same way as in **LP**. The disjunction of **L3A** and **L3B** coincide with **LP**'s one. **L3B**'s negation is the same as in **LP** also. **L3A**'s negation is the same as in Sette's logic $\mathbf{P^1}$ [18]. As mentioned in [3, 2], **L3A**'s and **L3B**'s conjunctions and disjunctions have the following properties:

(1) they are neoclassical, i.e. $A \wedge B$ is designated iff both A and B are designated, $A \vee B$ is designated iff either A or B are designated;

(2) they are **C**-extending, i.e. their restrictions on the set $\{1, 0\}$ are the same as classical conjunction and disjunction;

(3) they are commutative, i.e. $A \wedge B = B \wedge A$ and $A \vee B = B \vee A$.[4]

L3A's and **L3B**'s negations are not neoclassical, but they are **C**-extending. Totally, Béziau and Franceschetto [3] have found 4 three-valued genuine paraconsistent logics which satisfy the abovementioned conditioned. Two of them are already introduced **L3A** and **L3B**. The other ones are the $\{\neg, \wedge, \vee\}$-fragments of $\mathbf{P^1}$ [18] and $\mathbf{P^2}$ [5, 14], respectively. However, Béziau and Franceschetto exclude them into consideration in order "to minimize molecularization (molecular propositions behaving classically)" [3, p. 137]. **L3A** and **L3B** themselves are the $\{\neg, \wedge, \vee\}$-fragments of logics which are part of the family **8Kb** [6] of logics of formal inconsistency. However, the first paper which focus on **L3A** and **L3B** and deal explicitly with them is Béziau and Franceschetto's [3].

Let us also mention Hernández-Tello, Arrazola Ramírez, and Osorio Galindo's paper [9] devoted to implicational extensions of **L3A** and **L3B**. Although the authors suggest several suitable implications, they emphasize logics $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$ which are extensions of **L3A** and **L3B**, respectively, by Heyting's implication [10] (which is also known as Gödel's one [8] and was studied by Jaśkowski [12]). Heyting's implication is defined as follows:

$\rightarrow$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	1	0
0	1	1	1

[4] In [3, 2], this property is called symmetry and the second property is called "to be conservative connectives".

Clearly, $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$ are paraconsistent according to both Priest's and da Costa's approaches. However, they are paraconsistent according to Jaśkowski's approach also. Thus, these logics are even more genuine paraconsistent then **L3A** and **L3B** are.

One more extension of **L3B**, called **NH**, was studied by Caret [4]. **NH** has Sette's implication [18] and the following unary operator: $\circ A = \neg(A \wedge \neg A)$. **NH** is formalized via Hilbert-style calculus and analytic tableaux [4].

The aim of our paper is to present cut-free sequent calculi and natural deduction systems for **L3A** and **L3B** as well as $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$.

2 Sequent calculi for L3A and L3B

Sequent calculus $\mathfrak{S}_{\mathbf{L3A}}$ for **L3A** has the following axioms and inference rules (for each $A, B \in \mathscr{F}$ and $\Gamma, \Delta, \Theta, \Lambda \subseteq \mathscr{F}$):

$$(AX)\ A, \Gamma \Rightarrow A, \Delta$$

$$(\neg\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg A}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\vee \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\wedge \Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\neg\vee \Rightarrow)\ \frac{\neg A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \vee B), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg\vee)\ \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$$(\neg\wedge \Rightarrow_1)\ \frac{\neg A, \Gamma \Rightarrow \Delta \qquad \neg B, \Gamma \Rightarrow \Delta}{\neg(A \wedge B), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg\wedge)\ \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}$$

$$(\neg\wedge \Rightarrow_2)\ \frac{\Gamma \Rightarrow \Delta, B \qquad \neg A, \Theta \Rightarrow \Lambda}{\neg(A \wedge B), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\neg\wedge \Rightarrow_3)\ \frac{\Gamma \Rightarrow \Delta, A \qquad \neg B, \Theta \Rightarrow \Lambda}{\neg(A \wedge B), \Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

Sequent calculus $\mathfrak{S}_{\mathbf{L3B}}$ for **L3B** has the following axioms and inference rules (for each $A, B \in \mathscr{F}$ and $\Gamma, \Delta, \Theta, \Lambda \subseteq \mathscr{F}$): (AX), $(\vee \Rightarrow)$, $(\Rightarrow \vee)$, $(\wedge \Rightarrow)$, $(\Rightarrow \wedge)$, $(\neg\vee \Rightarrow)$, $(\neg\wedge \Rightarrow_1)$, $(\Rightarrow \neg\wedge)$ as well as the following ones:

$$(AX_{EM})\ \Gamma \Rightarrow \Delta, A, \neg A$$

$$(\neg\neg \Rightarrow^*)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg\neg)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$(\neg\wedge \Rightarrow_4)\ \frac{\Gamma \Rightarrow \Delta, A \wedge B \qquad \Theta \Rightarrow \Lambda, \neg A \wedge \neg B}{\neg(A \wedge B), \Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

$$\dfrac{\dfrac{\neg p \Rightarrow \neg p \qquad \neg q \Rightarrow \neg q}{\neg p \Rightarrow \neg q, \neg(p \vee q)}\,(\Rightarrow \neg\vee^*)}{\neg p \Rightarrow \neg q \vee \neg(p \vee q)}\,(\Rightarrow \vee)$$

Figure 1: Proof of $\neg p \Rightarrow \neg q \vee \neg(p \vee q)$ in $\mathfrak{S}_{\mathbf{L3B}}$.

$$(\Rightarrow \neg\vee^*)\ \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \neg B, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg(A \vee B)} \qquad (\Rightarrow \neg\vee^{**})\ \frac{\Gamma \Rightarrow \Delta, \neg B \qquad \neg A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg(A \vee B)}$$

The notion of proof in $\mathfrak{S}_{\mathbf{L}}$, where $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, is defined in a standard way. An example of proof in $\mathfrak{S}_{\mathbf{L3B}}$ is presented in Figure 1. Let $\mathfrak{S}^C_{\mathbf{L}}$, where $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, be an extension of $\mathfrak{S}_{\mathbf{L}}$ by the rule (Cut).

$$(Cut)\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

THEOREM 1. For each $\Gamma, \Delta \subseteq \mathscr{F}$ and $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, it holds that $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}_{\mathbf{L}}$ iff $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$.

Proof. To be written. □

THEOREM 2. For each $\Gamma, \Delta \subseteq \mathscr{F}$ and $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, it holds that $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}_{\mathbf{L}}$ iff $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}^C_{\mathbf{L}}$

Proof. To be written. □

3 Natural deduction for L3A and L3B

The set of all inference rules of natural deduction system $\mathfrak{N}\mathfrak{D}_{\mathbf{L3A}}$ for **L3A** is as follows ($A, B, C \in \mathscr{F}$):

$$(EFQ_\neg)\ \frac{\neg A \quad \neg\neg A}{B} \qquad (EM)\ \frac{}{A \vee \neg A} \qquad (\vee I_A)\ \frac{A}{A \vee B} \qquad (\vee I_B)\ \frac{B}{A \vee B}$$

$$(\vee E)\ \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \qquad (\wedge I)\ \frac{A \quad B}{A \wedge B} \qquad (\wedge E_A)\ \frac{A \wedge B}{A} \qquad (\wedge E_B)\ \frac{A \wedge B}{B}$$

$$(\neg \vee I)\ \frac{\neg A \wedge \neg B}{\neg(A \vee B)} \qquad (\neg \vee E)\ \frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad (\neg \wedge I)\ \frac{\neg A \vee \neg B}{\neg(A \wedge B)}$$

$$(\neg \wedge E_1)\ \frac{\neg(A \wedge B)}{\neg A \vee \neg B} \qquad (\neg \wedge E_2)\ \frac{B \quad \neg(A \wedge B)}{\neg A} \qquad (\neg \wedge E_3)\ \frac{A \quad \neg(A \wedge B)}{\neg B}$$

The set of all inference rules of natural deduction system $\mathfrak{N}\mathfrak{D}_{\mathbf{L3B}}$ for **L3B** consists of the following elements: (EM), $(\vee I_A)$, $(\vee I_B)$, $(\vee E)$, $(\wedge I)$, $(\wedge E_A)$, $(\wedge E_B)$, $(\neg \vee I)$, $(\neg \vee E)$, $(\neg \wedge E_1)$ as well as the following ones:

$$\dfrac{F \qquad \dfrac{\dfrac{\neg(q\wedge\neg r) \qquad \dfrac{[p\wedge\neg r]}{\neg r}\,(\wedge E_B)}{\neg q}\,(\neg\wedge E_3) \qquad \dfrac{\neg(p\wedge\neg q) \qquad \dfrac{[p\wedge\neg r]}{p}\,(\wedge E_A)}{\neg\neg q}\,(\neg\wedge E_2)}{\neg(p\wedge\neg r)}\,(EFQ_\neg) \qquad [\neg(p\wedge\neg r)]}{\neg(p\wedge\neg r)}\,(\sharp)$$

Figure 2: Derivation of $\neg(p \wedge \neg r)$ from $\neg(q \wedge \neg r)$ and $\neg(p \wedge \neg q)$ in $\mathfrak{ND}_{\mathbf{L3A}}$, where F is $(p \wedge \neg r) \vee \neg(p \wedge \neg r)$ and $(\sharp)$ is an abbreviation for "(EM), $(\vee E)$".

$$(\neg\wedge I_{\neg A})\ \frac{\neg A}{\neg B \vee \neg(A \vee B)} \qquad (\neg\wedge I_{\neg B})\ \frac{\neg B}{\neg A \vee \neg(A \vee B)}$$

$$(\neg\wedge E_4)\ \frac{\neg(A \wedge B) \quad A \wedge B \quad \neg A \wedge \neg B}{C} \qquad (\neg\neg I)\ \frac{A}{\neg\neg A} \qquad (\neg\neg E)\ \frac{\neg\neg A}{A}$$

The notion of derivation in $\mathfrak{ND}_{\mathbf{L}}$, where $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, is defined in a standard tree-format way. An example of derivation in $\mathfrak{ND}_{\mathbf{L3A}}$ is presented in Figure 2.

THEOREM 3. For each $\Gamma \subseteq \mathscr{F}$, $A \in \mathscr{F}$, and $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, it holds that $\vdash \Gamma \Rightarrow A$ is provable in $\mathfrak{S}^C_{\mathbf{L}}$ iff $\Gamma \vdash A$ is provable in $\mathfrak{ND}_{\mathbf{L}}$.

Proof. By induction on the depth of derivations. □

THEOREM 4. For each $\Gamma \subseteq \mathscr{F}$, $A \in \mathscr{F}$, and $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, it holds that $\Gamma \vdash_{\mathbf{L}} A$ is provable in $\mathfrak{ND}_{\mathbf{L}}$ iff $\Gamma \models_{\mathbf{L}} A$.

Proof. Theorem follows from Theorems 1 and 3. □

4 Proof theory for $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$

4.1 Sequent calculi for $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$

$\mathfrak{S}_{\mathbf{L_G}}$, where $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, is an extension of $\mathfrak{S}_{\mathbf{L}}$ by the following inference rules ($A, B \in \mathscr{F}_{\rightarrow}$ and $\Gamma, \Delta, \Theta, \Lambda \subseteq \mathscr{F}_{\rightarrow}$):

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Theta \Rightarrow \Lambda}{A \rightarrow B, \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow\rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$(\neg\!\rightarrow\ \Rightarrow_1)\ \frac{A, B, \Gamma \Rightarrow \Delta}{\neg(A \rightarrow B), \Gamma \Rightarrow \Delta} \qquad (\neg\!\rightarrow\ \Rightarrow_2)\ \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Theta \Rightarrow \Lambda, B}{\neg(A \rightarrow B), \Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

$$(\Rightarrow \neg\!\rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg B \qquad \neg A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg(A \rightarrow B)}$$

Let $\mathfrak{S}^C_{\mathbf{L}}$, where $\mathbf{L} \in \{\mathbf{L3A_G}, \mathbf{L3B_G}\}$, be an extension of $\mathfrak{S}_{\mathbf{L}}$ by the rule (Cut).

THEOREM 5. For each $\Gamma, \Delta \subseteq \mathscr{F}$ and $\mathbf{L} \in \{\mathbf{L3A_G}, \mathbf{L3B_G}\}$, it holds that $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}_\mathbf{L}$ iff $\models_\mathbf{L} \Gamma \Rightarrow \Delta$.

Proof. Similarly to Theorem 1. □

THEOREM 6. For each $\Gamma, \Delta \subseteq \mathscr{F}$ and $\mathbf{L} \in \{\mathbf{L3A_G}, \mathbf{L3B_G}\}$, it holds that $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}_\mathbf{L}$ iff $\vdash \Gamma \Rightarrow \Delta$ is provable in $\mathfrak{S}^C_\mathbf{L}$

Proof. Similarly to Theorem 2. □

4.2 Natural deduction for $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$

$\mathfrak{ND}_{\mathbf{L_G}}$, where $\mathbf{L} \in \{\mathbf{L3A}, \mathbf{L3B}\}$, is an extension of $\mathfrak{ND}_\mathbf{L}$ by the following inference rules ($A, B, C \in \mathscr{F}_\to$):

$$(\to I_1)\ \frac{\begin{array}{c}[A]\\ B\end{array}}{A \to B} \qquad (MP)\ \frac{A \quad A \to B}{B} \qquad (\neg{\to}I)\ \frac{\neg B}{\neg A \vee \neg(A \to B)}$$

$$(\neg{\to}E_1)\ \frac{\neg(A \to B)}{A \wedge \neg B} \qquad (\neg{\to}E_2)\ \frac{\neg(A \to B) \quad \neg A \quad B}{C}$$

THEOREM 7. For each $\Gamma \subseteq \mathscr{F}$, $A \in \mathscr{F}$, and $\mathbf{L} \in \{\mathbf{L3A_G}, \mathbf{L3B_G}\}$, it holds that $\vdash \Gamma \Rightarrow A$ is provable in $\mathfrak{S}^C_\mathbf{L}$ iff $\Gamma \vdash A$ is provable in $\mathfrak{ND}_\mathbf{L}$.

Proof. By induction on the depth of derivations. □

THEOREM 8. For each $\Gamma \subseteq \mathscr{F}$, $A \in \mathscr{F}$, and $\mathbf{L} \in \{\mathbf{L3A_G}, \mathbf{L3B_G}\}$, it holds that $\Gamma \vdash_\mathbf{L} A$ is provable in $\mathfrak{ND}_\mathbf{L}$ iff $\Gamma \models_\mathbf{L} A$.

Proof. Theorem follows from Theorems 5 and 7. □

5 Conclusion

To be written.

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