# Universal Logic III
# Estoril 2010
# Tutorial on Truth Values
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# Session 2, Part 1

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# **1**.
# What is meant by a “truth value”?

## Abstract

In this first part, the origin of the concept of “truth value” is reviewed.
The truth-tracking purpose of scientific discourse accounts for the restriction to bivalence; but the question-answer machinery of scientific activity makes room for further, non-bivalent answers. Some objections remain against the introduction of many-valuedness, given the nature of a truth value; they will be reviewed before turning to the second part of the session.

- Frege (1892): a theory of sense and reference

  The sense of a sentence is not its being true; otherwise, any false sentence couldn't be ever thought; hence a distinction between the sense of a sentence (proposition = *Gedanke* = thought) and its reference (truth value)

- Truth, falsity, and negation

*We are therefore driven into accepting the* truth value *of a sentence as constituting its reference. By the truth value of a sentence I understand the circumstance that it is true or false.* ***There are no further truth values****. For brevity I call the one the True, the other the False. Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the true or the false.*
(Frege (1892): 34)

*The scientific procedure usually includes a number of steps. A thought is conceived, first, which may be stated in an interrogative sentence; then, in the end of an inquiry, this thought is recognized as true. The recognition of truth is finally expressed in the form of the affirmative sentence.*
(Frege (1918): 62-3)

– p    thought that p
├ p    recognition of the truth of p

*An **interrogative** sentence contains a demand that we should either recognize the truth of the thought, or reject it as false. (…) The answer to a question is an assertion based on a judgment; this is so equally whether the answer is affirmative or negative.*
(Frege (1919): 143)

**Q**: "Is p true?"    **A**: Assertion: "Yes, p is true"
Rejection: "No, p is not true = p is false"

*We thus distinguish between: 1. the conception of the thought - thinking. 2. the recognition of the truth of a thought - judging. 3. the manifestation of this judgment - asserting.*
(Frege (1918): 62)

1. – p
2. "p is true"
3. ├ p

Frege assumes that rejecting the truth of p entails that p is false:
"p is not true" = "p is false"
And conversely (given negation):
"p is not false" = "p is true"

This restricts the scope of negation to definite informations (untrue is false)

- 2 questions about sentences and judgments (truth claims)

*Are there two different modes of judgment, the one being employed when the answer is* yes *and the other when the answer is* no*? Or is this the same judgment in both cases?*
(Frege (1919): 54)

*Does denial belong to the judgment? Or is denial a part of the thought that the judgment assumes?*
(Frege (1919): 54)

| | |
|---|---|
| $\nvdash p$ | negative judgment (1) |
| $\vdash \sim p$ | negative judgment (2) |

*Under the assumption of two ways of judging there must be:*
1. *the assertive force of affirmation.*
2. *the assertive force for denial, inextricably related with the word* false,
3. *a negative word like not in the sentences expressed without assertive force.*
*Should we adopt only one way of judging, then there must be:*
1. *the assertive force,*
2. *one negative word.*
*Such an economy is always the sign of a more penetrating analysis, thus yielding a clearer insight.*
(Frege (1919): 55)

$\vdash p$ affirmative judgment

$\nvdash p \Leftrightarrow \vdash \sim p$ negative judgment

Frege restricts the given answer to successful cases of truth-assignment:
either p is asserted, or p is denied and its negation is therefore asserted

- Beyond truth and falsity: further informations, further truth-values

| | |
|---|---|
| From $V^2$ = {F,T} to $V^3$ = {F,T,N} | Lukasiewicz's **Ł$_3$**, Kleene's **K$_3$** |
| From $V^2$ = {F,T} to $V^3$ = {F,T,B} | Priest's **LP** |
| From $V^2$ = {F,T} to $V^4$ = {F,T,N,B} | Belnap's and Dunn's **B$_4$** (**FDE**) |
| … | |
| From $V^2$ to $V^7$ | Jaina logic (**J$_7$**, see *infra*) |

- Truth values as subsets of elements: a **partition** of the universe of discourse

  Łukasiewicz's 4-valued logic **Ł$_4$**: necessarily true, true, false, necessarily false
  Avron's 5-valued logic **mCi**: necessarily true, necessarily false, contingently true, contingently false, inconsistent
  MacColl's 5-valued logic **MC$_5$**: certain, impossible, true, false, variable

$V^n$ is a partition of a basic set $V^{n-1}$, where $V^1$ is the singleton of truth {T} = ⊤

- 2 values: no less, no more?

Logic requires $n > 2$ for every non-trivial logic, to define logical consequence

Like Frege: Suszko endorses a "meta-version" of bivalence: $\mathbf{2} = \{D \cup V \setminus D\}$
Unlike Frege: the referent of a sentence is not a truth value, but a situation

*Thus, the* logical *valuations and* algebraic *valuations are functions of quite different conceptual nature. The former relate to the truth and falsity and the latter represent the reference assignments*.
(Suszko (1977): 378)

For every element $n$ of $V$, $n$ is a member of either of two subsets: **designated** values ($D$), or **non-designated** values ($V \setminus D$)
For every p, $v(p) \in D$ or $v(p) \in V \setminus D$, i.e. $v(p) \notin D$

Every element of $V$ is an algebraic value, while $D$ is a logical value: truth.
Every valuation is a bi-valuation $v^2(p) \mapsto \{D, V \setminus D\}$

Simons (1998) against Rescher's view that MacColl is a pioneer of “many-valued logic”: subsets of elements are not proper truth values (proper elements of *V*)

3 preconditions for many-valuedness (with $V^n$ s.t. $n > 2$)

A logic is properly many-valued iff:

**MV1** it contains at least one element *n* beyond truth and falsity
For some element x of *V*, $x \notin T$ and $x \notin F$

**MV2** all and only all its elements are pairwise exclusive and jointly exhaustive
For every elements $x_1,...,x_n$ of *V*, $(x_i \cap x_j) = \bot$ and $(x_1 \cup \ldots \cup x_n) = \top$

**MV3** its connectives are value-functional

- Given these 3 preconditions, subsets of elements are not proper truth-values and, accordingly, don't entail many-valuedness

**MV1** is violated by **$Ł_4$**, **mCi**, and **$MC_5$**:
Necessity (T) is included into truth (t), impossibility (F) is included into truth (t)

**MV2** is violated by **$Ł_4$**, **mCi**, and **$MC_5$**:
$(t \cap f) = \perp$ and $(t \cup f) = \top$

**MV3** is violated by **mCi** and Suszko's bivalent versions of many-valued logics

- According to Dubois (2008), proper many-valued logics (where truth-values are single elements of *V*) respect **MV3** but entail counter-intuitive inferences:

  Łukasiewicz's 4-valued modal logic **$Ł_4$** (1953)
  Belnap's 4-valued useful logic **FDE** (1977)

- non-classical truth values and modes of truth (modalities)

In **Ł**$_4$: $\Box p := v(p) = T$, and $\Box(p \vee q) \leftrightarrow (\Box p \vee \Box q)$

In **FDE**: $\not\models p \vee {\sim}p$, $\not\models {\sim}(p \wedge {\sim}p)$

- Dubois (2008): Belnap's values are not truth values but epistemic modalities

*It must be emphasized that* [{F}],[{T}]*, and* [{F,T}] *are not truth-values of propositions in* B. *They express what can be called epistemic values whereby the agent believes* p*, believes* ~p*, or is ignorant about* p *respectively. They are like* **modalities**. *Attaching the epistemic annotation* [{T}] *to* p *is like asserting* □p *using a* **necessity modality** *interpreted as* **belief** *or* **knowledge**. *Clearly, the negation of the statement* p is believed *(inferred from* B*) is not the statement* ~p is believed*, it is* p is not believed. *However, the statement* p is not believed *cannot be written in B because the syntax of classical logic does not allow for expressing ignorance in the object language. The latter requires a* **modal logic***, since in classical logic* ~p *means* □~p*, not* ~□p *(that cannot be expressed).*
(Dubois (2008): 201)

If so, then Belnap's "truth values" lead to untenable results:

- Either "told true" means "necessarily true", and B is absurd

| | |
|---|---|
| T | $\Box p$ |
| F | $\Box \sim p$ |
| B | $(\Box p \wedge \Box \sim p) = \bot$ |
| N | $\sim \Box p \wedge \sim \Box \sim p$ |

- Or "told true" means "possibly true", and N is absurd

| | |
|---|---|
| T | $\Diamond p$ |
| F | $\Diamond \sim p$ |
| B | $\Diamond p \wedge \Diamond \sim p$ |
| N | $(\sim \Diamond p \wedge \sim \Diamond \sim p) = \bot$ |

Wansing & Belnap (2009): Belnap's truth values are not epistemic modalities
they don't express **belief** states, but **information** states (data records)

*However, it is just not the case that* (...) *the truth values* T*,* F*, and* N *are interpreted as* "certainly true"*,* "certainly false"*, and* "unknown"*, respectively. The observation that* "belief is never truth-functional" (...) *is simply irrelevant for a discussion of* **FDE**. *The truth values of* $\mathbf{FOUR_2}$ *are neither modalities, nor are they interpreted in terms of belief. As the literature concerning the four values repeats from time to time, there is only one canonical and non-metaphorical account of the four values* {T,F,B,N}*: They are told values, representing what as a matter of fact the computer has been told.*
(Wansing & Belnap (2009): 5)

No wonder if $(p \vee {\sim}p)$ and ${\sim}(p \wedge {\sim}p)$ don't hold in **FDE**
Dubois' analogy between many-valuedness and modality relies upon a confusion between information states and belief states
**FDE** does not intend to be a modal many-valued logic (as $\mathbf{Ł_4}$ did)

- Truth values and information

A truth value is a value, i.e. a normative concept

not only the value (argument) of a sentential function

the reference (*Bedeutung*) of a sentence = its **significance** (with respect to its task)

The relevance of a sentence, in the framework of scientific research as a "truth-tracking" activity:

*It has already been said by E. Tugendhat that the meaning of the word* 'Bedeutung' *includes the meaning of the English word 'importance'. For this reason Tugendhat has proposed to translate* 'Bedeutung'*, in its Fregean use, as 'significance' and has argued that the* Bedeutung *of the parts of sentences is their (significant) "contribution to the truth-value of the sentences into which they may enter" (Tugendhat, p. 180), a contribution which Tugendhat calls"truth-value potential".* (Gabriel (1984): 372)

- The meaning of a sentence: its contribution to truth as an intended value
  any information that doesn't contribute to the knowledge of truth is irrelevant

  "*Truth* value" (and not "falsity value"!): truth as the intended norm of correction for any scientific judgment
  Truth and falsity are dependent from each other: falsity is the negation of truth

  "Russell's law": to deny p is to assert its negation ~p
  Compare with Frege's collapse: $\nvdash p \Leftrightarrow \vdash \sim p$

- Sentences without reference (truth value) are **irrelevant** in scientific research

  Example: the sentence "Zeus is immortal" has no reference (truth value), since one of its components ("Zeus") has no referent. (by Compositionality)
  "Neither true nor false" (N) doesn't denote a proper truth value, but a lack of truth value: no reference, no informational relevance

In a larger understanding of “significance”, a question-answer machinery purports to give informations about an initial source (the sentence)

The information can be gappy (no information) or glutty (too much information)

*Even if, **in itself**, a proposition cannot be but true or false, it may occur that a given person does not know the **answer**, at least at a given moment. Hence for this person, there is a third attitude in front of a proposition. This third attitude does not correspond to a third truth-value distinct from yes or no, but to the doubt between the **yes** and the **no** (as people, who, due to incomplete or indecipherable information, appear as of “unknown sex” in a given statistics. They do not constitute a third sex. They only form the group of people whose sex is unknown)”.*
(De Finetti (1936): 3)

**FDE** is not about truth values *per se*, but told truth values: $\mathbf{T} = \{T\}$ is not T!

the “truth-bearers” are not sentence, but statements about sentences

**Summary**:

- Truth-values are informations within a question-answer game, and Frege restricted the answers to only two significant ones: true, or false

- An extension to further informations is to be conceived, but restricted to some preconditions for many-valuedness

- This question-answer machinery can be enriched with more questions (beyond truth and falsity), or more answers (beyond **yes** and **no**)