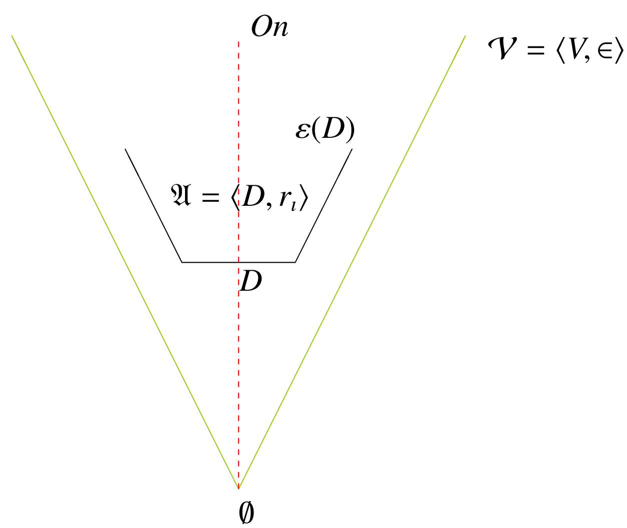
DÉCIO KRAUSE

LANGUAGES, STRUCTURES, AND MODELS OF SCIENTIFIC THEORIES



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To Mercedes.

"[Scientific concepts] are free creations of human mind and are not, however, it may seen, uniquely determined by the external world." (Einstein & Infeld (1938), p.33)

Preface

Ever since I was invited by Jean-Yves Béziau to present a tutorial on the structure of empirical sciences at the World Congress and School on Universal Logic III, I became quite enthusiastic but also worried about the way of approaching the subject. The first reason for such a worry was that I would like to present a course which would not be just a repetition of what can be found in the (already wide) literature on the subject. In fact, I would like to emphasize some points our research group in Florianópolis (the capital of the Santa Catarina State, in the South of Brazil), sponsored by the Brazilian *Conselho Nacional de Desenvolvimento Científico e Tecnológico*, CNPq, has been studying and has becoming to communicate to specialized audiences in conferences and papers. This could be a nice opportunity to give a general account of at least part of these studies, and so I tried to keep the text as closer as I could from our discussions. Our research group has been much influenced by Professor Newton da Costa, one of the outstanding today's philosophers of science. His insistence that any philosophical discussion on science must be not only speculative, but strongly grounded on science itself, should be taken in consideration. In agreement with this idea, I tried to be quite careful with the choice of the involved subjects, and I decided to advance some ideas on the importance in considering the logic and mathematical framework where the philosophical discussions (apparently) takes place. Really, in discussing philosophy of science, we usually speak of the structure of scientific theories, on the concept of truth, on the nature of the models of the theories, and so on. These are few examples of concepts that depend on the *place* where the discussion is performed, that is, they depend (in a certain sense) on the mathematical stuff which is usually only informally presupposed. Usually, a theory of sets is assumed without any discussion, and all happens as the mathematical objects we need *are there*, as the working scientist (even the mathematician) usually assume (as a kind of Planonism).[1] What are the consequences of these assumptions for the foundational discussion? We shall see some of them in this book.

Our discussions have gained in much from the participation of our colleague and friend Otávio Bueno, from the University of Miami, who is a member of our research group and who has contributed in much with the discussion on these subjects. Obviously that it is not possible to cover all the details neither

[1]Roughly speaking, as Fraenkel et al. explain, a Platonist is convinced that corresponding to each well-defined concept, say a monadic predicate or condition, there exists an object (a set or class) which comprises all and those entities that fulfill that condition, being an entity on its own right (Fraenkel et al. 1973, p.332).

in a short course like this nor in the corresponding notes that form part of this little book. Thus, this text must be considered as a first approach to the subject we intend to develop in a more extensive and detailed joint work in the near future.

The basic idea I would like to enlighten is the emphasis in some aspects of the notion of model of scientific theories, in a sense that have been not much discussed in the philosophical literature. In particular, I shall follow da Costa in arguing that the mathematical (set-theoretical) structures that are taken as models of the relevant empirical sciences are not *first-order* structures (or *order*–1 *structures*, as we shall call them), thus being not covered by standard model theory, as it is not uncommon to find philosophers make reference to.[2] Really, it is not uncommon to find philosophers referring to the models of scientific theories (such as some physical theories) as they were (order–1) structures of the kind

$$\mathfrak{A} = \langle D, R_1, \ldots, R_k \rangle, \tag{1}$$

where D is a non-empty set and the R_i ($i = 1, \ldots, k$) are n-ary relations on D, these relations having as 'relata' just elements of D, but not subsets of these elements or other 'higher' relations. If standard model theory they need to be order–1 structures, for *these* are the structures treated by standard model theory. But, as we shall see, most of the interesting structures

[2]For instance, see the definition of structure given in Hodges 1993, p.2. The recent second edition of the superb book on mathematical logic by Yuri Manin treats model theory in the same vein (Manin 2010, ch.10).

related to scientific theories are not order-1 structures of this kind. These imprecisions in the philosophical discourse seem to be raisen due to a confusion about the order of a structure and the order of the corresponding language where these structures are (in general) dealt with. Taking this fact into account, I have decided to address an introductory text dealing with some of these questions, trying to enlighten the necessary distinctions.

As far as I was writing these notes, I was realizing with no surprise that much more would be said, and that much more details would be presented with care. But I needed to end my text, and so some topics necessarily were left without the careful discussion they deserve. Thus this book should be taken as an introductory text on the theme, and I hope that the Bibliography provides additional informations for anyone interested in the subject. I also hope that these notes can be improved with suggestions and criticism, and I will welcome them.

Despite I have been much influenced by da Costa's ideas, I owe much also to Maria Luisa Dalla Chiara and Giuliano Toraldo di Francia, whose work on the foundations of physics I have followed (at least in part) ever since the 1980s, and to Patrick Suppes' approach, which has interested me so much over the years.

To be faithful to the scope of the Congress, I tried to be more 'universal' as I could. Universal Logic, as I understand it, can be seen as the general study of logic, without paying necessary attention to specific systems, which is in accordance (as it is well known) with the term ‘Universal Algebra’ used by Whitehead in his infamous book on the subject (from 1898). Thus, I tried to offer to the readers and to those who attend the tutorial an ‘universal’ discussion on structures, languages and models of scientific theories. Almost all the time I shall be following da Costa’s ideas and teachings. But of course I do not wish to compromise him with the way I present the subject here. Anyway, I would like to thank him for his kind patience for so long discussions and teachings. I also thank Jean-Yves for the invitation, so as my colleagues Otávio Bueno and Antonio M. N. Coelho, and my students Jonas Becher Arenhart Arenhart and Fernando T. F. Moraes for the interest and suggestions.

Florianópolis, February 2010
D.K.

Contents

Chapter 1

Introduction

In his book *Concepts of Force*, Max Jammer calls our attention to the fact that the concept of force is one of the "fundamental and primordial" notions of physical theories, and that "it seems [that it] has never heretofore been the object of a comprehensive analysis and critical investigation", and that "[i]n general, the concept is taken for granted and considered as sanctioned by its successful applications. In fact, he continues, standard textbooks, and even elaborated treatises, present little information, if any, of the nature of this concept; its problematic character is completely ignored in the maze of practical utilizations. " (Jammer 1962, p.5).

Then he acknowledges that in general scientists are little concerned with the history of concepts (ibid.). Of course he is right but, we should say, perhaps it not a task for the scientist to provide such an analysis, which is more concerned to the obligations of the historian of science. But the same, or at least quite similar things, can be said about the study of the underlying logic and mathematics the scientist employs in her work. And such an analysis, although may be also not required to be performed by the scientist, would be taken into account by the philosopher of science. Unfortunately, this is not what we see in most of the textbooks on the subject. Logic and mathematics are also 'taken for granted', to employ Jammer's words, in general being confused with (and 'considered as sanctioned' by) that what lay people call *classical logic* and *standard mathematics*, yet in most cases they are not able to give even a rough characterization of neither of them.

In this book we shall be concerned with precisely the logic and the mathematics that underly the most common physical theories. Although the results here outlined can be applied in general, we shall be more concerned with physics and, more specifically, with quantum theory. Just to tell you what we have in mind, let us recall that non-relativistic quantum mechanics, for instance, in its usual presentation, makes use of Hilbert spaces, expressing the states of the physical systems as vectors in an Hilbert space. Thus, one of these vectors may be written as a linear combination (or *superposition*, as the physicist uses to say) of an orthonormal basis formed by engeinvectors of a certain linear operator on such a space. But the existence of a basis for general spaces (not for a particular space, where in some cases we can just exhibit a basis) depend on the axiom of choice, hence of the underlying logic (see below about the meaning of this expression). Still concerning the axiom of choice, we recall that quantum mechanics makes use of unbounded operators, such as position and momentum, and the existence of these operators also depend on the postulates of the underlying mathematics (as we shall see at page 108). In the same vein, one of the most important theoretical results in quantum theory is Gleason's theorem (it does not matter us to formulate it here), which was stated by Gleason for separable Hilbert spaces (Chernoff 2009).[1] Robert Solovay extended Gleason's theorem also for non-separable Hilbert spaces, but at the expenses of introducing non-measurable cardinals, whose existence cannot be proven in standard set theories (such as ZF, the Zermelo--Fraenkel first-order set theory, supposed consistent). As we shall see, the consideration of the underlying mathematics (and logic) has relevance for certain kinds of foundational and philosophical discussion. Here it is another example of a more philosophical nature.

The so-called *semantic approach* to scientific theories claims, saying in short, that a theory is characterized by "identifying a class of structures as its models" (van Fraassen 1980, p.44). Without contextualization, this says practically nothing, so we need to go further. The notion of model, for most of the defenders of such a view, "derives from logic and metamathe-

[1] An Hilbert space is separable if t has a denumerable basis. We shall be back to this example later (see page 109).

matics", referring to "specific structures" (ibid.), and "[a]ny structure which satisfies the axioms of a theory (...) is called a *model* of that theory" (op.cit., p.43). That means that models are considered as such in the sense of logic, namely, as set-theoretical structures that satisfy the postulates of the theory. In other words, models are *sets*. But, although these philosophers aim at to concentrate themselves in the models, and not in the syntactical aspects of the theory, the axiomatization cannot be put completely aside. Models (in that 'logical' sense) are models of something, and this something can be seen as encapsulated in a set-theoretical predicate (Suppes 1967, 2002), later termed 'Suppes predicate' (da Costa & Chuaqui 1988), as we shall see. Hence, all the standard discussion is grounded on a concept of set, and of course it is not completely clear what philosophers mean by a set, as we shall see below.

Important to say that set theory is not the only possible mathematical framework we may use. Rudolf Carnap used higher-order logic in several examples he gave in his book from 1958 (Carnap 1958). Category theory could also be used instead. But set theory is the most common mathematical theory considered in these discussions, so we shall be restricted in considering them here. Models, them, as set-theoretical structures, depend on the notion of set, and we shall discuss also about some of the various meanings of the word *model* that appear in the foundational contexts.

Our strategy goes as follows. We begin with a brief presentation of the different meanings of the concept of model of a scientific theory, given in the chapter 2. The analysis is of course not exhaustive, but it provides the basic tools for the general discussion. Next, we recall some aspects of axiomatization and related issues. The distinction (generally not acknowledged in the philosophical literature) between order–1 structures and order–n structures ($n > 1$) is introduced, preparing the field for a more detailed discussion done at chapter 5. Chapters 3 and 4 continues discussing the different notions of the word 'model', and some simple cases are taken as illustrations. After having discussed the nature os scientific structures and their languages in chapter 5, all done within a 'classical' framework (such as our chosen Zermelo-Fraenkel (ZF) first-order set theory), we suggest a way of departing from this mathematical basis (and logic), by presenting a case study, motivated by quantum physics. Historical details are given, and a different 'set' theory is presented in details. Then, we analyse what would be understood by *working out* of 'classical' frameworks.

But a care must be taken just from the beginnings. The concepts of classical logic and of standard mathematics are also not precise. Surely most of the authors will accept that that what we call 'classical propositional logic' is part of classical logic, so it is the so-called 'classical first order predicate calculus'. But, what about 'classical' higher-order logic (type theory)? Should we consider it also as forming part of (classical) logic? It depends. As it is well known, Quine is one who rejects higher-order logic as being *logic*, claiming that it is mathematics. According to him, (classical) logic is first-order logic without identity and with a finite number of primitive predicates (then he can 'simulate' identity and so he doesn't need it as a primitive concept). But of course all of this is quite fuzzy. A right characterization cannot be done, but we can roughly characterize the field as follows (according to da Costa 1990). By classical logic we understand the so-called first-order predicate calculus with or without identity (see for instance Mendelson 1997), or some of its subsystems, such as the classical propositional calculus, or even some of its extensions, such as the classical higher-order logic (type theory) or some 'classical' systems of set theory, such as the Zermelo-Fraenkel set theory (ZF), von Neumann-Bernays-Gödel set theory (NBG), or even the system of Kelley-Morse (KM) among others. We think that standard category theory can be added to this schema. This is not a definition, for it is quite redundant, but at least guide our intuitions. In this vein, by *classical mathematics* we can understand that part of mathematics that can be erected within such a frameworks, for instance all mathematics encompassed in the books by Bourbaki, or standard category theory (Mac Lane 1998).

This book discusses how these basic frameworks are used in constructing scientific theories, and as we said above, we shall be more concerned with physics. In the final part of the text, a departure of this ‘classical’ schema is suggested.

In a quite general setting, what we sustain is a pluralistic approach to science and its philosophy. Up today we have a wide field of alternative logics, and distinct mathematics. The choice of a particular setting cannot be done out of pragmatic criteria, such as simplicity, expressive power, or even beauty. This of course brings to evidence a nice topic for the philosophical discussion, say in the sense of avoiding naïve relativism. The important fact, as we intend to submit to your the reader, is that today the philosopher of science cannot even more be completely distant from the foundational issues. Even without being a specialist in the field (say, in logic and set theory), she would be aware of their basic concepts and tools, for instance to avoid the claiming that the set-theoretical structures that are (mathematica) models of scientific theories are what we shall term *order*-1 structures. But let us begin our discussion.

Chapter 2

Models of scientific theories

The word *model* has several meanings and uses not only in general, but even in science. We can present a Lego 'model' of the Eiffel Tower for instance just for fun. An Earth globe can be used by kids for studying geography. Watson and Crick constructed a supposed 'model' of the DNA molecule with more sophisticated scientific interests, which contributed to give them the insights that lead them to win the Nobel Prize. Billiard balls are sometimes used to roughly exemplify the behaviour of molecule gases in classical mechanics. From another perspective, engineerings and other applied scientists generally make use of mathematical 'models' of certain situations they 'model' generally using mathematical devices, such as those which represent certain populations, say in biology, the heat transfer phenomena or the movement of a fluid. In this sense, a *mathematical model* is supposed to act as a simplified-mathematically treated portion of reality, acting more or less like a map of a city, which of course should be not confused with the city itself. There are also ‘models’ in logic. In this case, they usually are taken to be set-theoretical structures that satisfy the postulates of a theory; as put by Tarski, “[a] possible realization in which all valid sentences of a theory T are satisfied is called a *model* of T” (Tarski 1953, p.11). To some people, like P. Suppes, the notion of model in the sense (the ‘logic sense’) is the most fundamental one, and should prevail in the philosophical discussions on the foundations of science (Suppes 2002, pp.20-21).

For instance, take group theory. It studies some mathematical structures generally built in a set theory such as the Zermelo-Fraenkel set theory (ZF)[1] of the kind $\mathcal{G} = \langle G, * \rangle$, where G is a non-empty set and $*$ is a binary operation on G obeying well known postulates. As an example, let us take the additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$, where $\mathbb{Z}$ is the set of the integers and $+$ is the usual addition on such a set. But we could study the collection of all groups, which is not a set of ZF. From a certain perspective, it can be called a *category*. To study it, we use *category theory* (for a general view, see Marquis 2007). Here we will be interested in models that are set-theoretical structures, but we shall mention an important distinction between models of theories such as group theory and ‘models’ of theories such as ZF proper,

[1] If you don’t have familiarity with this theory, it would be better to read chapter 5 first, or at least its first part.

which cannot be constructed within ZF proper (supposed consistent). These considerations will bring us interesting philosophical questions regarding the mathematical stuff (say, ZF) we are using to built the structures that are the 'models' of scientific theories. The reasons for that will appear in due time. From now on, we shall use the word *model* without quotation marks for this 'logic sense'. Below the concept will become clear in some of its different aspects.

2.1 Axiomatization

Models, as set theoretical structures, are models of something, namely, of the postulates of a certain theory. There are no models *per se*. To speak of the postulates of a theory requires to speak of its axiomatization. As it is well known, the axiomatic method is an heritage from the ancient Greeks, and has influenced in much the Western culture, mainly mathematics. It is, in a sense, the method *par excellence* of presentation of a mathematical theory. The axiomatic method was used also outside mathematics, for instance by Spinoza, who tried to write his ethics *more geometrico*, that is, as the geometers do, namely, axiomatically.

Yet it had been used before, for instance by Newton,[2] the axiomatization of physical theories was proposed as the 6th

[2]According to Suppes, some people regard Archimedes' *On the Equilibrium of Planes* as the first book on mathematical physics, and its first chapter would be "an axiomatic analysis of measurement" (Suppes 1988, p.40).

of his celebrate list of 23 Mathematical Problems by David Hilbert during the Second International Congress of Mathematicians, held in Paris in 1900 (see Gray 2000), and this program was later extended to other disciplines as well, such as biology, by Woodger and others. During the XXth century, it became acknowledged by most specialists that there are two basic ways (essentially equivalent) to axiomatize a theory, namely, (1) to make explicit its underlying mathematical structure, in the sense of Bourbaki, that is, a *species of structures*, which characterizes structures of a certain species (say, groups) (Bourbaki 1968, chap.4), and (2) to construct a set-theoretical predicate that formalizes it (also called its *Suppes predicate*). Roughly speaking, the structures that satisfy the Suppes predicate of a certain theory are the theory's models. Both mathematical structures and set-teoretical predicates, for most of mathematical and scientific theories, are set-theoretical constructs (da Costa & Doria 2008, p. 70), as we have seen in the case of groups. Since we know the postulates of Zermelo-Fraenkel set theory, of vector spaces, Euclidean geometry and so on, we could think that we can apply the same procedure to all these theories. But it is necessary some care here, as advanced earlier, for there are important differences concerning models of a geometry and 'models' of a set theory such as ZF, to be explicated later. From the formal point of view, the only requirement a postulate has to obey is to be a formula of the theory's language (following the standard practice, we do not distinguish between axioms and postulates).[3] It is also well known that Hilbert's position against, for instance, Frege's, who said that the sentences we use in mathematics express nonlinguistic 'thoughts', so that there would be no sense in reinterpreting thoughts. Thus, to Frege, Hilbert's claim that there may be different interpretations that render the axioms true could be never accepted (see Blanchette 2009). But, in what regard physical theories, can we consider the postulates, say of non-relativistic quantum mechanics (in some formulation) as formulas of a certain language and no more? (From now on, when we say 'quantum mechanics' –QM– we mean 'non-relativistic quantum mechanics' plus some interpretation, which will be mentioned if necessary).

This brings us an interesting question. Take for instance one of the standard formulations of QM (Toraldo di Francia 1981, pp.270ff—see section 4.1.2). Provided that the states of a physical system are done by vectors in a Hilbert space, if A is an Hermitian operator representing an observable and $\{|u_i\rangle\}$ is an orthonormal basis constituted of eingenvectors of A, so that $A|u_i\rangle = a_i|u_i\rangle$, let $|\psi\rangle = \sum_i c_i|u_i\rangle$. We know from one of the postulates (see section 4.1.2) that the spectrum of A (the set of its eingenvalues) contains the possible results of a measurement of A. One of the postulates (the Projection Postulate) states that "[i]f we carry out the measurement [of A] and it

[3]But sometimes in logic we call 'postulates' the collection of axioms (or axiom schemata) and the inferences rules of a formal system.

gives the result a_i, the state vector $|\psi\rangle$ [as above] becomes $|u_i\rangle$ immediately after the measurement." (ibid., p.271).

How can we regard this postulate as a sentence of some formal language, say a first-order language? It would quite difficult, if not impossible, to state it to fulfill the above requirements. But we can accept that we can do it at least in principle. As da Costa says, "the standard postulates of quantum mechanics (so as those of quantum field theory) may be taken as rules for we to associate mathematical formalisms to certain physical systems. They can, with due qualifications, become axioms of an standard axiomatics (then we need to take, for instance, *physical system*, *states of physical system* and *observable* as primitive concepts). But they can also be taken as general principles that govern the 'models' of certain physical systems." (personal communication). Van Fraassen expressed himself the same way, when in his *The Scientific Image* he states that "the 'axioms of quantum theory'(...) don't look very much like what a logician expect axioms to look like; on the contrary, they form, in my opinion, a fairly straightforward description of a family of models, plus an indication of what are ro be taken as the empirical substructures of these models" (van Fraassen 1980, p.65).

Thus, although there is, as we see, a deep question regarding the very nature of the application of the axiomatic method to science, we shall assume that the words *postulates* and *axioms* of a physical theory are to be understood in the usual mathematical sense. But we may go deeper, by presenting not only the postulates of the theory themselves, but also the underlying logic, getting the *formalized* version of the theory. It would be relatively easy to formalize group theory this way, but non-relativistic quantum mechanics would demand a lot of work, and we really don't know if it would be advantageous. Really, it would be need to formalize all the theory of Hilbert spaces, differential equations, probability theory and so on. Although we can regard this task as possible, it seems more natural to suppose a set theory to begin with so that all these 'step theories' can in principle be developed. This strategy (essentially, Suppes' one) leads us directly with what if of interest, namely, the theory itself.

2.2 Models as kinds of structures

Some philosophers of science linked to the so-called *semantic approach* claim that to present a theory is to characterize a class of structures, the 'models' of the theory. The origins of this idea goes to the desire of going out of logical positivism, which describes a theory as a kind of calculus, usually said to be grounded on first-order logic, whose interpretations were given by some correspondence rules (see the introductory paper by Suppe in Suppe 1977).[4] The semantic approach, on

[4] Really, there seems to be not a consensus about the precise meaning of a theory according to logical positivism, for different philosophers arrived to different views. Furthermore, as Suppe shows, in some characterizations even modal logic was enabled.

the contrary, at least so it is claimed, focuses on 'models'.[5] Thus, we find van Fraassen, saying that "[t]o present a theory is to specify a family of structures, its *models*" (van Fraassen 1980, p.64). And he says that he agrees with Suppes in seeing how a theory is to be identified, namely, as a class of models (cf. ibid., p.66). Below we shall see that declarations such as this one need qualification, for we need to consider these models (as set theoretical structures) as given in some set theory, and them several interesting points shall appear. By the way, the approaches given by van Fraassen and by Suppes do not exactly fit one another. Suppes's suggestion is to present a set-theoretical predicate, a formula of the language of set theory (extended by some specific symbols, as we shall see later) which provides a description of the structures that satisfy the predicate (the models of the theory)—that it, Suppes is presenting a *species of structures* in the sense of Bourbaki, while van Fraassen focuses on the models themselves, in a certain sense trying to minimize the role of the theory's postulates.

But it should be insisted that there are not models which are not models of something. Thus, we need something to link the models as belonging to some 'class of models', and the

[5] Van Fraassen still distinguishes between set-theoretical models and what he calls 'model-types', which would stand for that 'models' the scientist makes use, say Hydrogen atoms for Bohr's theory of the atom (van Fraassen 1980, p.44). From the mathematical point of view, it is quite difficult to understand in what sense an Hidrogen atom may be a *model* of a scientific theory, for the semantic rules apply only to mathematical devices. Of course scientists take these concepts informally, but the philosophers would provide a careful analysis of these relationships (which of course is not van Fraassen's aim), for instance in creating a mathematical model of the Hidrogen atom first (the (MM) of the next chapter), and then to apply the semantic rules—if available.

'natural' option is to search for some postulates, perhaps in the form of a set-theoretical predicate such that the models are those structures that satisfy the predicate (which comprises the axioms/postulates). Thus, although we may be focused on the models, the axiomatization cannot be put completely aside, for it is the axioms that determine the class of structures that are the models of the theories being considered, and this forces us to consider languages. As we shall see below, to be aware of this intuitive fact is really quite relevant for the philosophical discourse about scientific theories. Anyway, even without a precise characterization of what is to be understood by a model, the *model-theoretical* or semantic approach to scientific theories became the paradigm of present day philosophy of science (van Fraassen ibid.; Suppe 1977; Suppes 2002).

We shall consider here those models of scientific theories that are set-theoretical structures in the Tarskian sense, and in this sense we agree with van Fraassen (op.cit., p.44). Thus, the DNA double helix is not a *model*, being regarded as an heuristic device to provide the grounds for the construction of (in principle) mathematized theories and 'truly' models. The axiomatic method was used first to provide a kind of polishing of a certain defined area. Later, mainly due to people such as Hilbert, it became a fundamental method of exploration. Even starting with a well known field of knowledge, once we obtain an axiomatic mathematical theory, it becomes abstract (we shall discuss this point below) and (being consistent) it may have models in the Tarskian sense, and some of them, of course, may fit — with reservations — the intuitive heuristic device which has originated the axiomatic system. For most theories, either in mathematics or in the empirical sciences, these models are set-theoretical structures but, as we shall see, this way of talking need to be qualified, for instance when the considered theory is set theory itself, for the 'models' of set theories in general cannot be constructed within themselves. The 'theory of models' began with Tarski in the 1920s, who arrived at the notion of model within the scope of ordinary mathematics (in the 'logical' sense posed above)—see Keisler 1977; Shoenfield 1967, ch.5. The theory of models studies certain types of mathematical structures (we shall term *order*–1 structures) and how we can operate with them and construct them. In model theory, we have some specific theorems dealing with certain (set-theoretical) structures, but let us insist that all of them refer to what we are calling *order*–1 structures. But in science we need to deal with *higher-order structures*, as we shall see, thus model theory in its usual sense must be taken carefously in these discussions (for instance, for higher-order structures built in higher-order languages, some basic theorems of model theory do not hold, such as compacity, completeness, and the Löwenheim-Skolem theorems, with important consequences). The advantages, as we shall see later, is that even higher-order structures can be built using first-order languages, although we make use of set theory.

The mathematical structures we usually use can be defined basically in three distinct ways (although there may be others—you can see for instance the suggestion by da Costa & French 2003, pp.26-27). Thus:

(i) We can make use of the resources of a set theory, as Bourbaki did (Bourbaki 1968).

(ii) Alternatively, we can make use of higher-order logic, as Carnap did (Carnap 1958).

(iii) Or then we can make use of category theory (Mac Lane 1998).

Notwithstanding, Tarski's approach and all subsequent model theory deal with *order*–1 *structures* only (we will not use the expression *first-order structures* for the reasons to be explained below; shortly, we introduce this terminology for not making any confusion with the order of a language). These are set-theoretical structures, n-tuples composed by a non-empty set D (the domain of the structure), plus a collection (possibly null) of distinguished elements of D, a collection of relations on elements of D and a collection (possibly empty) of operations on elements of D, that is, something which can be written as

$$\mathfrak{A} = \langle D, \{a_i\}_{i\in I}, \{R_j\}_{j\in J}, \{f_k\}_{k\in K}\rangle \tag{2.1}$$

where I, J, K are sets of indices. Sometimes we express ourselves differently, just by considering the distinguished elements, relations and operations instead of sets of them, as for instance in considering

$$\mathfrak{N} = \langle \omega, 0, s, +, \cdot \rangle \tag{2.2}$$

for first-order arithmetics, and not $\mathfrak{N} = \langle \omega, \{0\}, \{s, +, \cdot\} \rangle$.

As it is well known, we can consider just relations, for individual constants can be seen as 0-ary relations and n-ary operations (functions from D^n to D) can be seen as $n + 1$-ary relations. Thus our *Tarskian structures* become something like

$$\mathfrak{A} = \langle D, r_\iota \rangle, \tag{2.3}$$

where r_ι stands for a collection of relations on D, that is, subsets of D^n. Hence, the *relata* are individuals of D. Sometimes it is convenient to use operation symbols instead of relations. Thus we arrive at a structure of the form

$$\mathfrak{A} = \langle D, f_\iota \rangle, \tag{2.4}$$

where f_ι stands for a collection of n-ary functions from D^n in D. A typical case is a group, as we have seen before. A structure such as (6.3) is called an *algebra*, and we shall consider them in due time (see section 6.3).

Many philosophers of science report to model theory when speaking of the ‘models’ of scientific theories, which subsumes that these would be *order*–1 *structures*, although it is not always clear what they mean by a ‘model’. For instance, in Przelecki 1969 we find the author saying that her monograph intents "to apply some concepts, theorems, and methods of model theory (rather simple and elementary ones) to the semantical problems of empirical theories." (p.4). We of course are not claiming that this cannot be done, but it is important to acknowledge that the 'models' of scientific theories are in general not those dealt with by standard model theory: they are not order–1 structures. And we don't have yet an articulated 'theory of models' for order–n structures ($n > 1$), although some works have been advanced in this direction.[6]

The study of the mathematical structures that may be models (in the logic sense) of the scientific theories can be seen from several points of view. As recalled by da Costa & French (2003, p.22), for Beth and van Fraassen, "theory structures are captured in terms of state spaces, for Suppe they are understood as relational systems, and for Suppes and Sneed [one of the proponents of the *structuralistic* approach to science], they are regarded in terms of set-theoretical predicates" (see also van Fraassen op.cit., p.67). We shall not enter this discussion here, but simply adopt the third version (as these authors do), for it seems to be more general.

Important to realize that when we consider set-theoretical structures, we of course need to use a set theory. Without loss of generality, we shall consider here the first-order Zermelo-Fraenkel set theory (with the axiom of choice, ZFC); ZF stands

[6] This theory is called 'Generalized Galois Theory' by Newton da Costa—see da Costa & Rodrigues 2007.

for ZFC without the axiom of choice and ZFU for the corresponding theory with ur-elements. The choice of a set theory as a *meta*-framework may be important depending on the kind of analysis we intend to do. Really, the set-theoretical techniques developed by Gödel, Cohen and others enable us to build 'models' of set theory where some basic principles, such as the axiom of choice, do not hold, and this brings is important philosophical consequences we shall see below.

Chapter 3

Informal account to models

Even if we restrict our analysis to set-theoretical structures, we may consider different 'stages' in which these structures can be considered. We shall discuss this point in the sequence. To begin with, perhaps it would be interesting to follow Hilbert and Bernays in distinguishing between *formal axiomatics* and *concrete axiomatics* (see Kneebone 1963, p.201).

3.1 Formal vs. concrete axiomatics

From the historical point of view, the paradigm of the axiomatic method, as is well known, is Euclid's *Elements*, although today we are aware that he made deductions also from not assumed hypotheses.[1] The concept of axiomatization, as we accept it today, was put forward by the XIXth century, in

[1]For instance, the first proposition (theorem) of the Elements asks for the construction of an equilateral triangle on a straight line. In the proof, Euclid assumed that two circles intercept on a certain point, but this fact cannot be derived from the postulates.

great part due to Hilbert. Until those days, the axiomatic systems were 'material', always encompassing an intended interpretation. That is, the axiomatic systems involved a *material* content, for instance when we say that we have axiomatized evolution theory: the axiomatic system has an intended (generally intuitive) 'model' from its beginnings, and to axiomatize was taken sometimes as something deprived of real importance, serving more as a way to 'clean up' a certain already well know subject. The XXth century shown us that this is not so. Below we shall have a look at Suppes' theory of human paternity.

It was later realized that an axiomatic system, such as arithmetics, which has as the intended model what we call *the standard model*,[2] becomes a *formal* theory, with its symbols deprived of any interpretation. This step was of fundamental importance for the development of logic during the XXth century. Being so formal, a certain axiomatic theory may comport other interpretations, other *models* as well, sometimes not equivalent (not isomorphic) to the intended one, and being purely abstract mathematical models (structures that satisfy the postulates, without any appeal to 'reality'). First-order arithmetics provides a typical example, having also non-standard models as well, not isomorphic to the standard model.[3]

[2]Summing up, the domain is the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$ (the finite ordinals), endowed with an unary function of 'taken the sucessor', plus the standard concepts of addition and multiplication; see Enderton 1977.

[3]Informally speaking, in a non-standard model, there may exist 'natural numbers' which are none of the elements of the set ω mentioned previously.

In other words, formal axiomatics provides axiomatic systems which are independent of any content, or "factual knowledge" (Kneebone loc.cit.).

3.2 From 'the reality' to abstract models

If we consider the distinction between concrete and formal axiomatics, we should agree with Kneebone, when he says that

> "[i]in applying the *concrete* axiomatic method[4] we take some body of empirical knowledge with which we are already tolerably familiar, and we try to make this knowledge systematic by idealizing the concepts which it involves and picking out from among the known facts a small number of basic principles from which all else can be derived by deduction." (op.cit., p.201)

This 'empirical knowledge' may be understood as a previous theory or, as we shall prefer to call it here, *informal theory*. Then we get an axiomatized (or formalized, depending on the level of axiomatization we consider) version of this informal theory, which we shall term *theory* for short. This theory (there may be several—infinitely many—theories of a same informal theory), after its primitive concepts are deprived of

[4] Following Hilbert, we have used already the expression *material axiomatics* in these cases.

any content, becomes an abstract entity, and may be interpreted in different ways, giving rise to the distinct models of the theory. Let us put the things under a certain more general perspective.

We begin by accepting, as suggested by Bernard d'Espagnat, that there is an independent objective 'reality' (R). But, as he says, it remains *veiled* to us (d'Espagnat 2003, 2006).[5] The most we can experience is an *empirical reality* (ER) (or *phenomenalist* reality) given to us by our senses, experiments, or learning knowledge. Cultural aspects (sometimes even guided by ideological ideas, as Lysenko's 'Stalinistic-genetics' exemplifies) seem to me to be quite important for our way of forming a picture of some portion of (R) we are interested in.

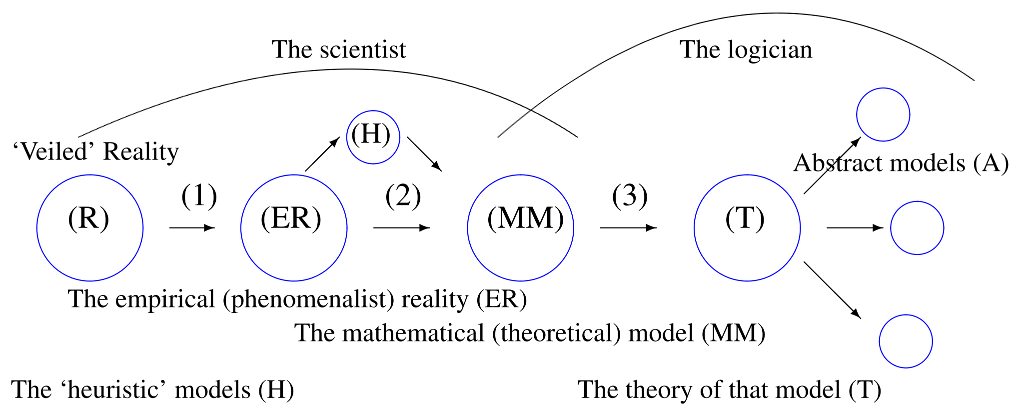

Figure 3.1: A general simplified schema for physical theories showing different senses of the word 'model'. I am not claiming that this figure traces a realistic picture of the scientific activity, being just a general idealized schema that can help us in approaching the different levels where models may appear in the scientific practice.

[5] But what I will say does not intent to be in total agreement with his ideas.

By (MM) I mean an informal mathematical structure (let's call it *model*$_1$) we construe based on technical expertise, background knowledge, insights, etc., generally elaborated taking (ER) into account, and by simplifying (or perhaps by 'putting some discipline' over if). Sometimes we make use of *heuristic models*, like the DNA double helix model$_1$ of Watson and Crick mentioned above to discipline the empirical data and organize our mind (the (H) in the figure). But scientists do not stop with these iconic models, for they need to get a *theory*, and in most advanced sciences, they need to consider mathematics. (I am not suggesting that those disciplines that do not make explicit use of maths, such as psychology, are not 'advanced'. Or aim is just to point out that the most *rigorous* ones are those that are closer to maths, such as physics.)

But, in general, at the level of (MM), we use the resources at our disposal (working within a kind of *normal science*, to make an analogy with Thomas Kuhn's ideas), and only rarely need to develop new resources. Anyway, sometimes this happens, as Newton's case shows paradigmatically (he developed Differential and Integral Calculus to deal with physical problems — yet in his time this distinction between math and physics did not exist).

Logic is used here only informally. Typical examples of (MM)s in this sense are all the non-axiomatized scientific theories. I think that it is not a mistake to say that most scientists work until this level; their models$_1$ are either iconic models$_1$

or informal mathematical models$_1$, as those used by engineers, applied mathematicians, economists, and so on. Scientists in general aim at to deal with a portion of (R)—or at least with (ER)—and do that by means of informal models$_1$ or structures ('informal' means neither axiomatized nor formalized).

Important to remark that there can be many (MM)s, depending on the scientist's expertise. Thus, the same phenomenalist reality may give rise to different and even incompatible mathematical models$_1$, depending on how the empirical data are interpreted. It is also possible that a further analysis shows that some informal theory (MM) is inconsistent, such as the well known case of the Bohr's theory of the atom (da Costa & French 2003, ch.5). In this case, the next step of getting the axiomatic or formalized version of (MM) needs to be taken quite carefully, for perhaps the consideration of a non-classical logic may be indicated. But it should be remarked that the (MM)s, once they are only informally stated, can be seen as inconsistent only when the contradictory theses appear quite explicitly, for in general no detailed analysis of their mathematical counterparts is advanced.

But logicians and philosophers of science want more. It is interesting to arrive at an axiomatized or even to a formalized version of the informal theory (MM). Let us call (T) an axiomatic or formal *theory* associated to (MM). For instance, a version of Darwinan theory of evolution was axiomatized by Mary Williams in the late sixties (see Magalhães

& Krause 2001); McKinsey, Sugar and Suppes gave an axiomatic version of classical particle mechanics, as we shall see soon; Walter Noll axiomatized continuum mechanics (Ignatieff 1996); Clifford Truesdell axiomatized thermodynamics; George Mackey presented an axiomatic version of non-relativistic quantum mechanics (Mackey 1963); Archimedes, Newton, and many others made use of the axiomatic method out of mathematics even before the XXth century. In speaking this way, it is useful to distinguish between an axiomatization of a certain informal theory, which means to make explicit some primitive concepts and postulates (equivalently, a set theoretical—or Suppes'—predicate), and a formalization of the same theory, which demands to make explicit of all the underlying logic apparatus.

Thus I am using the word *theory* to mean an *axiomatized* or even *formalized* version of (MM). But in model theory we usually say that a theory of a structure $\mathfrak{E}$, Th($\mathfrak{E}$), is the set of all sentences (of a language adequate for that structure—see below) true in $\mathfrak{E}$ (Keisler 1977, p.50). Here, the terminology is a little bit different, meaning—just to emphasize—the axiomatic/formal version of the informal theory (MM). Of course also here we may have different and even non equivalent theories (in this sense) of the same model$_1$, (MM), but let us fix one, and call it T. It is precisely here that set-theoretical predicates enter the scene, it we adopt (as we are suggesting by now) to do. As Suppes said, "to axiomatize a theory is to define a set-theoretical predicate" (Suppes 2002, p.30).

Thus, (T) is an axiomatic or even formal version of (MM). So, (T) is an abstract mathematical structure (mainly in its formal version) having no interpretation attributed to its primitive concepts. It is precisely to emphasize this that Hilbert told us that "[o]ne should always be able to say, instead of 'points, lines, and planes', 'tables, chairs, and beer mugs'." (Gray 2000, p.49). During the XXth century, we have learnt that an abstract formal or axiomatic theory may have several *abstract models* (A)—let us speak this way of those models which are set-theoretical structures that satisfy the postulates of the theory, which we shall term *models*$_2$. Most of these models$_2$ are just abstract mathematical structures, without (necessarily) any commitment to applications or even with some link with the realm that motivated the elaboration of the theory (think for instance of the non-standard models$_2$ of first-order arithmetics, although today they are studied also for applications).

The process of course does not end with the abstract models$_2$. After we have arrived to such stage, we may have learnt more about the domain of (R) we are interested in, so that a new look at (R) can be done, and the process starts again. This is a dynamical process, without end. Really, further investigations may suggest radically different approaches, which induce different (MM)s and (T)s.

From now on, we shall not make any further reference to the distinction between models$_1$ and models$_2$, leaving the differences implicit by the context. For sure the reader will recognize each case. The distinction will be made only when some emphasis is necessary.

Chapter 4

THE NATURE OF THE MODELS

WE MAY SAY that there are two basic ways to consider set-theoretical structures and models. The first is one raises from what Hilbert and Bernays termed 'formal' axiomatics, as we have seen before, and is more or less quite similar to what we usually teach in initial logic classes and resembles: we consider a language, say a first-order one, and then give it an interpretation, which means of provide a non-empty domain together with a mapping that interpret the chosen non-logical symbols. It may happens that this structure is a model for the axioms formulated in the considered language. This is achieved when the sentences that express the postulates of the theory are true (in the Tarskian sense) in the structure. The another alternative follows a roundabout way, and was termed by those authors 'concrete' (or 'material') axiomatic. We start with a structure which we suppose 'models' (in the sense of the models_1 of the previous chapter) a certain framework (say of empirical sciences), in which we put a set representing the elements of the domain, sets of these objects and so on, so as the relevant primitive relations we suppose to hold for such entities. That is, we start with an informal theory (MM). Then we look for *the language of that structure* (in the sense to be presented in the next chapter), in which we talk of it and formulate the axioms that give the operative characteristics of the used symbols, so that the structure itself turns to be a model of the axioms we use. The two approaches seen to be equivalent.

This second way seems to be closer to what scientist really do. In order to schematize or *to structure* a certain field of knowledge (the empirical sciences are our main goal here), we may consider certain individuals (ants in biology, say), which can be thought of as collected in a set, then we consider subsets (or properties) of them, relations and operations, etc. (such as those expressing their role in the hierarchy of the ant colony). Of course the basic domain may not comprise just one set, but several sets (as mentioned before, in the case of vector spaces), or even sets of sets of elements, as the case of topological spaces exemplifies. Classical particle mechanics, for instance, was obtained as a generalization of several distinct sample cases involving a set of massive bodies, an interval of time and some forces acting on them (as we shall see soon). All of this can be modeled as a structure, where we can collect the domain(s) and relevant relations holding on its elements. But in most of the applications in empirical science, we should use ZFU; strictly speaking, a collection of ants would not be considered as a set in the usual sense of the 'pure' set theories (without ur-elements). Furthermore, recall that in the empirical sciences, these structures are in general not order–1 structures. The elements we consider in the structure, out of the domain(s) are the *primitive elements* of the structure, and we shall suppose here the the structures comprise relations only (beyond the domain).

This schema immediately induces us to consider a language able to speak of this structure and, in general, of the whole scale based on its domain, obtained from the domain by using set-theoretical operations (as we shall see in the next chapter). In this language we introduce the basic rules and laws we suppose the elements and relations do obey, getting, first, an informal theory (MM). Later, when we feel that it is all right, we can present some postulates we suppose the chosen relations would satisfy, so providing a theory (T) of that field. This schema is not universal, for sometimes we go 'directly' to the postulates (or to the 'equations', as scientists generally prefer to call them), as Maxwell's electromagnetic theory exemplifies. Generally we do all of this without paying attention to the language, using it informally, but the postulates are formulated in a certain language, the language of the structure (which in general are not first-order languages), and of course it is important to realize its existence. In the next chapter we shall see the right definitions; here we shall proceed informally.

Let us give a simple example. Consider Suppes' theory of human paternity. We shall sketch a formal (first-order) version of this theory by supposing its language composed by the usual sentential connectives, quantifiers, punctuation symbols, individual variables and the equality symbol. The specific symbols are two unary predicate symbols A and M and a binary predicate P. The formulas are defined as usual, and $A(x)$ is read 'x is alive', $M(x)$ says that 'x is a male human being', and Pxy reads 'x is father of y'. An interpretation for this language may be a 4-tuple $\mathfrak{P} = \langle D, X, Y, R \rangle$, where D is a non-empty set (the set of human beings), $X \subseteq D$, $Y \subseteq D$ and R is a binary relation on D that stands for those order pairs such that the first element if father of the second one. X and Y interpret the predicates V and M respectively.

An important remark is the following. What does it mean to construct a language such as that of the above theory within ZFC?. The individual variables of $\mathcal{L}_\in$ (the language of ZFC) are thought as representing sets an the intended interpretation (the intuitive 'universe' of sets), but from a formal point of view, and ZFC is a formal theory, there is no interpretation associated to its language. Thus, all the primitive symbols of the above language are terms of $\mathcal{L}_\in$; if we associate to it the intended interpretation, these symbols are to be seen as names of sets. So, a symbol such as '(', the left parenthesis, is a name of a set, and so on. It would be quite difficult to proceed in explaining how certain sequences of symbols form the formulas, but we beg the reader to believe that this is possible.

As for the formalization, we can introduce, based on Suppes (1988, cap.8), the following postulates:

(L) A complete set of postulates for first-order logic with identity.

(P1) $\forall x \forall y (Pxy \to \neg Pyx)$

(P2) $\forall x (Ax \to \exists ! y Pyx)$

(P3) $\forall x (Ax \to \exists ! y (\neg My \wedge Pyx))$

It is easy to derive some theorems, such as $\neg Pxx$ (no human being is father of itself), among others. By definition, we can write $W(y)$ for $Pyx \wedge \neg My$ and call such an y the *mother* of x. It is clear that the intend interpretation provides an intuitive semantics for the system in an obvious sense. We can *see* that the above axioms are *true* with respect to this interpretation.

Abstract modes can also be constructed in the metatheory (of course we are supposing ZF here). For instance, take $D = \{\langle a, b\rangle, \langle b, c\rangle\}$, $X = \{b\}$ and $Y = \{a\}$. Then we can study the metamathematical properties of this system, for instance to look for a completeness theorem or to provide a study of these models, such as if there is a representation theorem. Informally speaking, a representation theorem will show that among the models of the theory there is a small class such that every model has in this class an isomorphic model. For instance, in group theory, every group is isomorphic to a group of permutations (Cayley's theorem), or that every Boolean algebra is isomorphic to a field of sets (Stone's theorem). In the same vein, a representation theorem for models of a scientific theory would help us in understanding them by acquaintance to this 'small class'. See Suppes (op.cit.) for more details.

But, as we have remarked above, there is no a 'model theory' for higher-order structures. Thus we need to be very careful.

4.1 Suppes predicates

Now let us show the another side of the coin, by approaching human paternity theory from a different alternative (really, you will not see much difference, but they exist). This is Suppes' approach of presenting a set theoretical predicate (later called 'Suppes' predicate' by da Costa and Chuaqui 1988). All the initial machinery involving logic, set theory and other possible step theories of which our main theory depends, are supposed to be subsumed into set theory. Thus, if a theory depends on Riemannian geometry, tensor calculus, differential manifolds, real numbers, differential equations and so on, as in the case of general relativity, we don't need to axiomatize (really, to formalize) all these step theories. As Suppes suggested, we can regard them as adequately given by set-theoretical resources. This enables us to go 'directly' to what interests, the theory proper, so we don't need to axiomatize also all these step theories. More precisely, let $\mathcal{L}_\in$ be the language of set theory. By a *predicate*, we understand a formula of $\mathcal{L}_\in$ (perhaps extended with other specific symbols) with only one free variable. If $\mathfrak{A}$ is a structure built in such a theory, let us consider a predicate P as formed by two parts (cf. da Costa & French 2003, p.37): the first part tells us how we can obtain $\mathfrak{A}$ from some basic sets (its *base sets*). The second part is the conjunction of the axioms of the structure. If $A_1, \ldots, A_n$ are the base sets of $\mathfrak{A}$, we may write the predicate

$$P(\mathfrak{A}; A_1, \ldots, A_n) \tag{4.1}$$

Then, the *species of structures* corresponding to $\mathfrak{A}$ is, by definition, the predicate

$$\mathbb{P}(x) \leftrightarrow \exists x_1 \exists x_2 \ldots \exists x_n P(x; x_1, \ldots, x_n). \tag{4.2}$$

In this sense, $\mathfrak{A}$ is a structure of species $\mathbb{P}$, and $\mathbb{P}$ is called a *Suppes predicate*.

For instance, we may consider human paternity theory as modeled by a structure such as

$$\mathfrak{P} = \langle D, X, Y, R \rangle, \tag{4.3}$$

as above, and we define a set-theoretical predicate (a formula of the language of set theory) of the kind

$$\begin{aligned} \mathcal{P}(x) = \exists D \exists X \exists Y \exists R (x = \langle D, X, Y, R \rangle \wedge (D \neq \emptyset) \\ \wedge (X \subseteq D) \wedge (Y \subseteq D) \wedge (R \subseteq D \times D) \\ \wedge (P1) \wedge (P2) \wedge (P3)), \end{aligned}$$

where (P1), (P2) and (P3) are adequate formulations of the three axioms above. Notice that in this formulation we are using the language of set theory supplemented with additional parameters which correspond to the primitive elements of the theory, and in this case, the formula above has x as its only free variable, hence the formula is *transportable* in the sense of Bourbaki (informally, it says that the definitions does not depend upon any specific property of the involved sets, which enables us to consider a class of structures of the kind (4.3) satisfying it). The structures x that satisfy the predicate $\mathcal{P}$ are the models of the set-theoretical predicate, and they constitute the class (generally, it is not a set) of the models of the theory. Of course there are infinitely many models most of them just abstract models, having no (in principle) any commitment to applications.

Here is another important example: group theory. As it is well known, a group can be seen as a structure $\mathfrak{G} = \langle G, *, e \rangle$ where G is a non empty set, $e \in G$ and $*$ is a binary operation on G, that is, $* : G \times G \mapsto G$. Alternatively, we can consider $*$ as a ternary relation on G, that is, it is an element of $\mathcal{P}(G \times G \times G)$ satisfying the additional condition of being a function. Thus, this structure has only one basis set, G. Starting with it, we get $G \times G$, then $G \times G \times G$ and finally $\mathcal{P}(G \times G \times G)$. So we can chose the desired element satisfying the desired properties (namely, $*$ obeying the standard group axioms). The Suppes predicate will be

$$\mathbb{G}(x) \leftrightarrow \exists x_1 \exists x_2 \exists x_3 (x = \langle x_1, x_2, x_3 \rangle \wedge x_2 \in x_1 \wedge x_3 \in \mathcal{P}(x_1 \times x_1 \times x_1) \wedge G1 \wedge G2 \wedge G3),$$

where $G1, G2, G3$ stand for suitable formulation of the group axioms. Of course this is quite artificial and we prefer to use the standard mathematical practice in saying that a group is a structure of the kind $\mathfrak{G} = \langle G, *, e \rangle$ where G is a non empty set, $e \in G$ and $*$ is a binary operation on G, so that (G1) $*$ is associative, (G2) e is the identity element of $*$ and (G3) any element of G has an inverse relative to $*$ in G. This is of course an equivalent way to present the theory, and we shall follow it in our examples below.

Thus, in a terminology to be introduced in the next chapter (but which is informally described here) an axiomatic theory T can be abbreviated by presenting the following items (based on da Costa & Rodrigues 2007, p.28):

(i) The language $\mathcal{L}_\in$ is supplemented with additional terms, the primitive terms of T.

(ii) A *species of structures* (in the sense of Bourbaki 1968, ch.4), that is, a class of structures with relations of the same order,[1] described in the extended language. One way of specifying the species of structures is by presenting a Suppes predicate, as above.

[1] We say that these structures have the same *signature*.

(iii) A set (finite or infinite) of sentences of a suitable language,[2] which are the axioms of T.

Of course we suppose that the structures of the mentioned class satisfy the axioms, that is, they are models of the axiomatics. Below we shall sketch some examples, but without all the details; we just present the class of structures, leaving the language implicit.

4.1.1 Classical particle mechanics

Thus, to present a set-theoretical predicate, or a Suppes predicate, is equivalent to present the postulates of the theory, as we do usually—really, in practice we of course *do not* write something like (4.4). As a simple case in science, let us take Classical Particle Mechanics (CPM) as formulated by McKinsey, Sugar and Suppes in 1953. Simply put (details in Suppes 1957, chap.12; Suppes 2002, chap.7),[3] a CPM is a structure of the form (that is, this schema specifies a species of structures)

$$\mathsf{CPM} = \langle P, T, s, m, f, g \rangle \tag{4.4}$$

where P is a finite non-empty set (the 'particles'), T is an interval of real numbers, standing for the instants of time, s is a function from $P \times T$ to $\mathbb{R}^3$ (the position of the particle $p \in P$ at time $t \in T$ is written as a vector $s(p,t)$), m is a function

[2] More precisely, the language is $\mathcal{L}^{\omega}_{\omega\omega}(\{D\} \cup \mathsf{rang}(\mathsf{r}_\iota))$, where D is the domain of the structures which are of the form $\mathfrak{A} = \langle D, r_\iota \rangle$—see the next chapter.

[3] These formulations are not the same; the second one is more precise.

which associate a positive real number to each particle (the numerical value of its mass), for each $p, q \in P$, $f(p, q, t)$ is the force which the particle q exerts on p, and $g(p, t)$ is the resultant external force acting on p at $t \in T$. All these concepts are subjected to certain postulates given below (here we follow Suppes 1957, p.294).

(1) For $p \in P$ and $t \in T$, $s(p, t)$ is twice differentiable on T.

(2) For $p, q \in P$ and $t \in T$, $f(p, q, t) = -f(q, p, t)$.

(3) For $p, q \in P$ and $t \in T$,

$$s(p, t) \times f(p, q, t) = -s(q, t) \times f(q, p, t).$$

(4) For $p \in P$ and $t \in T$,

$$m(p).\frac{d^2 s(p, t)}{dt^2} = \sum_{q \in P} f(p, q, t) + g(p, t).$$

Other theories can (in principle) be treated in the same way.

4.1.2 Non-relativistic quantum mechanics

Now we shall describe in a rather simple way a Suppes predicate for non-relativistic quantum mechanics, by adapting Toraldo di Francia's postulates (1981, pp.270-1). Of course we shall suppose that the reader has some understandings of the subject.

Working in ZF, we say that a non-relativistic quantum mechanics ($\mathsf{QM_{NR}}$) is a structure

$$\mathsf{QM_{NR}} = \langle S, \{H_i\}, \{A_{ij}\}, \{T_{ik}\}\rangle_{i\in I, j\in J, k\in K} \tag{4.5}$$

where S is a set of physical systems, $\{H_i\}$ is a collection of Hilbert spaces, $\{A_{ij}\}$ is a collection of Hermitian operators on the space H_i and $\{T_{ik}\}$ is a collection of unitary operators on H_i, $\{T_{ik}\} \subset \{A_{ij}\}$, where the following guidelines (usually called 'axioms') are satisfied:

(i) For each physical system $s \in S$, we associate a complex Hilbert space $H_s \in \{H_i\}$. The vectors $|\psi\rangle$ of this space represent the *states* of the physical system. It is called the *state vector* of the system, and stands for all we know about it. The state vectors are normalized, for $k.|\psi\rangle$ (for any compex number k) represents the same state as $|\psi\rangle$.

When we have a system composed by several elements of S, we associate to it the tensor product of the Hilbert spaces of the composing systems (in some order). If the cardinal of the subset of systems is n (call them $s_1, \ldots, s_n$), the Hilbert space is

$$\mathcal{H} = \mathcal{H}_{s_1} \otimes \ldots \otimes \mathcal{H}_{s_n}.$$

A typical vector of this space is written $|\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$, or simply $|\psi_1\rangle \ldots |\psi_n\rangle$ for short. When the systems are considered to be indiscernible, we make $\mathcal{H}_i = \mathcal{H}_j$ for any i and j.

(2) Let $|\psi(t)\rangle$ represent the state at time t. Then, for each $|\psi\rangle$ we associate an unitary operator T_s such that for any instant of time t, we have that

$$|\psi(t)\rangle = T_s(t).|\psi(0)\rangle, \tag{4.6}$$

where $|\psi(0)\rangle$ is the state at time $t = 0$. Ths represents the *unitary evolution* (in time) of the vector state, and it is called the *Schrödinger equation*.

(3) The eingenvalues of A, that i, those (real) scalars a_i such that $A|\psi_i\rangle = a_i.|\psi_i\rangle$ are the possible results of a measurement of A. It is assumed that the Hermitian operators represent observable physical quantities that can be measured on the system at a certain state. Sometimes we distinguish between the observable (such as mass, energy, momentum, number of particles, etc.) from the corresponding Hermitian operators by writing A for the observable and $\hat{A}$ for the operator. We think that we don't need this distinction here.

(4) It is know that any Hermitian A is diagonalizable, what means that we can find a basis $\{|\alpha_i\rangle\}$ for the considered Hilbert space formed by engeinvectors of A. Thus, for any state $|\psi\rangle$, we can write $|\psi\rangle = \sum_i c_i|\alpha_i\rangle$, where $c_i = \langle\alpha_i|\psi\rangle$ are the Fourier coeficients. Thus, $|c_i|^2 = P_i$ represents the probability that the measurement of A gets the value a_i. This postulate is known as *Born rule*.

(5) If a measurement of A gives the result a_i, the state vector

$|\psi\rangle$ becomes $|\alpha_i\rangle$ immediately after the measurement. This is known as the *collapse* of the vector state.

If we consider the observable 'position of the (considered) system along the x axis' and A the corresponding operator, then its engeinvalues (call them x_i) are the possible positions of the system. Let us write $|x\rangle$ for the corresponding eingenvector of the eingenvalue x. Thus, from (4), the probability of finding the system in the state $|\psi\rangle$ at the position x is $|\langle x|\psi\rangle^2$. In doing that, Dirac wrote $\psi(x) = \langle x|\psi\rangle$, and called $\psi(x)$ the *wave function* of the system. For any time t, if H is the Hamiltonian of the system (energy operator), the wave function $\psi(x, t)$ obeys the Schrödinger time dependent equation in the form

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t). \tag{4.7}$$

The two ways to write the Schrödinger equation are equivalent. We shall not pursue more details here, for we are just exemplifying the way a set-theoretical predicate can be done. But it should be remarked that the above formalisms stands for just one physical system. Composite physical systems (say, a two electrons system) are dealt with by associating to each part (each electron) one Hibert space as above and to the composite system the tensor product of the Hilbert spaces. If the systems are considered indiscernible, we use the same Hilbert space for all of them. We shall speak more on quantum physics in chapter 8.[4]

[4] In order to see this schema applied to biology, look to the synthetic theory of evolution as formulated

in (Magalhães & Krause 2001).

Chapter 5

The mathematical basis

We shall outline the mathematical framework where the mathematical counterpart of scientific theories can be discussed, namely, the Zermelo-Fraenkel set theory, ZF. We have chosen to treat it as a first-order theory, and its language, presented below (section 5.2), will be termed $\mathcal{L}_{\in}$.[1] But before describing ZF, we need to discuss a little bit about the construction of axiomatic/formal systems.

5.1 The Principle of Constructivity

In his book *Ensaio sobre os Fundamentos da Lógica* ('Essay on the Foundations of Logic'),[2] da Costa calls 'the Principle

[1] In fact, we could use a different language, say by using other primitive symbols and postulates (although apparently equivalent), and it would be a debatable question to say that these formulations would be different theories. But there is also the possibility of constructing a theory grounded on second-order logic, as Zermelo preferred—see Moore 1982, pp. 267ff—(or still higher-order) logic, and in this case in fact we should have different theories, for their properties would be distinct, as we shall see soon.

[2] There is a French version of this book, translated by Jean-Yves Béziau, called *Logique Classique et Non-Classique*, Paris, Flammarion, 1996.

of Constructivity' the following methodological rule:

> "The whole exercise of reason presupposes that it has a certain intuitive ability of constructive idealization, whose regularities are systematized by intuitionistic arithmetics, with the inclusion of its underlying logic." (da Costa 1980, p.57)

What does he mean? The explanations began earlier in his book, and we base our argumentation on his ideas, with the risk of leaving out much of his rich and clear account. In this section, when we make reference to a certain page only (say, by writing 'page 51'), it refers to the mentioned book by da Costa. To speak a little bit of such matters is of course quite important, for we intend to present a formal system (ZF) using expressions such as 'infinite set of individual variables', for instance, which presupposes a previous notion of 'infinite', hence, it apparently presupposes a mathematics it intends to base.

But let us 'listen' to da Costa:

> "Formal disciplines are essentially discursive. But the discourse develops itself in different levels, and each one of them must be understood, or intuited, as already noticed by Descartes. Even if one reasons symbolically and formally, the different elementary steps of the evolution of the discourse need to be clear and evident, for in the contrary there would be no

> reasoning, and one would not know what she is doing." (p.50).

In trying to systematize the 'experimental reality' (the ER of page 79), we make use of our capacity of reasoning. We use intuition and other devices, such as learned theories, personal experiences, memory, imagination, expertise, insights. In a certain sense, all of this depends on the present day state of the evolution of science, so as of our own biological and cultural characteristics, as we have seen before. But intuition is not enough. We need to systematize our intuitions, and in mathematics, the axiomatic method became the (apparently) best methodological tool, being extended to science, as we have seen already. Well developed sciences, or disciplines, yet not completely either axiomatized or formalized, can be treated, from a mathematical perspective the same way as the formal sciences, being also essentially discursive. In other ways, scientific knowledge, being essentially conceptual knowledge, needs discourse, and our 'discourse' needs language and symbolism (p.35).

As is well known, in intuitionistic mathematics we have an intuitive 'visualization' of the entities that interest us (p.50). This is essentially an *intellectual intuition*, an expression that intends to capture the idea that "there cannot be immediate and evident knowledge without contemplation, without a look to the objects that interest us or, at least, of the conceptual relations which define them; in an analogous way, there is no intellectual contemplation which does not enable us to formulate direct judgments, linked to different levels of evidence" (p.51). The reference to intuitionistic mathematics is grounded on the fact that, according to da Costa, it provides an intuitive 'visualization' of the entities that interest us (p.52), contrarily to standard mathematics, where there may be not is such an intuition (p.54). He exemplifies with examples of the following kind: we have no a clear 'vision' neither of transfinite cardinals, nor of the totality of the real numbers, but only an intuition of the system of relations that implicitly define these concepts by means of axiomatic systems (p.54). This 'intuitive visualization' provides us with an *intuitive pragmatic nucleus* and based on this nucleus we articulate a kind of algebra (da Costa doesn't use this word in this context) that enables us to compose them, operate with them and so on, going to more sophisticated and sometimes not intuitively evident conceptualizations. This way, we go out of the intuitive nucleus, and in flying so high, our 'autopilot' is the axiomatic method.

In this sense, we begin by describing a formal system using this intuitive nucleus, of finitist and constructive nature; as da Costa says, exemplifying, "it is today universally accepted that there cannot be formalized arithmetics without intuitive arithmetics" (p.57). It is this informal and intuitive manipulation with symbols and concepts that, at first glance, enable us to make reference to the tools we need to characterize our axiomatic/formalized theories. Thus, it is in this sense that we formulate the language of ZF described below; in saying, for instance, that we have a denumerable infinite collection of individual variables, we could say that our language encompasses two symbols, x and $'$, and the variables would be expressions of the form $x, x', x'', \ldots$, and the same for the individual constants. This way, we avoid speaking of idealized concepts (in Hilbert's sense, such as 'denumerable many') and keep ourselves in a constructive fashion. Enough for this kind of discussion. Let us turn now to the ZF set theory.

5.2 The ZF set theory

The language $\mathcal{L}_{\in}$ of ZF has the following categories of primitive symbols:

(i) The propositional connectives: $\neg$ and $\rightarrow$

(ii) The universal quantifier: $\forall$

(iii) Two binary predicates: $=$ and $\in$

(iv) Auxiliary symbols: left and right parentheses: (and)

(v) A denumerable infinite set of individual variables: $x_1, x_2, \ldots$

(vi) A denumerable infinite set of individual constants (or 'parameters'): $a_1, a_2, \ldots$ (most formulations don't use individual constants, so the *terms* are just the individual variables).

This vocabulary forms the primitive alphabet of ZF. Other symbols can be introduced by abbreviative definitions, such as the other sentential connectives $\wedge$, $\vee$ and $\leftrightarrow$, the existential quantifier $\exists$ and 'specific' symbols of set theory, such as $\subseteq$ (subset), $\mathcal{P}$ (for the power set) and all other mathematical machinery. But all of this presupposes that we know the syntax of $\mathcal{L}_\in$, and later we shall speak of its semantics. In the *intended* intuitive interpretation (as we shall see below), the individual variables range over a collection of objects we call *sets*. The individual constants name particular sets we use for a purposes, say by terming '$\mathbb{R}$' the set of the real numbers. Both individual variables and individual constants are called *terms* of the language.

The syntax of $\mathcal{L}_\in$ can be described without difficulty, as done in standard books of logic and set theory (see Franco de Oliveira op.cit., pp.196ss). We shall use $x, y, z, \ldots$ for terms of $\mathcal{L}_\in$ in general, either individual variables and constants, but in the formulas $\forall x\alpha(x)$ and $\exists x\alpha(x)$, x is always a variable.

Let $\alpha(x)$ be a formula of $\mathcal{L}_\in$ in which x is the only free variable. We shall call this formula a *condition* In the intuitive theory of sets (which admits as 'sets' whatever collection of objects you wish), we have the following principle, called the *Axiom of Comprehension for Classes*, namely, given a condition $\alpha(x)$, there exists a collection (called a *class*) which consists exactly of those elements that satisfy the condition. We denote this class by

$$\{x : \alpha(x)\}. \tag{5.1}$$

Not all classes are sets of ZF (ZFC, ZFU). For instance, Russell's class $\{x : x \notin x\}$ is not a set of none of these theories (supposed consistent). The same happens with some 'large' classes, such as the class of all groups, of all vector spaces, of all models of a scientific theory, of a set theory such as ZF (ZFC, ZFU). These collections are very large to be sets of these theories (but they can 'exist' for instance in some stronger set theories). So, what is a set? It depends on the axioms we use. Below we shall see the postulates that determine what are the sets of ZFC.

5.2.1 The postulates of ZF

Let $F(x)$ be a formula of $\mathcal{L}_\in$ where x is free. The collection of the objects satisfying $F(x)$ is written $\{x : F(x)\}$ and is called a *class*. The question is: what classes are sets? The answer depends on the axioms we use. In ZF (ZFC, ZFU) we shall have some that will deserve to be named *sets*, while others do no. Thus, the postulates below determine what are the sets of ZFC.[3] There are different set theories (really, potentially infinitely many). One of the most well-known is the system NBG, the von Neumann, Bernays and Gödel set theory. In

[3] For instance, the so-called 'Russell's class', namely, $R\{x : x \notin x\}$ is not a set of ZFC, but it 'exists' in some paraconsistent set theories—see da Costa et al. 2007.

this theory, the basic objects are classes. But some of them are sets, namely, those classes that belong to other classes. Those classes that do not belong to other classes were termed *proper classes* by Gödel. The sets of NBG and the sets of ZF are essentially the same objects (NBG is a conservative extension of ZF).

The logic postulates of ZF are those of the first order logic with identity, that is, being α, β, and γ formulas:

(1) $\alpha \to (\beta \to \alpha)$

(2) $(\alpha \to (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

(3) $(\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha))$

(4) $\alpha, \alpha \to \beta/\beta$ (Modus Ponens)

(5) $\forall x\alpha(x) \to \alpha(t)$, where t is a term free for x in $\alpha(x)$

(6) $\forall x(\alpha \to \beta(x))/(\alpha \to \forall x\beta(x))$, if x does not appear free in α

(7) $\forall x(x = x)$

(8) $u = v \to (\alpha(u) \to \alpha(v))$, where u and v are distinct terms of $\mathcal{L}_\in$.

The specific postulates are the following ones:

(ZF1) [Extensionality] $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$

(ZF2) [Pair] $\forall x\forall y\exists z(\forall w(w \in z \leftrightarrow w = x \vee w = y))$

(ZF3) [Separation Axioms] If $\alpha(x)$ is a formula of $\mathcal{L}_\in$ with only x as a free variable, then for each α, the following is an axiom: $\forall w \exists x \forall y(y \in x \leftrightarrow y \in w \wedge \alpha(y))$

Taking $\alpha(x)$ as $x \neq x$, and applying the separation schema on a set w whatever, we get a set with no elements, which using the axiom ZF1 we can show is unique. This set is called 'empty set' and denoted by $\emptyset$ (we can suppose that this symbol is one of the individual constants, and the same happens for other introduced symbols naming sets). The set x of postulate ZF3 is written (in the metalanguage) as $x = \{y \in w : \alpha(y)\}$. This poses a fundamental difference between this set and a class as given at equation (5.1). Here, the elements that belong to x are taken from an already given set w, while in (5.1) they are not coming from a previous set. Thus, axiom (ZF3) is called the postulate of the limitation of size (of sets), and it is due to Zermelo.

(ZF4) [Union] $\forall x(\forall y(y \in x \to y \neq \emptyset) \to \exists w(\forall z(z \in w \leftrightarrow z \in y \wedge y \in x)))$. The set w is written $\bigcup x$, and sometimes $u \cup v$ if x has just to elements, u and v.

(ZF5) [Power set] $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$, where $u \subseteq v := \forall w(w \in u \to w \in v)$. The set y is written $\mathcal{P}(x)$, the power set of x.

(ZF6) [Substitution Axioms] Let $F(x, y)$ be a formula in which x and y are distinct and free variables. Let u and v

be distinct variables from both x and y. Then, the following expression is an axiom:

$$\forall x \exists ! y F(x, y) \rightarrow \forall u \exists v \forall y(y \in v \leftrightarrow \exists x(x \in u \wedge F(x, y)).$$

In saying that $\forall x \exists ! y F(x, y)$ holds, we are assuming that F is y-functional.

(ZF7) [Infinity]

$$\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x).$$

(ZF8) [Choice]

$$\forall x(\forall y(y \in x \rightarrow y \neq \emptyset) \wedge \forall t \forall z(t \in x \wedge z \in x \wedge t \neq z \\ \rightarrow t \cap z = \emptyset) \rightarrow \exists t(t \subseteq x \wedge \forall u(u \in x \rightarrow \exists v(t \cap u = \{v\}))).$$

We could formulate these axioms without recurring to defined symbols such as $\subseteq$, $\cup$ and others. But we have used them here to keep the text more readable. The axiom of choice is considered as the second most famous axiom of all mathematics, being overcome only by Euclid's parallel postulate. There are excellent books on its history and importance (see for instance Fraenkel et al. 1973, ch.2, §4).

5.2.2 Informal semantics of $\mathcal{L}_\in$

From the formal point of view, the symbols of $\mathcal{L}_\in$ have no meaning. But, intentionally, we usually accept that they make reference to the objects of a certain intuitive non empty domain of objects we think as representing *sets*, but in principle they could be of any 'nature' we wish. If D is such a domain of sets, we interpret the predicates $=$ and $\in$ as being respectively the identity of D and in a 'membership relation' of D. Furthermore, the expressions $\forall x \alpha(x)$ and $\exists x \alpha(x)$, where $\alpha(x)$ is a formula of $\mathcal{L}_\in$ in which x is the only free variable, mean 'for all elements of D' and 'there exists an element of D' such that $\alpha(x)$ holds respectively. In this sense, an interpretation for $\mathcal{L}_\in$ is an ordered pair (intuitively speaking) $\mathfrak{A} = \langle D, \xi \rangle$, where D is a non empty set and ξ is a binary relation (in the intuitive sense) on D. The elements of D are called *sets* and ξ interprets the symbol $\in$ of $\mathcal{L}_\in$.[4] To each individual constant a_i of $\mathcal{L}_\in$, we assume that the interpretation associates (by means of an intuitive function) a particular individual of D; the individual constants act as *names* of particular sets.

One of the outstanding problems is to find an interpretation that makes the postulates of ZF 'true' (in the intuitive sense). To begin with, let us reproduce here some examples of possible interpretations for $\mathcal{L}_\in$.[5] The first interpretation takes D as the set $\mathbb{Z}$ of the integers, and ξ as the relation $<$ on the integers. Thus, the integers are now our sets, and $x \in y$ means $x < y$. Of course this is an interpretation as we have described informally; to the predicate $=$ of identity we associate the set of all

[4]All of this is quite well described in Franco de Oliveira op.cit., pp. 209ff.
[5]Based on Franco de Oliveira op.cit., p.210.

couples $\langle x, x\rangle$ with $x \in \mathbb{Z}$, and the connectives and quantifiers are interpreted as usual ($\exists x$ means 'there exists an integer x so that ...' and so on).

But we could take another interpretation, just taking $D = \mathbb{N}$ (the set of the intuitive natural numbers) and ξ being the order relation $<$ on such a domain. This is also an interpretation. Now let us consider the sentence (a formula without free variables)

$$\forall x_0 \exists x_1 (x_1 \in x_0). \tag{5.2}$$

The reader should recognize that (5.2) stands for a formula of $\mathcal{L}_\in$. In the metalanguage, we could write $\forall x \exists y(y \in x))$. According to the first interpretation, it means that *for any integer there exists an integer which is least than it*, and it is true (in the intuitive sense). But, according to the second interpretation, it means that *for any natural number there exists an integer which is least than it*, and it is false. Thus, a certain sentence may be true according to one interpretation, but false according to another one. Franco de Oliveira still gives us another example, concerning the axiom of the unordered pair. Or course we could provide other examples than his ones, but his book is so well written and clear that it would be really a mistake do not acknowledge it.

Take the sentence of $\mathcal{L}_\in$ that expresses the axiom of extensionality, namely (written in the metalanguage)

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y). \tag{5.3}$$

This sentence is intuitively true in both interpretations above, as it is easy to see (for any two integers or natural numbers a and b, if $a < b$ iff $b < a$, then of course $a = b$. But now take the sentence that expresses that there exists an empty set, namely,

$$\exists x \forall y (y \notin x), \tag{5.4}$$

which is false in the first interpretation but true in the second one.[6] This would not be surprising, for the two given interpretations, although trully ‘interpretations’ of $\mathcal{L}_\in$, are not adequate for expressing our supposed intuitive universe of sets. How can we thought of interpretations which are *models* of ZF, then? We shall discuss this important topic next.

5.3 ‘Models’ of ZF

A model of a formal theory, as we now already, is an interpretation that makes its postulates true (in the Taskian sense). Can we think of models of ZF in this sense? As we shall see, we can, but the corresponding structures will be classes which are not ZF-sets (if we suppose ZF consistent).

The consistency of ZF can be proven only relative to another stronger theory, whose consistency is then put also into question, and to answer whether this stronger theory is by its turn consistent, we will need an still stronger theory, and so on.

[6]Here $y \notin x$ abbreviates $\neg(y \in x)$. The empty set can be obtained from the above axioms as follows. Given a set z whatever, consider the formula $F(x) \leftrightarrow x \neq x$ and apply the separation axioms. Then we get a set with no elements, which is unique by extensionality; we usually write $\emptyset$ to represent such a set.

Important to recall that there are two notions of consistency, one syntactical and another 'semantic'; the syntactic definition says that a theory T, whose language contains a negation symbol $\neg$, is consistent iff there is no formula α such that both α and $\neg\alpha$ are theorems of T. The semantic definition says that a theory is consistent iff it has a model (an interpretation that satisfies the axioms of T). Semantic consistency implied syntactic consistency, but not the other way around.

The reason we cannot prove the consistency of ZF within itself is grounded on the way we have formulated this theory. Built from a recursive denumerable language, and being sufficiently strong to cope with elementary arithmetics, it is subject to Gödel's first incompleteness theorem (see Smullyan 1991), which, roughly speaking, says that no theory whose set of axioms is recursively enumerable encompassing elementary arithmetics (really a certain 'part' of it) can be at the same time consistent and complete (complete in the syntactical sense of proving either α or $\neg\alpha$ for any formula α of its language).[7]

Being consistent, ZF has 'models'. In the 1940s, Gödel studied the so-called 'universe of constructible sets', L (constructed in a stronger theory) which constitutes an *inner model*, a transitive class of sets containing all the ordinals, and which results to be a 'model' of the ZF axioms plus the axiom of choice (ZFC) and for the generalized continuum hypothesis

[7]By recursively enumerable, we intuitively means that there is a an algorithm (a computer program) that, in principle, can enumerate all the axioms of the theory, but not the sentences that are not axioms.

(that is, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α).[8] The technique of constructing inner models became one of the important tools in mathematical logic, but we are not concerned with them here, although we shall see later what does it mean to say that the axioms of ZF are true in an inner model M; it suffices to emphasize that ZF has ‘models’ (if consistent), and that these models are not ZF-sets.

Thus, we may ask for a description of an universe of sets in which we can show that the axioms of ZFC are ‘true’ (in the intuitive sense). One of these possible universes is the *cumulative hierarchy of sets*, which informally speaking means that a set is ‘constructed’ only after we have ‘constructed’ its elements. Sets are elaborated in stages, starting from a basic collection of ur-elements, entities that are not sets but which can me members of sets, and performing the set-theoretical operations.

For instance, given the objects (sets of ur-elements) a and b, we can form the sets $\{a\}$, $\{a, b\}$, then $\{\{a\}, \{a, b\}\}$ (which we identify with the ordered pair $\langle a, b\rangle$, and so on. Zermelo’s

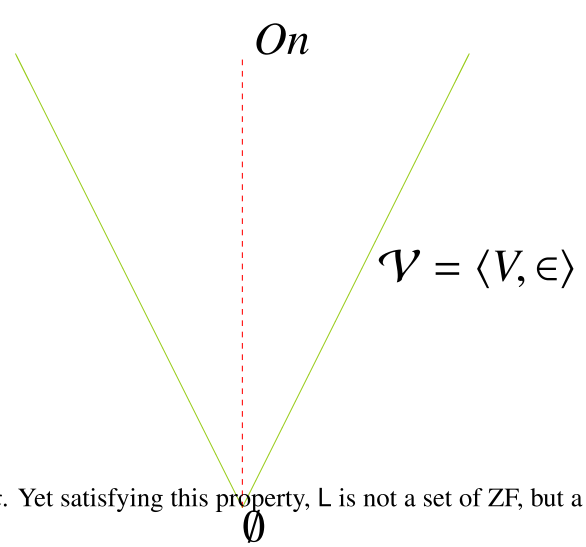

Figure 5.1: The well-founded ‘model’ of ZFC without ur-elements.

[8] A set A is transitive if $x \in A \to x \subseteq A$ for any x. Yet satisfying this property, L is not a set of ZF, but a proper class.

original *universe of sets* was of this kind, although his axiomatization didn't capture it in full.[9] Later, Fraenkel has shown that the ur-elements are not necessary for the foundations of mathematics, and his ideas (so as Skolem's) have conduced to a 'pure' set theory, whose stages begin from the empty set. Figures 5.1 and 5.2 exemplify the universes of set theory without and with ur-elements respectively.

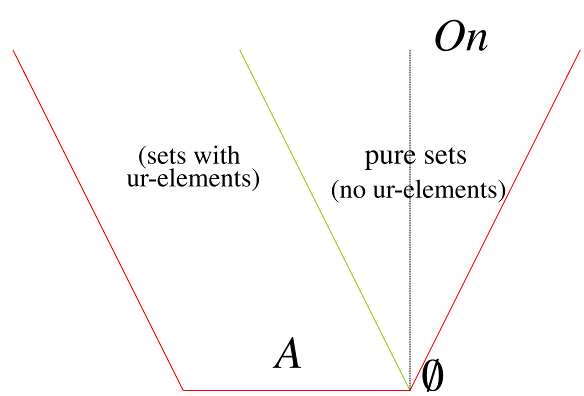

Figure 5.2: The universe of sets with ur-elements. *On* is the class of ordinals, and *A* is the set of atoms (ur-elements).

This intuitive 'universe of sets' can be described using the ordinals for indexing the stages, getting V_0, V_1, and so on. Thus, we are conduced to the so-called *von Neumann universe*, termed V. We sketch both descriptions, with and without ur-elements next. Important to notice that these descriptions afe done in the meta-language, as we shall justify below. Just to mention, the sequence of ordinals can be writ-

[9] We shall not discuss this point here, but this is due to the fact that is axioms are compatible with the existence of *extraordinary sets*, such as sets that belong to themselves. See Krause 2002. There are good discussions on the construction of sets in stages, as in Shoenfield 1977 and Franco de Oliveira 1982.

ten as follows:

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega 2, \omega 2 + 1, \ldots, \omega^2, \ldots \tag{5.5}$$

In this sequence, there are ordinals β such that there exist an ordinal α and $\beta = \alpha + 1$. These are called *sucessor ordinals*. For instance, all natural numbers (the finite ordinals) are sucessors, so as are $\omega + n$, $\omega 2 + n$ for any natural number n. But ω, $\omega 2$, etc. are not sucessors, for they do not have an 'immediate predecessor'. They are called *limit ordinals*.

Thus, the von Neumann cumulative hierarchy is defined as follows (corresponding to Figure 5.1):

$V_0 = \emptyset$

$V_1 = \mathcal{P}(V_0)$

$\vdots$

$V_{n+1} = \mathcal{P}(V_n)$

$V_\lambda = \bigcup_{\beta<\lambda} \mathcal{P}(V_\beta)$, for λ a limit ordinal,

$\vdots$

$V = \bigcup_{\alpha \in On} V_\alpha$, where On is the class of ordinals (which is not a set of ZF).

The word 'cumulative' means that every *stage* V_α contains all objects of the lower levels V_β, for $\beta < \alpha$. We have also seen some of the consequences of this assumptions, and now we shall see some other results. If we assume the existence of a set A of ur-elements to begin with, we pose (see figure 5.2)

$$V_0 = A$$

$$V_1 = V_0 \cup \mathcal{P}(V_0)$$

$$\vdots$$

$$V_{n+1} = V_n \cup \mathcal{P}(V_n)$$

$$\vdots$$

$V_\lambda = \bigcup_{\beta<\lambda} \mathcal{P}(V_\beta)$, for λ a limit ordinal,

$$V_{\lambda+1} = V_\lambda \cup \mathcal{P}(V_\lambda)$$

$$\vdots$$

$V = \bigcup_{\alpha \in On} V_\alpha$, where On is the class of ordinals (which is not a set of ZF).

The remarks that follow will make reference to both universes. First of all, it should be noticed that any V_α is a transitive class. Now let us consider the following question: given a certain ordinal α, which axioms of ZFC are satisfied in $\langle V_\alpha, \in \rangle$, where $\in$ the membership relation relativized to V_α (that is, $x \in y$ means $x \in y$ and $x, y \in V_\alpha$)? Interesting is that we can prove the following results:[10]

[10]Let M be a class (it may be a set) and F a formula of $\mathcal{L}_\in$. The formula F^M is called the *relativization* of F to M if it is obtained by substituting $\exists x \in M$ for $\exists x$, and $\forall x \in M$ for $\forall x$. Thus, it says exactly what F says but concerning elements of M only. The results mentioned below are proven in Franco de Oliveira 1981, pp.298ff, Enderton 1977, pp.249ff; Fraenkel et al. 1973, p.289.

(i) Any V_α is a model of the following axioms: Extensionality, Separation, Union, Power Set, Choice, Regularity.

(ii) For the Pair axiom, we need a limit ordinal. This can intuitively be understood for the set of two objects is of a higher level that the levels of the objects.

(iii) For the infinity axiom, we need that α be a limite ordinal greater that ω, for instance, $\omega 2$.

(iv) For the Replacement Axioms, we need more, for instance, an *inaccessible cardinal*.

Usually we call Z the Zermelo set theory, which is ZF without the substitution axioms. All the levels V_α can be constructed in Z as well. But, by the above results, we see that $V_\omega 2$ is a 'model' of Z, since $\omega 2$ is a limit ordinal. Then, if consistent, Z cannot admit $V_{\omega 2}$ as one of its sets, for in this case we would be against Gödel's second incompleteness theorem. Reasoning in the same vein, we cannot prove the existence of inaccessible cardinals within ZFC. Thus, we can't prove the existence of V, the whole universe of sets, within ZFC. Of course we could think that the universe can be constructed within a stronger theory. This is true, but we would just transfer the problem to this another theory, for we also don't know what *its* notion of set means. This problem brings interesting philosophical questions, due to the fact that, if consistent, a set theory such as ZFC (formulated as a first-order theory) is not *categorical*.[11]

We have made such a digression on set theory and on its 'models' just to show to the reader that there are differences in speaking of certain structures as models of certain theories and 'models' of the set theories themselves. The models of both mathematical and scientific theories, such as groups, vectors spaces, Euclideian geometry, metric spaces, differentible manifolds, classical mechanics, relativity theory, etc. can be constructed within Z, ZF, ZFC or ZFU or in another theory. But the 'models' of these theories themselves cannot be constructed within themselves (supposing them consistent). Below we shall discuss a little bit a particular case involving quantum mechanics.

5.4 The different 'models' of ZF

We have remarked above that ZF (Z, ZFC, ZFU), formulated as as first-order theory, is not categorical. Let us comment a little bit on this claim in order to see its consequences, even to physical theories. We shall mention just a case, among several possible others which could be obtained by using techniques such as Cohen's forcing, but which are out of the level of this text. The importance of the 'models' of set theory for the philosophy of science may be explained as follows. A formalized theory (even of in the empirical sciences) say by a

[11] A theory is categorical if its models are isomorphic.

Suppes predicate is generally elaborated to cope with a certain informal given theory, as we have seen before. But in order to achieve this aim, we need to provide an *interpretation* to our formalism, generally by associating to it an structure built in set theory or, what it is the same, in a 'model' of the set theory we use as our metatheory. But a set theory such as ZF (ZFC, ZFU) has different models. If axiomatized as a first-order theory, it has even a denumerable 'model' due to the Löwenheim-Skolem theorem, and since in general we never know in what model we are working, we really will have difficulties to state that the interpretation we provide fits the intended concepts. We remark once more that despite we can think in sets and related concepts as done by the 'standard model' of ZF, we cannot be sure that this is really so, for we have no grounds for proving that such a 'model' exists. Let us consider a particular example concerning denumerable 'models' only.

Real numbers can be used in physics for representing time, probability measure, eingenvalues of self-adjunct operators, to parametrize certain functions, and so on. The physicist has an intuitive idea of what real numbers are, so as what are *sets*. If pressed, a physicist may mention even some definition of real numbers from the rationals, say by convergent sequences (Cauchy sequences) or by Dedekind cuts (see Enderton 1977, pp.111ff; Franco de Oliveira 1981, pp. 87ff). These constructions are done within ZF, say. But suppose ZF is consistent. Then it has 'models'. The first we can imagine may be called the *standard model*, which is more or less identified with an 'universe of sets' such as the *V* seen before. But, as we have seen, we cannot prove in ZF that such a 'model' exists, so we don't have any evidence that our intuitive account to sets, that is, our intuitive idea of a set, is captured by the elements of such an universe. But there is more. As a first-order theory, ZF is subjected to the Löwenheim-Skolem theorem, which implies two things: firstly, since any model of ZF must satisfy the axiom of infinite, ZF will have an infinite 'model' (recall that 'model' is being written in quotation marks to emphasize that is is not a set of ZF, but a proper class). Then, by the *upward* Löwenheim-Skolem theorem, ZF has models of any infinite cardinality, being non isomorphic. But let us fix in the another result entailed by this theorem: by the *downward* Löwenheim-Skolem theorem, having models, ZF has a denumerable model.[12] What for science?

Let us suppose that our theory T demands the real number system, as most physical theories do. Then, it seems 'natural' to suppose that the set $\mathbb{R}$ of the real numbers is not denumerable, as Cantor shown in the XIX century with his famous diagonal argument. But, in such a denumerable 'model', the set that (in the 'model') corresponds to the set of real numbers must be denumerable. This is of course puzzling, but there is not a contradiction here. As remarked by Skolem, this

[12] Roughly speaking, the *downward* version of the theorem says that if a consistent theory T in a countable language has a model, then it has a denumerable model. The *upward* theorem says that such a theory, having infinite models, has models of any infinite cardinality.

result, known as the ‘Skolem paradox’ is not a formal ‘paradox’, but just something against our intuition, showing that what there is no *within* ZF (that is, a ZF-set) is a bijection between $\mathbb{R}$ and ω, but this does not entail that such a bijection does not exists *outside* ZF, that is, as something which is not a ZF-set. Thus, if a physical theory demands a mathematical concept which, if standardly defined, presupposes the set of real numbers, how can we grant that it is non-denumerable? In other words, how can we ensure that, working within ZF we are making reference to some ‘model’ that fits our intuitive claims? Unfortunately, this is not possible, and the most we can do is to fix a particular ‘model’ of ZF when we need to give an interpretation of concepts, but we will be always subject to questionings. Really, as remarked, we can’t show that even the so-called ‘standard’ (intended) model of ZF does exist!

In what concerns scientific theories, as remarked by da Costa, “we should never forget that set theories, supposed consistent, have non-standard ‘models’, thus any theory founded on them will have non-standard models too." (personal communication). This entails that if we try to provide an understanding for the concepts we use (that is, by given them an interpretation), we need to consider ‘models’ of ZF and this poses us an impasse: we never know in which model we are working, so we really never know what our concepts really mean. All we can do is to *suppose* we are working in the standard ‘model’, where (apparently) sets are as we imagine, that the real numbers cannot be enumerated, that there are basis for our vector spaces, and so on. The look for non-standard models for physical theories is still a novel domain of research, and of course it will be a rich one.

Chapter 6

Structures, and languages

We recall that we have chosen to work in first-order ZFC and will be following (da Costa & Rodrigues 2007) for the mathematical definitions. Within ZFC we can, at least in principle, construct particular structures for mathematics and for empirical sciences —sometimes we may be in need to strength ZFC, say with universes (which enables us to deal also with category theory, although we shall be restricted here to set-theoretical structures). As we shall see, the language of ZFC, termed $\mathcal{L}_\in$, yet taken here to be a first-order language, is so powerful that using it we construct even languages which are not first-order and structures which are not order–1 structures (these definitions shall be introduced in what follows). To go to some details, we need to introduce a few basic definitions we present in the next sections.

The general idea can be seen with an analogy with group theory. Starting with a non-empry set G, by using the set-theoretical operations we obtain $\mathcal{P}(G \times G \times G)$, and then we may chose an element of this set (which is a set whose elements are collections of 3-uples of elements of G) satisfying the required properties (the group axioms of course).[1] The corresponding structures (namely, the *groups*) are order–1 structures.

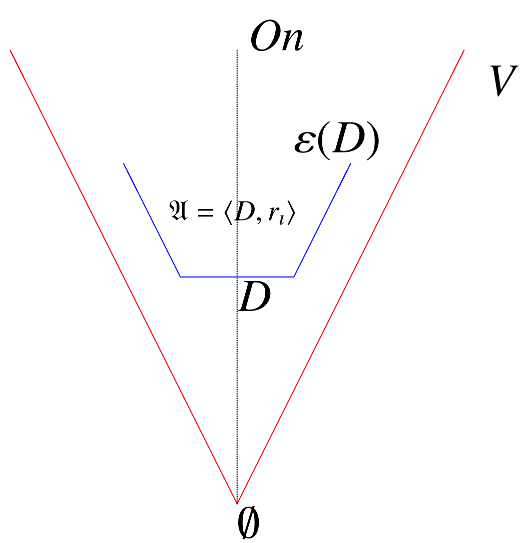

Figure 6.1: The well-founded 'model' $\mathcal{V} = \langle V, \in \rangle$ of ZF, a structure built within ZF, and the scale based on the domain D. For details, see the text.

Summing up, what we are going to see is that a mathematical structure, such as those used in mathematics and in the empirical sciences, can be constructed within a set theory such as ZFC (with or without ur-elements). The figure below provides a schema of this situation, using 'pure' set theory (without ur-elements) with an axiom of foundation, as in standard first-order ZFC. It should be remarked that while the structure $\mathfrak{A}$ (to be defined below) is a set, the whole universe V is not, as we have seen in the last chapter.

[1]The binary group operation $*$ is a certain function from $G \times G$ in G or, what is the same, it is a certain ternary relation on G, that is, a collection of triples $\langle a, b, c \rangle$ of elements of G, with $c = a * b$.

This is a first point usually ignored in the philosophical discussions: there are fundamental differences in saying that a certain structure (a set) $\mathfrak{A}$ is a *model* of some collection of sentences (written in the language of the structure, as we shall see soon) and that a certain class $\mathcal{M} = \langle M, E\rangle$ is a *model* of ZF (or either of ZFC or ZFU), in the sense we have discussed in the last chapter. These diverse uses of the word 'model', as we have already referred to above, need to be understood by the philosopher of science. This is the main reason we propose the discussion we shall start now.

6.1 Basic definitions

From now on, except when we explicitly mention another situation, we shall be working in first-order ZFC set theory. The definitions follow da Costa & Rodrigues 2007.

Definition 6.1.1 (Types) *The set $\mathbb{T}$ of types is the least set satisfying the following conditions:*

(a) $i \in \mathbb{T}$ (i is the type of the *individuals*)

(b) if $t_1, \ldots, t_n \in \mathbb{T}$, then $\langle t_1, \ldots, t_n\rangle \in \mathbb{T}$

Thus, i, $\langle i\rangle$, $\langle i, i\rangle$, $\langle\langle i\rangle, i\rangle$, $\langle\langle i\rangle\rangle$ are types. Intuitively speaking, in this list we have types for individuals, for sets (or properties) of individuals, for binary relations on individuals, binary relations whose relata are properties of individuals and individuals, and properties of properties of individuals.

DEFINITION 6.1.2 (ORDER OF A TYPE) *The order of a type,* $\mathsf{Ord}(t)$*, is defined as follows:*

(a) $\mathsf{Ord}(i) = 0$

(b) $\mathsf{Ord}(\langle t_1, \ldots, t_n \rangle) = \max\{\mathsf{Ord}(t_1), \ldots, \mathsf{Ord}(t_n)\} + 1.$

Thus, $\mathsf{Ord}(\langle i \rangle) = \mathsf{Ord}(\langle i, i \rangle) = 1$, while $\mathsf{Ord}(\langle i, \langle i \rangle \rangle) = 2$.

Relations will be understood here as both extensional sets (collections of n-tuples) and being of finite rank (that is, having finite elements only). *Unary* relations are sets.

DEFINITION 6.1.3 (ORDER OF A RELATION) *The order of a relation is the order of its type.*

Thus, binary relations of individuals are order–1 relations, and so on. We shall introduce a function t_D as follows:

DEFINITION 6.1.4 (SCALE BASED ON D) *Let D be a set. Then,*

(a) $t_D(i) = D$

(b) If $t_1, \ldots, t_n \in \mathbb{T}$, then $t_D(\langle t_1, \ldots, t_n \rangle) = \mathcal{P}(t_D(t_1) \times \ldots \times t_D(t_n))$.

(c) The scale based on D is the union of the range of t_D, and it is denoted by $\varepsilon(D)$.

Let $t = \langle t_1, \ldots, t_n \rangle \in \mathbb{T}$, $t \neq 0$. The elements of $t_D(t)$ are relations of degree or rank n. For instance, a binary relation on D is an element of $t_D(\langle i, i \rangle) = \mathcal{P}(t_D(i) \times t_D(i)) = \mathcal{P}(D \times D)$.

According to definition 6.1.3, the type of a binary relation on D is $\langle i, i\rangle$, as intuitively expected. If $n = 1$, we speak of unary relations (or 'properties') or sets of individuals of D, and we speak of distinguished individuals of D as order–0 relations. Since a binary operation on D (as the group operation) can be seen as a ternary relation on D, we can deal with it as follows. Having the scale $\varepsilon(D)$, we just take an alement of $t_D(\langle\langle i, i, i\rangle\rangle) = \mathcal{P}(D \times D \times D)$ satisfying well known conditions (the group postulates, written in this 'relational' notation). In this sense, it can be shown that we can map Bourbaki's *echelon construction schema* (Bourbaki 1968, chap.4) within this schema.

Definition 6.1.5 (Structure) *A structure $\mathfrak{E}$ based on a set D is an ordered pair*

$$\mathfrak{E} = \langle D, r_\iota\rangle \tag{6.1}$$

where $D \neq \emptyset$ and r_ι represents a sequence of relations of degree n belonging to $\varepsilon(D)$. These relations are called the primitive elements of the structure.

For instance, a structure such as

$$\mathfrak{K} = \langle K, +, \cdot, 0, 1\rangle$$

can be used for representing fields (let us recall once more that although in the definition of structure we have mentioned relations only, we shall use operation and distinguished elements as well, according to the standard mathematics practice for, as we have seen, n-ary operations are $n + 1$ relations and distinguished elements are 0-ary relations—later we shall turn to consider *algebras*). For vector spaces over a field $\mathfrak{K}$, we usually write $\mathfrak{V} = \langle \mathcal{V}, \mathfrak{K}, +, \cdot \rangle$, where $\mathcal{V}$ is the set of vectors. In this case, in order to conform it to the above definition, we can write

$$\mathfrak{V} = \langle D, V, S, +_V, \cdot_{SV} \rangle,$$

where $D = \mathcal{V} \cup K$ (being K the domain of the structure of field $\mathfrak{K} = \langle K, +, \cdot \rangle$), where V and S are new unary relations for 'being a vector' and 'being a scalar' respectively, and the new operations $+_V$ and $\cdot_{SV}$ are adequate binary operations representing the vectors addition and the product of vectors by scalars respectively. In the same vein (although with more difficulties) we can consider structures for Euclidean geometry, differential manifolds, particle mechanics and so on.

We may also have structures of infinite order. If we call κ_D the cardinal associated to $\mathfrak{E}$, defined as

$$\kappa_D = \sup\{|D|, |\mathcal{P}(D)|, |\mathcal{P}^2(D)|, \ldots\},$$

being $|X|$ the cardinal of the set X, then, if κ_D is an infinite cardinal, we can construct in $\varepsilon(D)$ all ordinals less than κ_D. Since in such a construction we are taking all ordinals less than a certain ordinal, the ordinals here should be understood in Frege's sense, and not a la von Neumann.

Definition 6.1.6 (Order of a structure) *Let $\mathfrak{E} = \langle D, r_\iota \rangle$ be a structure. Its order,* $\mathsf{Ord}(\mathfrak{E})$*, is defined as follows: if there is* *a greatest order of the relations in r_ι, then the order of the structure is that greater order, and it is ω otherwise.*

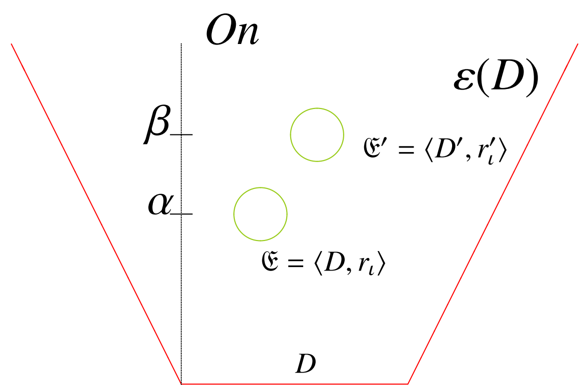

Figure 6.2: A scale based on D and two structures; $\mathfrak{E}'$ if of a higher order than $\mathfrak{E}$. On is the corresponding segment of the class of ordinals, and ordinals α and β are such that $\alpha < \beta$, so the order of $\mathfrak{E}'$ is greater than the order of $\mathfrak{E}$ (see the text below).

If the primitive relations of the structure are just relations having individuals of D as relata, we say that the structure is of *order*–1. The above structures for fields and vector spaces are order–1 structures, for their relations have as relata just elements of the domains. The same happens for groups and other standard algebraic and order structures (partial order, linear order, etc.). But we could suppose that there are structures whose relations relate not only individuals of D, but subsets of elements of D (equivalently in extensional contexts, properties of individuals). In this case we speak of *order*–2 structures. In this way, we can define *order*–*n* structures for any $n \in \omega$ without difficulty.

Thus, *order*–1 *structures* (as those dealt with by model theory) consider only order–1 relations, that is, those which have as relata the elements of the domain only, but not its subsets or other higher relations. More sophisticated structures necessary for treating empirical sciences (and mathematics) are not order–1 in this sense. Really, suppose the axiomatization of classical particle mechanics by McKinsey, Sugar and Suppes we have considered above. There, the primitive elements of the structure do not deal with just particles (elements of the set P—see section 4.1.1), but involve derivatives and other functions. So, a structure such as CPM (see equation 4.4) is not an order–1 structure. Yet, the structures for empirical science can be constructed *within* 'first-order' ZFC, that is, the Zermelo-Fraenkel set theory grounded on elementary logic, as put forward by Bourbaki in what respects mathematical structures (Bourbaki 1968, ch.4). So, the reader should realize that the order of a language and the order of a structure are distinct concepts, and this is why we are introducing the terminology *order–n* structures and relations to differentiate them. So, 'first-order' ZFC stands for that theory which has elementary logic as its underlying logic, while order–1 structures are those structures that have just order–1 relations as primitive elements. Particularly (let us emphasize this), the above definitions show that *we can consider order–n* ($n > 1$) *structures within the scope of first-order languages*.

As remarked by da Costa and Rodrigues [op.cit.], the usual mathematical structures can be reduced to structures embedded in this schema, yet sometimes this can be very difficult. The above definition of structure may encompass also structures having several sets as domains of individuals (just take D to be their union). Finally, sometimes we need to deal with relations of infinite degree, which (according to these authors) can also be reduced to the structures in the above sense.

6.2 Languages

An important remark is the following. Recall that, as we have said, we are working *within* ZFC. Hence, all we have at our disposal are the devices of $\mathcal{L}_\in$. As we have seen already (see the last chapter), in the intended interpretation the terms of $\mathcal{L}_\in$ stand for *sets* which, from an intuitive viewpoint, are collections "into a whole of definite and separate [*i.e.*, distinct] elements of our intuition or of our thought", as said Cantor. (see Fraenkel 1966, p.9). Then, when we construct a certain language, as those we shall consider below (for instance, the language $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ to be considered next), what kind of entities are their symbols? The answer is that they are terms of $\mathcal{L}_\in$, that is (in the intended interpretation), names for sets of ZFC.

What, then, is a language? Da Costa put the things clearly when he says that we usually believe that without language there is no logic, for the logic operators are usually applied to linguistic items such as sentences (da Costa 2007; we shall follow him in this section, with adaptations—but see also Gratzer 2008, and Barnes & Mack 1975). However, he says, sometimes we may wish to apply logic to handle other items that do not seen as linguistic, but as arbitrary objects, "such as believes and assumptions which sometimes do not involve language or, at least, well defined languages".[2] Thus, the 'linguistic' approach to logic is sometimes not so useful, and we can find a more general account (as he proposes) by means of Abstract Logics (sometimes called Universal Logics). Remember that the expression 'universal algebra' comes from Whitehead's book (from 1898) *Universal Algebra*, which means, roughly speaking, the study of algebraic structures (or *algebras*—see below) from a general ('universal') point of view, departing from the study or particular algebraic systems. Thus, we can see a language as an algebra, which can be defined within ZFC. Let us sketch this idea a little bit.

6.3 Algebras

So far we have considered a structure as something that can be written as $\mathfrak{E} = \langle D, r_\iota \rangle$. But of course this is not the only possible notation. In this section, we shall change our notation by considering structures of the kind

$$\mathfrak{A} = \langle A, f_\lambda \rangle, \tag{6.2}$$

where A is a non empty set and f_λ stands for a family of operations of rank n on A. That is, each member f of this family

[2] He also recalls that a similar thesis was proposed for instance by Russell.

associates one and only one element of A to each n-tuple of elements of A. If $\langle x_1, \ldots, x_n\rangle$ denotes one of these n-tuples and y the corresponding element, we write

$$y = f(x_1, \ldots, x_n)$$

to indicated that. If $n = 2$, we usually write $x_1 \, f \, x_2$ instead of $f(x_1, x_2)$ —as in elementary algebra, where we write $a + b$ and not $+(a, b)$. If $n = 0$, we say that f determines a fixed element of A.

Definition 6.3.1 (Algebra) *An algebra, or algebraic structure, is a structure*

$$\mathfrak{A} = \langle A, f_\lambda\rangle, \tag{6.3}$$

where $A \neq \emptyset$ and f_λ is a sequence of operations defined on A, each one of them of a definite finite rank n.

A *subalgebra* of $\mathfrak{A}$ is an algebra $\mathfrak{B} = \langle B, f'_\lambda\rangle$ such that $B \subseteq A$ and the f'_λ are the restrictions of the corresponding f_λ to B. Now let $X \subseteq A$ and let $[X]$ denote the collection of all subalgebras of $\mathfrak{A}$. Since $X \in [X]$, then $[X] \neq \emptyset$. The intersection $\bigcap[X]$ is again a subalgebra of $\mathfrak{A}$ and contains X. It is called the subalgebra *generated* by X. It is to be the smallest algebra over $\mathfrak{A}$ that contains X, sometimes denoted by $\langle X\rangle$ or by $\langle X\rangle_{\mathfrak{A}}$ if there is some risk of confusion.

Really, we should be quite careful here. Suppose we have a group, seen as an algebra $\mathfrak{G} = \langle G, *\rangle$, where $G \neq \emptyset$ and $*$ is a binary operation on G. If we simply take a subset $G' \subset G$ and an operation $*' = *|_{G'}$, the restriction of $*$ to G', may be that $\langle G', *' \rangle$ is not a group. But this does not happen if we describe a group by mentioning all the basic operations, namely, as $\mathfrak{G} = \langle G, *, ^{-}, e \rangle$, where $^{-}$ is an unary operation on G that associates to each $a \in G$ its inverse $\bar{a} \in G$ and e is the identity element of the group. In this case, $\mathfrak{G}' = \langle G', *|_{G'}, ^{-}|_{G'}, e \rangle$ (provided that $e \in G'$) is a group. From now on, our subalgebras will be always of this kind, that is, they shall 'preserve the structure'.

Any algebra has a *type of similarity*, or *signature* (please do not make confusion with the concept of type given before). Although we shall not introduce the definition here, it is quite intuitive. Take for instance the field of reals, $\mathcal{R} = \langle \mathbb{R}, +, \times, 0, 1 \rangle$. By considering the rank of the involved operations, we may say that this structure has type of similarity $(2, 2, 0, 0)$. Similarly, the structure $\mathcal{N} = \langle \omega, s, 0, +, \times \rangle$ for elementary arithmetics has type of similarity $(1, 0, 2, 2)$ —notice that only the rank of the operations are mentioned. Two algebras of the same type of similarity are called *similar*. An homomorphism from an algebra $\mathfrak{A} = \langle A, f_\lambda \rangle$ into a similar algebra $\mathfrak{B} = \langle B, g_\lambda \rangle$ is a mapping $h : A \mapsto B$ such that for all $a \in A$, $h(f(a)) = g(h(a))$. If h is bijective, then it is called an *isomorphism*, and in this case we write $\mathfrak{A} \equiv \mathfrak{B}$.

The following definition is attributed to G. Birkhoff (see Gratzer op.cit., p.162).

Definition 6.3.2 (Free algebra) *Let $\mathcal{K}$ be a class of algebras of the same signature and $\mathfrak{A} \in \mathcal{K}$. Let $X = \{x_i\}_{i \in I}$ be a subset* *of elements of A such that X generates $\mathfrak{A}$, and let $\phi : I \mapsto A$ such that $\phi(i) = x_i$. Then we say that $\mathfrak{A}$ is a free algebra over $\mathcal{K}$ with the elements of X as free generators it for any algebra $\mathfrak{B} = \langle B, g_\lambda \rangle \in \mathcal{K}$, and any mapping $\psi : I \mapsto B$, there is an homomorphism h from $\mathfrak{A}$ into $\mathfrak{B}$ such that $h(x_i) = \psi(i)$.*

The following diagram intends to clarify the definition using an easier notation.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{matrix} I \\ i \end{matrix} \arrow[r, "\phi", "\phi(i) = x_i"'] \arrow[dr, "\psi"', "\psi(i) = y_i"] & \mathfrak{A} \arrow[d, "h"', "h(x_i) = y_i"] \\
& \mathfrak{B}
\end{tikzcd}$$

Figure 6.3: Free algebras: $h \circ \phi = \psi$, that is, the diagram commute.

We can prove that the homomorphism h, when exists, is unique, and there are also necessary and sufficient conditions for the existence of free algebras which do not interests us here. Let us see now (although without full details) how we can identity formal languages with free algebras.

6.4 Languages as free algebras

Let us suppose a simple case to exemplify how can we see what a language is from an algebraic point of view. In order to make the ideas clear, I shall consider in parallel an example, taking the language of classical propositional logic, $\mathcal{L}_{\mathsf{CPL}}$.

Thus, suppose we have a set $X = \{x_i\}_{i \in I}$ (this set may be infinite) and another set f_λ whose elements will be denoted generically by f. I shall use for this set the same notation we used above to define an algebra. This second set plays the role of the operations of the algebra we shall define. In our sample case, $X = \{p_0, p_1, \ldots\}$ is the set of propositional variables, and $f_\lambda = \{\neg, \rightarrow\}$ are the propositional connectives (we could use whatever adequate set of connectives of course). We still suppose that there is a mapping $\mathsf{ar} : f_\lambda \mapsto \omega$, the arity function, which assigns a natural number to each element of f_λ, called its arity. For instance, $\mathsf{ar}(\neg) = 1$ and $\mathsf{ar}(\rightarrow) = 2$.

Now we define by induction a set $\mathcal{F}$ as follows:

(1) $F_0 := X \cup \{\langle f, x_1, x_2, \ldots, x_{\mathsf{ar}(f)}\rangle : f \in f_\lambda \wedge x_i \in X\}$. This is the set of the *atomic formulas*. The $(\mathsf{ar}(f) + 1)$-tuple $\langle f, x_1, x_2, \ldots, x_{\mathsf{ar}(f)}\rangle$ is written $f x_1 x_2 \ldots x_{\mathsf{ar}(f)}$. In our example, the atomic formulas are the propositional variables and the expressions of both forms $\neg p_i$ and $\rightarrow p_i p_j$. This last one may be abbreviated by $p_i \rightarrow p_j$.

(2) Now let α_i, $i = 1, \ldots$ denote atomic formulas. Then, the set of *complex formulas* is $F_1 = \{\langle f, \alpha_1, \alpha_2, \ldots, \alpha_{\mathsf{ar}(f)}\rangle : f \in f_\lambda \wedge \alpha_i \in F_1\}$, and again the tuples are abbreviated by $f \alpha_1 \alpha_2 \ldots \alpha_{\mathsf{ar}(f)}$. For instance, in our example, we may have as complex formulas expressions such as $\neg\neg p_i$, $\rightarrow \neg p_1 \rightarrow p_2 \neg p_3$, this last one being abbreviated by $\neg p_1 \rightarrow (p_2 \rightarrow \neg p_3)$, according to a standard notation. In our case study,

we can introduce other operational symbols by definition, such as the other standard connectives $\wedge$, $\vee$ and $\leftrightarrow$.

(4) In a general way, let $F_n := \{\langle f, \alpha_1, \alpha_2, \ldots, \alpha_{\mathsf{ar}(f)}\rangle : f \in f_\lambda \wedge \alpha_i \in F_{n-1}\}$, then

(5) $\mathcal{F} = \bigcup_{n\in\omega} F_n$.

We can prove that $\mathcal{A} = \langle \mathcal{F}, f_\lambda\rangle$ is a free algebra by adapting the proof given in Barnes & Mack (1975, p.5). We call a free algebra constructed this way a *language*.

Thus we can see how a language can be constructed in ZFC. The cases of first-order usual languages, higher-order languages and even infinitary languages can be treated the same way, although they will demand more details. Anyway, they can be treated as being certain free algebras. Thus, when we mention certain languages to speak of structures in the next section, they can be considered as constructed within ZFC.

6.5 Languages for speaking of structures

Of course we aim at to speak of structures and of all the objects of a scale. To do that in an adequate way, we consider two basic infinitary languages, termed (cf. da Costa and Rodrigues) $\mathcal{L}^{\omega}_{\omega\omega}(R)$ (or simply $\mathcal{L}^{\omega}(R)$) and $\mathcal{L}^{\omega}_{\omega\kappa}(R)$.

In general, an infinitary language $\mathcal{L}^{\eta}_{\mu\kappa}$, with $\kappa < \mu$ being infinite cardinals (or ordinals) and $1 \leq \eta \leq \omega$, enables us to consider conjunctions and disjunctions of $n \leq \mu$ formulas and blocks of quantifiers with $m < \kappa$ many quantifiers. The superscript η indicates the order of the language (first-order, second order, etc.). In both cases R is the set of the constants of the language. Thus, in $\mathcal{L}^{\omega}_{\mu\omega}(R)$ $(\omega < \mu)$ we may have infinitely many conjunctions and disjunctions of formulae, but blocks of quantifiers with finitely many quantifiers only. $\mathcal{L}^{\omega}_{\omega\omega}(R)$ is a higher-order language, suitable for type theory (higher-order logic). Standard first-order languages are of the kind $\mathcal{L}^{1}_{\omega\omega}$, so is $\mathcal{L}_{\in}$. Put in a more precise way,

Definition 6.5.1 (Order of a language) *A language $\mathcal{L}^{n}_{\mu\kappa}$, with $1 \leq n < \omega$, is called a language of order n. A language $\mathcal{L}^{\omega}_{\mu\kappa}$ is said to be or order ω.*

A language of order n contains only types of order $t \leq n$ and quantification of variables of types having order $\leq n-1$ (da Costa & Rodrigues 2007, p.8).

In order to exemplify how we can define a higher-order language using a frst-order language (such as $\mathcal{L}_{\in}$), let us sketch the language $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$, but we could consider whatever infinitary language $\mathcal{L}^{\omega}_{\mu\kappa}$, provided that the involved cardinals exist in ZFC (for instance, we couldn't use an inaccessible cardinal).[3]

The primitive symbols of $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ are the following ones:[4]

(i) Sentential connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\bigwedge$, and $\bigvee$.

[3] As we have seen in the last chapter, there are cardinals whose existence cannot be proved in ZFC, provided this theory is consistent. Inaccessible cardinals belong to this class.

[4] Of course we could use the above schema of free algebras to characterize this language, but this would demand a lot of artificiality and will not conduce to nothing really relevant. The important thing is to acknowledge that the languages we will consider can be treated as free algebras.

(ii) Quantifiers: $\forall$ and $\exists$

(iii) For each type t, a family of variables of type t whose cardinal is ω.

(iv) Primitive relations: for any type t, a collection of constants of that type (possibly some of them may be empty). The collection of these constants form the set R.

(v) Parentheses: left and right parentheses (‘(’ and ‘)’), and comma (‘,’).

(vi) Equality: $=_t$ of type $t = \langle t_1, t_2 \rangle$, with t_1 and t_2 of the same type.

Variables and constants of type t are *terms* of that type. If T is a term of type $\langle t_1, \ldots, t_n \rangle$ and $T_1, \ldots, T_n$ are terms of types $t_1, \ldots, t_n$ respectively, then $T(T_1, \ldots, T_n)$ is an *atomic formula*. If T_1 and T_2 are terms of the same type t, then $T_1 =_{\langle t_1, t_2 \rangle} T_2$ is an *atomic formula*. We shall write $T_1 = T_2$ for this last formula, leaving the type of the identity relation implicit. If α, β, α_i are formulas ($i = 1, \ldots$), then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \to \beta$, $\bigwedge \alpha_i$, and $\bigvee \alpha_i$ are formulas. Then, we are able to write formulas with denumerably many conjunctions and disjunctions. Furthermore, if X is a variable of type t, then $\forall X\alpha$ and $\exists X\alpha$ are also formulas (and only finite blocks of quantifiers are allowed). These are the only formulas of the language. The concepts of free and bound variables and other syntactic concepts can be introduced as usual.

Now let $\mathfrak{E} = \langle D, r_\iota \rangle$ be a structure, where $r_\iota \in R$, that is, the primitive relations of the structure are chosen among the constants of our language $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$. Then, $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ can be taken as a language for $\mathfrak{E} = \langle D, r_\iota \rangle$, provided that $\kappa_D = \omega$ (recall that κ_D is the cardinal associated to $\mathfrak{E}$). Still working within (say) ZFC, we can define an interpretation of $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ in $\mathfrak{E} = \langle D, r_\iota \rangle$ in an obvious way, so as what we mean by a sentence S of $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ (a formula without free variables) being *true* in such a structure in the Tarskian sense, that is,

$$\mathfrak{E} \models S. \tag{6.4}$$

In the same vein, we can define the notion of *validity*. A sentence S is valid, and we write

$$\models S,$$

if $\mathfrak{E} \models S$ for every structure $\mathfrak{E}$.

Important to emphasize that we are describing the language $\mathcal{L}^{\omega}_{\omega_1\omega}(R)$ using the resources of some set theory such as ZFC. This way, we can speak of denumerable many variables, for instance, in a precise way. In this sense, any symbol of $\mathcal{L}^{\omega}_{\omega\kappa}(R)$, as we have remarked already, can be seen as a name for a set. Thus, '(' (the left parenthesis), for example, names a set.

6.5.1 The language of a structure

Now let $\mathfrak{E} = \langle D, r_\iota \rangle$ be a structure, while $\mathsf{rng}(r_\iota)$ denote the range of r_ι. Remember that r_ι stands for a sequence of relations of the scale $\varepsilon(D)$, that is, it is a mapping from a finite ordinal into a collection of relations in the scale. Thus $\mathsf{rng}(r_\iota)$ stands for just the set of these relations. So, $\mathcal{L}^{\omega}_{\omega\omega}(\mathsf{rng}(r_\iota))$ (= $\mathcal{L}^{\omega}(\mathsf{rng}(r_\iota))$) *the basic language of the structure* (it is not the only one, for other stronger languages encompassing it could be used instead). In this case, we can interpret a sentence containing constants in $\mathsf{rng}(r_\iota)$ in $\mathfrak{E} = \langle D, r_\iota \rangle$, and to define the notion of truth for sentences of this language according to this structure in an obvious way.

Digression For certain applications in science, sometimes it is better to consider *partial relations* as the primitive relations of a certain structure. A relation (say, a binary one) R on a set A is partial if there are situations where we cannot assert neither that aRb not that $\neg(aRb)$ (see da Costa & French 2003 for all the philosophical discussion on this topic). In this case, the notion of truth is changed to *partial truth*, a concept that generalizes Tarski's approach and seems to be more adequate for empirical sciences. But we shall not touch this point here (but see da Costa & French 2003).

6.6 Definability and expressive elements

Now we wish to understand when an object of a scale $\varepsilon(D)$ is definable in a structure $\mathfrak{E} = \langle D, r_\iota \rangle$ by a formula of $\mathcal{L}^{\omega}(\mathsf{rng}(r_\iota))$

so as when an element of the scale is expressible in the structure with respect to a sequence of objects of the scale.

Definition 6.6.1 (Definability of a relation) *Let R be a relation of type $t = \langle t_1, \ldots, t_n \rangle$ and $\mathfrak{E} = \langle D, r_\iota \rangle$ a structure. We say that R is definable in $\mathfrak{E}$ if there exists a formula $F(x_1, x_2, \ldots, x_n)$ of $\mathcal{L}^\omega(\mathsf{rng}(r_\iota))$ whose only free variables are $x_1, \ldots, x_n$ of types $t_1, \ldots, t_n$ respectively, such that in $\mathcal{L}^\omega(\mathsf{rng}(r_\iota)) \cup \{R\}$, the formula*

$$\forall x_1 \ldots \forall x_n (R(x_1, \ldots, x_n) \leftrightarrow F(x_1, x_2, \ldots, x_n))$$

is true in $\varepsilon(D)$.

For instance, for each type t we can define an identity relation $=_t$ as follows. Let Z be a variable of type $\langle t \rangle$, then we can easily see that for suitable structures and scales, the following is true:

$$\exists! I_t \forall x \forall y (I_t(x, y) \leftrightarrow \forall Z(Z(x) \leftrightarrow Z(y))).$$

We may call I_t the identity of type t, and write $x =_t y$ for $I_t(x, y)$, what intuitively means that identity is defined by Leibniz Law (see chapter 8), as it is usual: just re-write the above definition as follows:

$$x =_t y := \forall Z(Z(x) \leftrightarrow Z(y)).$$

Usually, we suppress the index t and write just $x = y$, leaving the type implicit (= is of type $\langle t \rangle$, while x and y are both of type t). This kind of definability, which involves structures and scales is called *semantic definability*, and goes back to Tarski.

Here it is another example. Suppose the language $\mathcal{L}_\in$ of 'pure' set theory ZF. As is is well known, this language has $\in$ as its only non-logical constant. If we aim at to define the subset relation $\subseteq$, we can do it in the extended language $\mathcal{L}_\in \cup \{\subseteq\}$ by showing that the formula

$$\forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y))$$

is true in any structure built in ZF. Below we shall give some further examples.

Another important case is the next one, also involving a semantic definability.

Definition 6.6.2 (Definability of an object) *Let $\mathfrak{E} = \langle D, r_\iota \rangle$, $\varepsilon(D)$, and $\mathcal{L}^\omega(\mathsf{rng}(r_\iota))$ as above. Given an object $a \in \varepsilon(D)$ of type t, we say that it is $\mathcal{L}^\omega(\mathsf{rng}(r_\iota))$-definable or definable in the strict sense in $\mathfrak{E} = \langle D, r_\iota \rangle$ if there is a formula $F(x)$ in the only free variable x of type t such that*

$$\mathfrak{E} \models \forall x (x =_t a \leftrightarrow F(x)). \tag{6.5}$$

The case of the well-order on the reals, mentioned above, shows that, taking into account the last definition, the least element of (0,1), that cannot be definable by a formula. Let us consider a 'positive' example. Let $\mathfrak{N} = \langle \omega, +, \cdot, s, 0 \rangle$ be an of order–1 structure for first-order arithmetics. In order to define a natural number (any one) we need just a finitary language, say $\mathcal{L}^{\omega}_{\omega\omega}(R)$ with $R = \{+, \cdot, 0, s\}$. Then, it is easy to see that (in an obvious abbreviate notation)

$$\mathfrak{N} \models \forall x(x = n \leftrightarrow x = ss \ldots s(0)),$$

If we consider a suitable infinitary language, say $\mathcal{L}_{\omega_1\omega}$ (where we can admit denumerable infinite conjunctions and disjunctions), we can inside the parentheses the formula we abbreviate by

$$x \in \omega \leftrightarrow x = 0 \vee x = 1 \vee \ldots, \tag{6.6}$$

which permits us to define not a particular natural number, but the notion of 'being a natural number'. An important remark: the expression (6.6) is not a formula strictly speaking (for the dots do not make part of the language), but abbreviates a formula of $\mathcal{L}_{\omega_1\omega}$).

An illustrative case is the following one. We know that within ZFC (supposed consistent), the set $\mathbb{R}$ of the reals is not denumerable. This means that we cannot find a mapping (a set in ZFC) that maps the reals onto the natural numbers. Thus, using standard denumerable languages, we do not have sufficient *names* for the reals, and . But if we use a suitable infinitary language $\mathcal{L}_{\mu\kappa}$ (for suitable ordinals μ and κ) we can find a name for each real, so we can define all of them by the condition given in definition (6.6.2). This shows that definability and other related concepts depend on the employed language.

Here it is another interesting case related to the above definitions. Using the axiom of choice we can show that every set is well ordered (by the way, let us insist, this sentence is equivalent to the axiom of choice). For instance, the set ω of natural numbers is well ordered by the usual less-than relation $\leq$. We can define the usual order $\leq$ as follows:[5] $a \leq b := \exists c(b = a + c)$. This *usual* order relation, notwithstanding, does not well order the set $\mathbb{Z}$ of the whole integers, for the subset $\{\ldots, -2, -1, 0\}$, for instance, has not a least element. But if we order $\mathbb{Z}$ by writing it as $\{0, -1, 1, -2, 2, \ldots\}$, then it is well-ordered by this order, termed $\leq_1$, which can be defined by $a \leq_1 b := (|a| < |b|) \vee (|a| = |b| \wedge a \leq b)$, where $\leq$ is the usual order relation and $<$ is defined as $a < b := a \leq b \wedge a \neq b$.

Obviously the 'usual' relation (defined on $\mathbb{R}$) does not well order the set of reals (for instance, an open set (a, b), with $a < b$, has not a least element). But, can we find a well order on such a set? According to the axiom of choice, we can assume that this order does exist, since its existence is consistent with ZFC (supposed consistent). The problem is that it can be proven that the ZFC axioms (plus the so-called generalized continuum hypothesis) are not sufficient to show that this order can be definable by a formula of its language.[6] In the same vein, we cannot define the least element of a certain subset of reals (say, the open interval $(0, 1)$) in the sense of definition

[5] Of course this definition can be conformed to definition 6.6.1.

[6] The interested reader can check theorem 4.11 of S. Feferman's paper and the remark at p.342, just after the proof of the theorem; see `http://matwbn.icm.edu.pl/ksiazki/fm/fm56/fm56129.pdf`.

(6.6.2). All of this show that the notions of definability and expressibility, among others, depend on the language and on the theory we are supposing.

6.7 On the new symbols

In our formalization of ZFC, we have used individual constants to facilitate our mentioning of certain particular sets, for in doing that we would have a denumerable set of 'names' to them. This is a matter of choice. Most authors don't use individual constants, but just individual variables. Since it is supposed that this move doesn't change the set of theorems, we regard the resulting theories as being the same, yet their languages differ. In this section, we shall make some few remarks we regard are of philosophical importance which, although they are well known by the logician, can be not so well understood by the general philosopher interested in foundational issues. So, let us suppose for a moment that our language $\mathcal{L}_\in$ has no individual constants. To avoid any confusion, we shall term it $\mathcal{L}_{\in^-}$.

Due to our new convention, the only non-logical symbol of $\mathcal{L}_{\in^-}$ is $\in$. So, how can we refer to a particular relation R? (the same holds for a particular object whatever, such as a structure named $\mathfrak{A}$ for instance). This is a common practice in the mathematical discourse. Really, in geometry we usually say "Let A be a point in a line r", making use of individual letters for naming particular entities, the point and the line. How can we explain this move of naming objects with constants that do not appear as primitive concepts of the employed languages? There are two answers to this challenge.

The first is that we simply extend the language $\mathcal{L}_{\in^-}$ with additional constants to name the objects we intend to make reference to. Thus, we can extend $\mathcal{L}_{\in^-}$ with symbols of three kinds: (1) additional individual constants, (2) new predicate symbols, and (3) new operation symbols. In whatever situation, we must be sure that the so-called *Leśniewski's criteria* are being obeyed (Suppes 1957, ch.8), namely, the *Criterion of Eliminability* and the *Criterion of Non-Criativity*. The first says, in short, that the new symbols can be eliminated. That is, the formula S introducing the new symbol must be so that whenever a formula S_1 with the new symbol occurs, there is another formula S_2 without this symbol such that $S \to (S_1 \leftrightarrow S_2)$ is a theorem of the preceding theory (without the new symbol). The second criterion says that there is no formula T in which the new symbol does not occur such that $S \to T$ is derivable from the preceding theory, but T is not so derivable. In other words, no new theorem previously unproved, and stated in terms of the primitive symbols and already defined symbols only can be derived. In our case, we can add the desired symbols, say 'R' for the relation in the above definition of definability of a relation (see definition 6.6.1), once we grant that the Leśniewski's conditions hold (which of course we suppose here).

The another alternative is to work with $\mathcal{L}_{\in^-}$ proper, and regard all other symbols as metalinguistic abbreviations. This is 'more economic', and we usually do it for instance when defining (in $\mathcal{L}_{\in^-}$) the concept of subset, posing for instance that $A \subseteq B := \forall x(x \in A \to x \in B)$. The new symbol $\subseteq$ does not make part of the language $\mathcal{L}_{\in^-}$, but belongs to its metalanguage, and the expression $A \subseteq B$ simply abbreviates a sentence of $\mathcal{L}_{\in^-}$, namely, $\forall x(x \in A \to x \in B)$. We can understand this move as enabling us to use an *auxiliary constant* (say, 'R'), provided that the object which it will name exists (the proof of its existence is called *theorem of legitimation* by Bourkaki (1968, p.32).[7] In the example, given two sets A and B, we realize that all elements of A are also elements of B, hence we are justified to write $A \subseteq B$ for expressing that (as a metalinguistic abbreviation). In logic, we usually express that by the so called *method of the auxiliary constant*, which may be formulated as follows. Let c be a constant that does not appear in the formulae A or B. Assume that we have proven that $\exists xA$ (the theorem of legitimation). If we have also proven that $A[^x_c] \vdash B$, where $A[^x_c]$ stands for the formula obtained by the substitution of c in any free occurrence of x, then $\vdash B$ as well.

But we need some care even here. Suppose we wish to refer to real numbers. We cannot name all of them in the

[7] This is essentially what Mendelson calls 'Rule C'; see Mendelson 1997, p.81.

standard denumerable language $\mathcal{L}_{\in^-}$. But in general we can name a particular real number, say by calling it *zero*. But there are real numbers that cannot be named even this way, as we have seen above when we have discussed the well-order of the reals. This poses an interesting question regarding empirical sciences. In constructing a mathematical model of a physical theory, we suppose we *represent* physical entities, say quantum objects and the properties and relations holding among them, in the mathematical framework we have chosen, say ZFC set theory. How can we ensure that this really makes sense? For instance, in some common interpretations of quantum mechanics (the Copenhagen interpretation), we really cannot (with any sense) make reference to quantum entities out of measurement. Out of measurement, quantum entities are no more than sets of potentialities or possible outcomes of measurement, to use Paul Davies' words in his Introduction to Heisenberg 1989 (p.8). Let us give an example we shall discuss also below. Suppose we are considering the two electrons of an Helium atom in the fundamental state.

The anti-symmetric wave function of the join system can be written as

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle). \tag{6.7}$$

where $|\psi_i\rangle$ $(i = 1, 2)$ are the wave functions of the individual electrons. Notice that we need to label them by '1' and '2' for our languages are *objectual* —we speak of objects.[8] But we need to make this labeling to be not compelled with individuation, so we use (in this case) anti-symmetric functions, for the if $|\psi_{21}\rangle$ stands for he wave-function of the system after a permutation of the electrons, then $\||\psi_{12}\rangle|^2 = \||\psi_{21}\rangle|^2$, that is, the relevant probabilities are the same. For now, what import is that this function cannot be factored, giving a particular description of the electrons. Only after a measurement, say of the component of their spin in a given direction, is that the wave function *collapses* either in $|\psi_1\rangle|\psi_2\rangle$ or in $|\psi_2\rangle|\psi_1\rangle$, then indicating, say, that electron 1 has spin up in the chosen direction, while electron 2 has spin down in the same direction or the other way around. But, before the measurement, nothing can be said of them. This implies that when we say, for instance, that there are two electrons, one *here* and another *there*, we are already supposing a measurement, thus begging the question concerning their individuation.

Situations such as this one are puzzling if we consider the underlying mathematics as the classical one. For suppose we aim at to define the electron of an Helium atom that has spin up in a given direction. According to definition (6.6.2), we need to find a formula $F(x)$ so that, if we denote that electron by a, we can prove that the following formula

$$\forall x(x = a \leftrightarrow F(x)) \tag{6.8}$$

[8]Toraldo di Francia says that *objectuation* is a primitive act of our mind; see his 1981 book, p.222.

is true in an adequate 'quantum structure'. But, what would be taken to be $F(x)$?

Really, according to standard set theory (ZF), when we say that there is an electron here and another there, they are already distinct entities, and we are presupposing a kind of realism concerning these entities. In other words, in supposing *two* entities within ZF, we are really begging the question concerning their individuation.[9]

These remarks, we think, point to an interesting philosophical problem of studying the definability of physical 'objects' and relations, but this point will be not discussed here too.

[9]This is, in my view, the mistake made by Muller and Saunders (2008) and by Muller and Seevincki (2009) in discerning quantum entities; in assuming the mathematical framework as being ZF, they are already assuming that the represented entities are either identical (that is, they are the very same entity) or that they are distinguishable, and this is not a characteristic of quantum objects proper, but of the mathematics they have employed.

Chapter 7

Back to empirical sciences

Going back to empirical science, we think that it is time to address some few comments on general philosophy of science where the notion of mathematical structures seem to be relevant. The discussion of course cannot be detailed, so we begin by considering just a case, but an important and updated one, namely, the recent discussions on structural realism. After this discussion, we turn to consider the possibility of leaving 'classical frameworks'.

7.1 'Full-blooded' structural realism

Philosophers have recently considered two versions of structural realism: the epistemological version (ESR), roughly speaking, says that all we can know (about the world) are structures, while the ontological version (OSR) says that all there is are structures (for details, see French & Ladyman 2003). Here we would like to advance an alternative view, perhaps closer to the epistemological version: that we *construct*, at a rigorous mathematical level according to the above schema, in order to get a view of the world, are structures, mathematical structures. Let us call this view *Full-blooded Structural Realism (FSR).*[1] And here, contrary to what happens with OSR, where it is difficult to say what a structure is (see below), we really can consider that the structures we elaborate are as those as described above. But of course much need to be done to make these ideas clear.

It should be remarked that according to the proponents of OSR, since all things would be structures, we need a concept of mathematical structure whose relations do not involve relata other than structures themselves. In other words, what is demanded are relations without the relata. But it easy to see why such a definition cannot be done within standard set theories like (extensional) ZFC. Suppose we wish to define a binary relation on a (given) set A such that a and b are

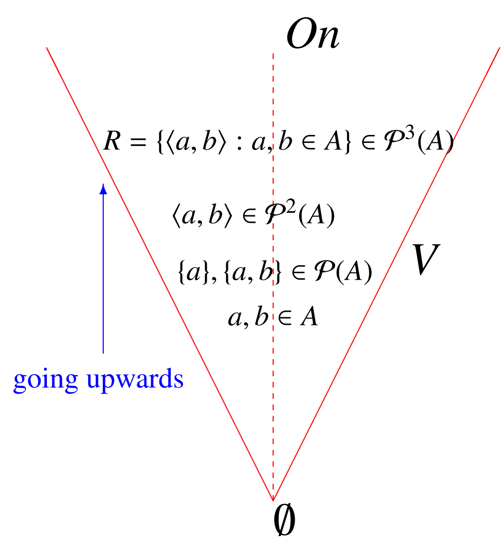

Figure 7.1: In ZF, the relations on a set A are constructed from the elements of A going upwards in the hierarchy.

[1]This terminology was suggested by Otávio Bueno.

elements of A (we are working in an extensional set theory, where the axiom of extensionality holds by hypothesis—our theory is ZFC). Surely a and b could be ur-element (if we work in ZFU). It doesn't matter which is the case. In both theories (ZFC and ZFU, formulated as usual), given a and b (see figure 7.1), we may get by the pair axiom the set $\{a, b\}$, so as the singletons $\{a, a\} = \{a\}$ and $\{b, b\} = \{b\}$. The next step (note that we are going upwards in the universe, getting sets of greater ranks) we get (again by the pair axiom) the set $\{\{a\}, \{a, b\}\}$. But this set is written (Wiener-Kuratowski's definition) $\langle a, b\rangle$. We can do this with other elements of A of course, and we can, with other axioms of ZFC (or ZFU), take collections of these ordered pairs, that is, a binary relation on A. In other words, from A, we get in the scale based on this set the sets $A \times A$, $\mathcal{P}(A \times A)$, and then we chose an element of this last set satisfying some desired properties, such as being reflexive (all pairs $\langle x, x\rangle$ must be there, for all $x \in A$).

This shows that in set theories such as those we are supposing here (and which are assumed in the standard literature), there are no relations without the relata, that is, 'pure' relations as the proponents of the OSR wish. The problem of finding a right definition of structure that fits their claims remains open (I guess that perhaps Tarski's set theory grounded in his theory of relations—see Tarski & Givant 1987—can be used here, but this is still an open problem).

But, by adopting PSR, we don't need to go so far as Max

Tegmark when he says that "our universe is not just described by mathematics—it is mathematics" (Tegmark 2007). I believe we really don't know what the universe is, but all approaches we provide are (*in potentia*) mathematically driven models, elaborated in terms of set-theoretical structures.[2] Of course this doesn't imply neither that all we know are structures (epistemologial structural realism) nor that all there is are structures (ontological structural realism), but that we know the world, or at least we so suppose, by constructing structures. A slogan for FSR would be read as saying that all we know in science is through structures, that is, in order to understand, we elaborate conceptual networks about our descriptions of the world by means of (in principle) mathematical structures that save the appearances or, as we prefer to say, which are quasi-truth (da Costa & French 2003). In this sense our position has resemblance but departs from van Fraassen's empiricists structuralism, according to which "all we know *through* science is structure" (van Fraassen 2008, p.238). In fact, according to van Fraassen, "scientific theories give us information only about the structure of processes in nature, or even that all we know is structure" (van Fraassen 1997), while in FSR we claim that we know the processes in nature by means of (mathematical) structures linked to some interpretation. And, when we try to provide an interpretation to the

[2]Of course I am ignoring here higher-order logics and category theory, but they could be offered as alternatives.

symbols we used to elaborate our theories, the very nature of the models of set theory enter the scene. But of course much would be done to develop such a philosophical view, so we postpone the issue to a future work.

7.2 Leaving the classical frameworks

So far we have been restricted to what can be constructed within ZFC, its underlying logic included, which we report as 'the classical' framework. But of course we could investigate some ways to go out of this schema. The possibility of even to (wittingly) think of a departure from standard mathematics (or logic) is possible only if we have our theories put axiomatically. Ever since the XIXth century we were faced with several kinds of geometries distinct from Euclidean geometry, as it is well known. For instance, in Euclidean geometry, we can prove (using the so called postulate of Archimedes) that two triangles having the same bases and the same altitude have equal areas. This is false in non-archimedian geometry, as Hilbert has shown (Hilbert 1950, p.40). But we wish to consider this question from another point of view, namely, from a possible exchange of the set theory employed, where the relevant structures are built. Let us consider this possibility a little in this section.

Non-classical logics were developed during the XXth century, as is well known, and although they quite important, we shall not consider them here. Even in set theory the rise of 'non-cantorian' frameworks are today widespread. Really, due to results such as those of Gödel and Cohen, among others, we know today a series of techniques that enable us to construct (potentially) infinitely many 'models' of a set theory such as ZF in which some of the 'classical principles', such as the axiom of choice, and the continuum hypothesis, do not hold. These *non-cantorian set theories* (Cohen's words) are today quite common in discussions involving set theory proper, but they are still far from the interest of the general philosopher of science. In this section we shall present some few details on this direction, trying to show that the consideration of non-classical logics and alternative mathematics would be part of the philosophical discussion. Of course I need to beg the reader to understand that I can't provide here all the details, so I would be imprecise in some aspects. But I hope the general ideas can be understood.

Let's begin with a fact involving the foundations of quantum mechanics.[3] In quantum theory, position and momentum operators are quite relevant. In the Hilbert space $L^2(\mathbb{R})$ of the equivalence classes of square integrable functions, they are unbounded operators. Just to recall, if A is a linear operator, then A is unbounded if for any $M > 0$ there exists a vector α such that $\|A(\alpha)\| > M\|\alpha\|$. Otherwise, A is bounded. But the american logician Robert Solovay proved that if ZF is

[3] I refer to non-relativistic quantum mechanics when no further indications are given.

consistent, and if DC stands for a weakened form of the axiom of choice entailing that a 'countable' form of the axiom of choice can be obtained. (In particular, if $\{B_n : n \in \omega\}$ is a countable collection of nonempty sets, then it follows ZF plus DC has a model in which each subset of the real numbers is Lebesgue measurable (using the full axiom of choice, we can show that there are subsets of the set of real numbers which are not Lebesgue measurable).[4] Let us call 'Solovay's axiom' (AS) the statement that "Any subset of $\mathbb{R}$ is Lebesgue measurable". Then, in the theory ZF+DC+AS (termed 'Solovay's set theory'), it can be proven that any linear operator is bounded —see Maitland-Wright 1973. Thus, if we use Solovay's theory instead of standard ZF to build quantum structures, how can we consider unbounded operators in quantum mechanics?

Here is another example also involving Robert Solovay. One of the fundamental theorems in quantum mechanics is Gleason's theorem (it does not matter us here to formulate it). The theorem shows the existence of certain probability measures in separable Hilbert spaces. Solovay obtained a generalization of the theorem also for non-separable spaces, but it was necessary to assume the existence of a *gigantic orthonormal basis* whose cardinal is a measurable cardinal. But the existence of measurable cardinals cannot cannot be proven in ZF set theory (supposed consistent). Thus, in order to get the generalization,

[4] It does not matter for us here what a Lebesgue measure is. Intuitively speaking, it generalizes the usual notion of measure –lengths, areas, etc.

we need to go out of ZF. These examples show that for certain considerations, it is extremely important to consider the mathematical framework we are working in.

These examples show that there are alternative mathematical (and or course, logical) frameworks in which the philosophical discussions could take place, and depending on the theory used, the results may change. It is worthwhile to emphasize that philosophers of science should pay attention to this plurality of logics (recall the variety of non-classical logics there are) and mathematical frameworks at our disposal. Then, in considering these alternative mathematical frameworks, we should agree with da Costa and Doria when they say that

> "[i]t is therefore natural to enquire whether the use of one or another of those models of set theory [they refer to those 'models' of set theory built with Gödel and Cohen's techniques] have consequences for the general theory of mathematical structures; in particular, whether the employment of a certain kind of model has consequences for the axiomatization of physical theories." (da Costa & Doria 2008, p.71)

In chapter 8 we present a more elaborated example taken from present day philosophy of physics. But before that, let us make some comments on the nature of the 'models' of ZFC.

Chapter 8

Quantum physics and the Leibniz's principle

Leibniz's Principle of the Identity of Indiscernibles (henceforth, PII) plays an important role in present day philosophy of physics. Those philosophers who have considered it in connection with quantum physics are divided up into two classes: those who think that quantum physics violate the principle (at least in some of its forms–see below) and those who sustain that PII remains intact even with the consideration of *indiscernible* (or *indistinguishable*) quantum objects. In this chapter we shall consider this topic in order to show a possible way to go out of standard set theories to deal with some questions, such as this one. Our discussion will be not exhaustive, and for further historical and philosophical details, we suggest French & Krause 2006. The mathematical theory presented here (termed *quasi-set theory*) and all the related discussion is taken from French & Krause 2010 (forthcoming).

The first problem, arisen from the discussion we have made above, is to say precisely of what we are speaking about. What we understand by Leibniz's principle? Of which 'quantum mechanics' are we speaking about? We shall address some remarks on these topics in what follows.

8.1 The Leibniz's principle

Leibniz's metaphysics is of course a difficult subject. We shall not discuss it here, but just make some remarks to what became known as his *Principle of the Identity of Indiscernibles*, PII for short.[1] When he developed his ideas, present day mathematical logic was not known, although he is regarded as the first philosopher that had the intuition that Aristotelian logic was not adequate to give us an account of the kind of reasoning done within mathematics. In what respects PII, we can found in several parts of his writings references to the idea that there cannot be two absolutely indiscernible objects, two objects which, as he referred to, differ *solo numero*.

In recent literature on the philosophy of physics, there is a lot of discussion on the validity of Leibniz's principle in quantum physics. The fact that quantum entities may be indiscernible is against the thesis that there are no indiscernible objects, which is the essence of the principle. The problem be-

[1] There are good recent works on Leibniz's notions of individuals and individuation, as for instance Nachtomy 2007, and Cover & O'Leary-Hawthorne 2004.

gin with the definitions of the involved concepts. Informally speaking, indiscernible things are those entities that partake all their properties, or attributes, while identical things are the very same entity (although physicists call 'identical' just what we have termed the indiscernible particles).[2] Leibniz claimed that indiscernible things are identical. But the precise meaning of the concepts of 'identity', 'indiscernibility', 'property' etc. are not clear and depend on the employed language and logic. We shall not discuss this topic here, for we have addressed much of this discussion in French & Krause 2006. Here, we will just mention some basic facts in order to give to the reader an idea for the understanding of what follows.

In $\mathcal{L}_\in$, how could we define indiscernibility and identity? ZF encompasses a theory of identity, given by its logical axioms (7) and (8) of page 56 and the principle of extensionality (ZF1) (the same axioms hold for ZFC and for ZFU with a suitable modification in the axiom of extensionality, for it holds only for sets, and not for the ur-elements).[3] Thus, if 'two' sets have the same elements, they are identical. The converse is an immediate consequence of axiom (8). If there are ur-elements, they are identical if they belong to the same sets. Since for any set or ur-element a, the only element that belongs to the set $\{a\}$ is a itself, it results that in ZF, ZFC or ZFU any object is identical with itself, and with no other object. Summing up,

[2]More precisely, 'identical' particles are those particles that agree with respect to all their 'intrinsic' (independent of the state) properties, according to Jauch (1968, p.275).

[3]The axiom of extensionality of ZFU says that *sets* that have the same elements are identical.

there are no indiscernible objects. Of course we can speak of indiscernibility only relative to a certain group of properties, which we identify with sets: any property determines a class, and in particular a set. The axiom of extensionality says that we don't need to go beyond sets to conclude that two objects are identical, it suffices that they belong to the same sets. So, in ZF, ZFC and ZFU all objects are, in a sense, individuals, if we define an individual as an object which is indistinguishable only from itself. Thus, any object described by these set theories are individuals. But quantum physics seens to present a challenge to this conclusion. Bosons, that is, those elementary particles that may be in the same state, hence being absolutely indiscernible by all means, may be indiscernible, but they are of course the same object. There are philosophers who sustain that this result holds also for fermions (see French & Krause 2006, ch.4). This is disputable, and there is no philosophical or logic argumentation that can prove that, for it is a physical question. But let us suppose that we have the grounds for sustaining that the principle does not hold in the quantum realm, that is, that there may be indiscernible but not identical objects (quantum entities). How can we deal with them within standard mathematics and logic? The answer is straightforward. We 'confine' the discussion to a deformable (non-rigid) structure, where there are automorphisms other than the identity function. This is, in a certain sense, what we do when we use symmetric and anti-symmetric vectors/wavefunctions to represent the relevant states of composite systems in quantum physics. We start with distinct things, and then we postulate that their states are described by symmetric or anti-symmetric functions, so that no distinction can be noticed (even in the case of fermions, which due to the Pauli's exclusion principle cannot be in the same state, we never can say which is which). This is of course a trick, and physics goes well using it. In the case of non-rigid structures, we know already that in ZF any structure is substructure of a rigid structure, so that the initial indiscernibility revels the very nature of the individuals in the extended structure. No way. To deal with legitimate indiscernible things, we must proceed as the Pytagorians, when they realized that it was necessary to leave the realm of rational numbers: we need to leave standard logic and classical set theories and to develop a new kind of mathematics, and below we present one.

8.2 Quasi-set theory

Quasi-set theory was proposed to handle collections of indistinguishable (or indiscernible) objects. Informally speaking, these objects would be so that they can form aggregates (quasi-sets) having a (quasi-)cardinal (perhaps greater than one) but so that they would be not entities that 'maintain' their distinguishability; once mixed among others, one would no more be able to tell which is which, and so that it would be not possible to attach them a label, a proper name. This idea is grounded on Schrödinger's claim that "you cannot mark an electron, you cannot paint it red" (Schrödinger 1953), although this analogy should be pursued with care (see below). For the sake of intuitiveness, the 100 Smiths of the film Matrix Reloaded provide an example, for they are 'all the same'. Is there any similarity with electrons, protons and other quantum objects?

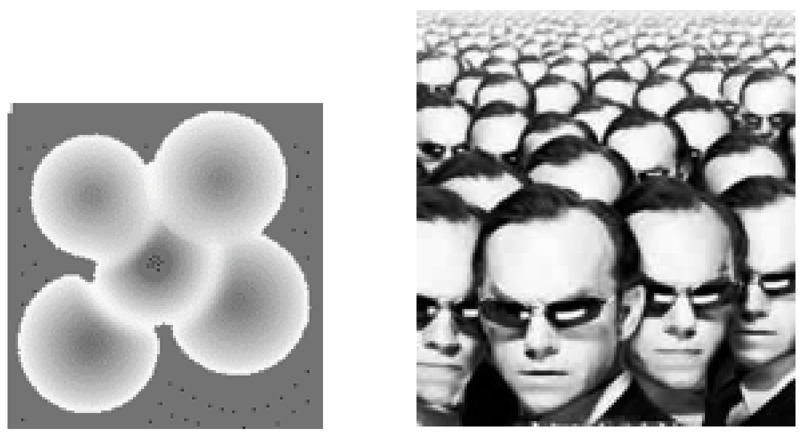

Figure 8.1: Are there differences either among cold atoms or among the 100 Smiths?

Important to say that the theory does not compromises us with an interpretation of particles in quantum theory. Whatever 'entities' that can be considered as indiscernible (this concept must be explained) can (in principle) be covered by the theory. The motivation to build such a theory is grounded on a possible reading of the behaviour of quantum objects, at least with respect to non-relativist quantum mechanics (QM for short)—although it can be extended to cope with certain aspects of quantum field theory (QFT) as well (French & Krause

2006, ch.9; Domemech et al. 2008, 2009). The aim of quasi-set theory is to pursue the indiscernibility of the basic objects supposed by the theory *right from the start*, as suggested by Heinz Post (see French & Krause op.cit. for the related history). Thus, since there is a sense in saying that all objects represented in ZF (ZFC, ZFU) are individuals, for they can always be (in principle) distinguished from any *other* individual,[4] to cope with Post's ideas we must go out of ZF and find an alternative theory. We propose the theory $\mathfrak{Q}$ sketched below as such a theory.

8.2.1 Language and basic definitions

The theory $\mathfrak{Q}$ is based on ZFU–like axioms, but encompasses two kinds of ur-elements and not only one, as in standard ZFU. Its language $\mathcal{L}_{\mathfrak{Q}}$ has the following categories of primitive symbols: (i) propositional connectives $\neg$ and $\rightarrow$, (ii) the universal quantifier $\forall$, (iii) a denumerable collection of individual variables $x_1, \ldots, x_n$ (we shall use $x, y, \ldots$ to refer to them), (iv) two binary predicates $\equiv$ and $\in$, (v) three unary predicates m, M and Z, and (vi) an unary functional symbol qc. The terms of $\mathcal{L}_{\mathfrak{Q}}$ are the individual variables (which are thought as ranging over an universe of quasi-sets and atoms), and the expressions of the form $qc(x)$, being x an individual variable. Intuitively

[4]The Excluded Middle principle holds, hence, for any a and b, we have that $a = b \vee a \neq b$ is a theorem of ZF, yet in some cases we cannot decide which is the case. But, if we can name them and $a \neq b$, then we can of course distinguish them, for only a belongs to the singleton $\{a\}$ (or, alternatively, obeys the property $x = a$ —here, a property is a formula with just one free variable.

speaking, $qc(x)$ stands for 'the quasi-cardinal of x'. The formulas are defined in the standard way; informally, $m(x)$ says that x is a micro-object, $M(x)$ says that x is a macro-object, $Z(x)$ says that x is a set. We read $x \equiv y$ as 'x is indiscernible (or indistinguishable) from y' and $x \in y$ as 'x belongs to y'. The propositional connectives $\wedge$, $\vee$ and $\leftrightarrow$, so as the existential quantifier $\exists$, are defined as usual.

The basic idea is that the M-atoms have the properties of standard *Urelemente* of ZFU, while the m-atoms may be thought of as representing entities which may be 'absolutely indiscernible' (indistinguishable *solo numero*), and the elementary basic entities of quantum physics seem to be a good example. Really, they form our 'intended interpretation'. Within $\mathfrak{Q}$, as within ZF, we can construct structures, formulate scientific theories and so on. In fact, all the discussion given before can now be extended to cope with more general structures encompassing indiscernible entities. Our concern here is to discuss the possibility of defining not rigid (deformable) structures that cannot be extended to rigid (undeformable) ones. Thus, contrarily to what happens in ZF, in defining such structures we would be able to discuss quantum theories that consider indiscernible objects that even outside of the considered structures cannot be discerned one each other.

Some additional intuitive ideas related to the theory might be useful at this point. A quasi-set (qset for short) x is something which is not an *Urelement* (ur-element). A qset x may have a cardinal (termed its *quasi cardinal*, denoted by $qc(x)$), but the idea is that the theory does not associate an ordinal with certain qsets, since there will be quasi sets which cannot be ordered (since their elements are to be absolutely indistinguishable m-atoms, expressed by the relation $\equiv$). The concept of quasi cardinal is taken as primitive, since it cannot be defined by the usual means (that is, as particular ordinals). This fits the idea that quantum particles cannot be ordered or counted, but only aggregated in certain amounts. In fact, if we cannot discern among the elements of a certain collection, we cannot define a function from an ordinal and having this set as the counter-domain. Nevertheless, given the concept of quasi cardinal, there is a sense in saying that there may exist a certain quantity of m-atoms obeying certain conditions, although they cannot be named or labeled.

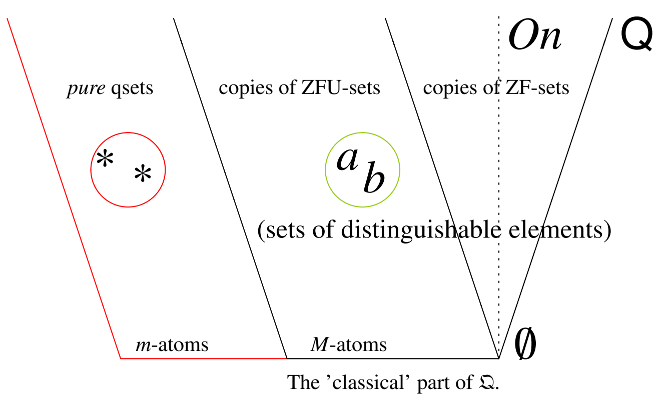

Figure 8.2: The Quasi-Set Universe. *On* is the class of ordinals, defined in the 'classical' part of the theory (in back), and there are two kinds of ur-elements.

Figure 8.2 provides an intuitive view of the 'universe of quasi-sets' (recall our intuitive picture of the well-founded universe of sets with ur-elements, given at page 64). There are 'copies' of both ZF and ZFU; really, the theory 'reduces' to these theories if we keep out the ur-elements or the m-atoms respectively.

The first group of axioms of $\mathfrak{Q}$ are the postulates, that is, axiom schema and inference rules, of classical first-order logic without identity (see Mendelson 1997, section 2.8). It should be noted that $\mathfrak{Q}$ is a formal theory; thus the postulates provide the operational meaning of the relevant concepts (in Hilbert's sense, the postulates implicitly define the concepts).

Definition 8.2.1

(i) $Q(x) := \neg(m(x) \vee M(x))$ (x is a qset)

(ii) $P(x) := Q(x) \wedge \forall y(y \in x \rightarrow m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \equiv z)$ (x is a pure qset, having m-atoms as elements, not necessarily indiscernible from one each other)

(iii) $D(x) := M(x) \vee Z(x)$ (x is a *Ding*, a 'classical object' in the sense of Zermelo's set theory, namely, either a set or a macro *Urelemente*).

(iv) $E(x) := Q(x) \wedge \forall y(y \in x \rightarrow Q(y))$ (x is a qset whose elements are qsets)

(v) $x =_E y := (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \leftrightarrow y \in z))$ (extensional identity)—we shall write simply $x = y$ instead of $x =_E y$ from now on.

(vi) $x \subseteq y := \forall z(z \in x \rightarrow z \in y)$ (subqset)

8.2.2 Specific postulates

The list of the specific postulates of $\mathfrak{Q}$ is presented next.

($\equiv_1$) $\forall x(x \equiv x)$

($\equiv_2$) $\forall x \forall y(x \equiv y \to y \equiv x)$

($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

($=_4$) $\forall x \forall y(x = y \to (\alpha(x) \to \alpha(y)))$, with the standard restrictions (recall that $x = y$ means $x =_E y$ and that it does not hold for m-atoms).

($\in_1$) $\forall x \forall y(x \in y \to Q(y))$ If something has an element, then it is a qset; in other words, the atoms have no elements (in terms of the membership relation).[5]

($\in_2$) $\forall_D x \forall_D y(x \equiv y \to x = y)$ Indistinguishable *Dinge* are extensionally identical. This makes $=$ and $\equiv$ to be the same relation for this kind of entities.

($\in_3$) $\forall x \forall y[(m(x) \wedge x \equiv y \to m(y)) \wedge (M(x) \wedge x = y \to M(y)) \wedge (Z(x) \wedge x = y \to Z(y))]$ In words, an indistinguishable of an m-atom (M-atom, a set) is also an m-atom (an M-atom, a set).

[5]As French & Krause advanced in their book, to the postulates of $\mathfrak{Q}$ we could add a certain mereology relating M-objects and m-objects. But if this mereology ought to reflect quantum physics in some sense, it would overcome some typical problems, as for instance quantum holism—according to which the 'whole' would not be properly 'the sum' of its parts, so as that a whole would be formed by indistinguishable parts. We think that to pursue such a *quantum mereology* is an interesting research program, despite its apparent difficulty.

($\in_4$) $\exists x \forall y(\neg x \in y)$ This qset will be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from extensionality. Thus, from now own we shall denote it, as usual, by '$\emptyset$'.

($\in_5$) $\forall_Q x(\forall y(y \in x \to D(y)) \leftrightarrow Z(x))$ Informally speaking, this postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m-atoms. That is, *sets* in $\mathfrak{Q}$ are those entities obtained in the 'classical' part of the theory (see figure 8.2 once more).

($\in_6$) $\forall x \forall y \exists_Q z(x \in z \wedge y \in z)$ Usually (that is, in ZF), we formulate the axiom in an equivalent way, saying that given two individuals a and b, there exists a set that contains they both as elements and nothing else. In ZF, the two formulations are equivalent, but not here. We shall see why in a moment. The suggestion of using the form $\in_6$ in $\mathfrak{Q}$ is due to Jonas Becker Arenhart, and it is also used by Halmos in his book from 1974.[6]

($\in_7$) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x(x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

The qset y of the schema ($\in_7$), the Separation Schema, is denoted by $[x \in z : \alpha(x)]$. When this qset is a set, we write,

[6]But, while in standard ZF this axiom is equivalent to that which is most used, namely, that given two objects a and b there is a set w which contains just a and b and nothing else, in $\mathfrak{Q}$ this is not true. In fact, although $w = [a, b]$ contains 'all' indiscernible elements of either a or b (and probably this collection should not be regarded as a legitimate qset), it follows from our axiom ($\in_6$) that we can take only the indiscernible elements from either a or b which belong to an already given qset z.

as usual, $\{x \in z : \alpha(x)\}$.

($\in_8$) $\forall_Q x(E(x) \rightarrow \exists_Q y(\forall z(z \in y \leftrightarrow \exists w(z \in w \wedge w \in x)))$. The union of x, writen $\cup x$. Usual notation is used in particular cases.

As we see, axiom ($\in_6$) says that, given x and y, there is a qset z which contains both of them, although z may have other elements as well. But, from the separation schema, using the formula $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, we get a subqset of z which we denote by $[x, y]_z$, which is the qset of the indiscernibles of either x or y that belong to z. When $x \equiv y$, this qset reduces to $[x]_z$, called the *weak singleton* of x. This qset is not the collection of *all* indiscernibles from x, but it contains as elements just those indiscernible from x that belong to an already given qset z. Later, with the postulates of quasi-cardinal, we will be able to prove that this qset has a subqset with quasi-cardinal equals to 1, which we call a *strong singleton* of x (in z), written $[[x]]_z$ (sometimes the sub-indice will be left implicit; really, depending on the considered qset, there may exist more than one element in $[[x]]_z$, that is $qc([[x]]_z \geq 1)$. A counter-intuitive fact is that, since the relation $\equiv$ is reflexive and the strong singleton of x (really, *a* strong singleton, for in $\mathfrak{Q}$ we cannot prove that it is unique) has just one element, we can think that this element *is* x. But this cannot be proven in the theory, for such a proof would demand the identity relation, which cannot be applied to m-atoms to prove that a certain y

is identical to x (as we shall see soon, the only fact we can prove is that $\neg(x \equiv y)$, the negation of this result). Anyway, we can informally reason *as if* the element of the strong singleton $[[x]]_z$ *is* x, although this must be understood as a way of speech. Perhaps the better way to refer to this situation is (informally) to say that the element of $[[x]]_z$ is *an object of the kind x*. In this sense, the sentences that contain strong singletons might be seen as sentences containing *place-holders* for entities of a certain kind.[7]

As remarked above, we are using just one sort of variables, for we think we can circumvent some of the problems that may appear due to this option. For instance, although we have originally motivated quasi-set theory with the Schrödingerian claim that the standard concept of identity would not apply to quantum entities (here represented by the m-atoms), definition 8.2.1 makes things a little bit different from the formal point of view. Really, since we are using a monosorted language, if $m(x)$, then by the definition, we get $\neg(x = y)$ for any y. In particular, $\neg(x = x)$ for any m-object x. The same happens if x is a qset having m-atoms in its transitive closure, that is, being x a qset which is not a set (in the sense of not obeying the predicate Z). That is, if $Q(x)$ and $\neg Z(x)$, then $\neg(x = y)$ for any y, and in particular $\neg(x = x)$. Anyway, there are no (as far as we know) formal problems concerning these facts, for we

[7] In certain forms of structuralism, objects are regarded as merely placeholders for the relevant relations and in this sense perhaps quasi-set theory may be useful to formalise these forms of structuralism.

have only 'deduced', say, that $\neg(x = x)$ for an m-object x, but we can't get its identity (that is, $x = x$ is not a theorem of $\mathfrak{Q}$ which is supposed to be consistent). Although the third middle law holds, for the underlying logic of $\mathfrak{Q}$ is classical logic, that is, yet $\neg(x = y) \vee (x = y)$ is a theorem of $\mathfrak{Q}$, we never can prove that $x = y$ in the case of m-objects, and this of course does not entail that $\neg(x = y)$. Intuitively, perhaps we can say that, since the concept of identity should make no sense for m-objects, it would be quite natural to state that they cannot be identical to themselves.

Let us go back to the postulates of $\mathfrak{Q}$.

($\in_9$) $\forall_Q x \exists_Q y \forall z (z \in y \leftrightarrow w \subseteq x)$, the power qset of x, denoted $\mathcal{P}(x)$.

($\in_{10}$) $\forall_Q x(\emptyset \in x \wedge \forall y(y \in x \to y \cup [y]_x \in x))$, the infinity axiom.

($\in_{11}$) $\forall_Q x(E(x) \wedge x \neq \emptyset \to \exists_Q y(y \in x \wedge y \cap x = \emptyset))$, the axiom of foundation, where $x \cap y$ is defined as usual.

Definition 8.2.2 (Weak ordered pair) *Being z a qset to which both x and y belong, we pose*

$$\langle x, y \rangle_z := [[x]_z, [x, y]_z]_z \tag{8.1}$$

Then, $\langle x, y \rangle_z$ takes all indiscernible elements from either x or y that belong to z, and it is called the 'weak' ordered pair, for it may have more than two elements. Sometimes the subindice z will be left implicit. (Of course this definition could be generalized, taking x and y from discernible qsets, but this most limited axiom seems to suffice for all applications of the theory.)

Definition 8.2.3 (Cartesian Product) *Let z and w be two qsets. We define the cartesian product $z \times w$ as follows:*

$$z \times w := [\langle x, y\rangle_{z \cup w} : x \in z \wedge y \in w] \tag{8.2}$$

Functions and relations cannot also be defined as usual, for when there are m-atoms involved, a mapping may not distinguish between arguments and values. Thus we provide a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities. Thus, we introduce the following concepts.

8.2.3 Quasi-relations and quasi-functions

Definition 8.2.4 (Quasi-relation) *A qset R is a binary quasi-relation (q-relation) between to qsets A and B if its elements are weak ordered pairs of the form $\langle x, y\rangle_{A \cup B}$, with $x \in A$ and $y \in B$.*

In symbols, R is a q-relation iff

$$Q(R) \wedge \forall z(z \in R \rightarrow \exists u \exists v(u \in A \wedge v \in B \wedge z =_E \langle u, v\rangle_{A \cup B}).$$

More general quasi-relations (n-ary) can be easily defined. Thus, a quasi-function (mapping) between z and w is a binary quasi-relation f between them such that if $\langle x, y\rangle \in f$ and

$\langle x', y' \rangle \in f$ and if $x \equiv x'$, then $y \equiv y'$. In other words, a quasi-function maps indistinguishable elements indistinguishable elements. It is easy to define the corresponding concepts of injective, surjective, and bijective quasi-functions. The definitions are as follows.

Definition 8.2.5 *Let A and B be qsets. A binary quasi-relation R between A and B is q-injective, q-surjective and q-bijective if respectively the following conditions holds (in a simplified notation):*

(i) (q-injection) $R(x, y) \wedge R(x', y') \wedge y \equiv y' \rightarrow x \equiv x'$

(ii) (q-surjection) $(\forall y \in B)(\exists x \in A)R(x, y)$

(iii) (q-bijection) R is q-injective and q-surjective.

Intuitively speaking, a q-injection doesn't map two distinct (yet indiscernible) qsets into the same qset. The meaning of 'two' here concerns the notion of quasi-cardinal. That is, the domain of the q-function has not two indiscernible subsets which are mapped in the same qset of the counter-domain.

Here we can see a distinctive characteristics of m-objects. Suppose we have a set with 8 'classical' elements, ordered as $a_1, \ldots, a_8$. Of course a permutation between two of them, say the permutation π_{13} which exchange a_1 and a_3 leads to a different arrangemen, from $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ to $a_3, a_2, a_1, a_4, a_5, a_6, a_7, a_8$. But this would not happen if they were indistinguishable m-atoms, for the two arrangements would be taken as 'the same'. Concerning quantum objects, physicists have used certain 'queues' of Calcium ions to transmit information (see figure 8.3 below), and of course the information is independent of the position of the particular Calcium ions. Differently, a queue with John, Mary, and Tom in this order (say, to by tickets to the best places in a theatre) is different from another with Mary, Tom, and John in this order (mainly if there are only two tickets available).

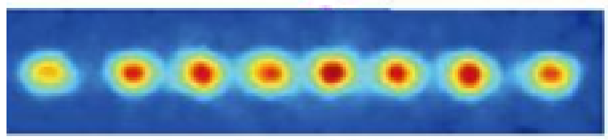

Figure 8.3: Eight Calcium ions in a 'W state'. (From *Scientific American*, Mach 2006). For any considerations, the order of the ions it is not relevant, and any 'permuted' arrangement acts the same way.

8.2.4 Postulates for quasi-cardinals

In standard set theories, as in the usual formulations of ZF, a cardinal is a particular ordinal.[8] Hence, if we follow this standard tradition, in order to go to cardinals, we must have the ordinal concept defined first (of course there are alternatives to define cardinals out of ordinals, but in standard set theory —with the axiom of choice—, any set can be (well)

[8] In the presence of the axiom of choice, the cardinal of a set A is the least ordinal α such that there is a bijection between A and α. There are other ways of defining cardinals; see for instance Enderton 1977, p.222 we do not use the axiom of choice, but relies on regularity.

ordered). In $\mathfrak{Q}$, the idea is that a pure qset of indiscernible objects may have a cardinal (its *quasi-cardinal*), but not an associated ordinal. Hence, in quasi-set theory the concept of quasi-cardinal is taken as primitive, subjected to adequate postulates that grant the operational character of the concept. In the first versions of $\mathfrak{Q}$, every qset has a cardinal. But Domenech and Holik (2007) have argued that when we consider relativistic quantum physics, sometime we can't associate a cardinal with every collection. They are right, and the axiom (qc_1) below enables us to say that *some* qsets have an associated cardinal only.[9] The postulates are as follows, where we use α, β, etc, for naming cardinals (defined in the classical part of the theory); $Cd(y)$ says that y is a cardinal and $card(x)$ stands for the cardinal of x:

(qc_1) $\forall_Q x(\exists_Z y(y = qc(x)) \rightarrow \exists! y(Cd(y) \wedge y = qc(x) \wedge (Z(x) \rightarrow y = card(x)))$ If the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the 'classical' part of the theory) and coincides with the cardinal of x stricto sensu if x is a set.

(qc_2) $\forall_Q x(x \neq \emptyset \rightarrow qc(x) \neq 0)$. Every non-empty qset has a non-null quasi-cardinal.

(qc_3) $\forall_Q x(\exists_Z \alpha(\alpha = qc(x)) \rightarrow \forall \beta(\beta \leq \alpha \rightarrow \exists_Q z(z \subseteq x \wedge qc(z) = \beta)))$ If x has quasi-cardinal α, then for any cardinal

[9] This has obvious significance, since may take 'numerical diversity' or 'countability' as indicators of individuality; yet quasi-sets of non-individual objects may also be regarded as 'countable' in this sense.

$\beta \leq \alpha$, there is a sub-qset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Q_{qc}} x$ (or $\exists_{Q_{qc}} x$) for quantifications over qsets x having a quasi-cardinal. (Here, $Fin(x)$ means that x is a finite qset, that is, one having a natural number as its quasi-cardinal).

(qc_4) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(y \subseteq x \rightarrow qc(y) \leq qc(x))$

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(Fin(x) \wedge x \subset y \rightarrow qc(x) < qc(y))$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(\forall w(w \notin x \vee w \notin y) \rightarrow qc(x \cup y) = qc(x) + qc(y))$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(qc_6) $\forall_{Q_{qc}} x(qc(\mathcal{P}(x)) = 2^{qc(x)})$

Informally speaking, this axiom enables us to reason as if there are $2^{qc(x)}$ sub-quasi-sets of x. If $qc(x) = \alpha$, then it is consistent with the theory to assume (in the presence of axiom qc_3) that there are sub-quasi-sets with quasi-cardinals 0, 1, 2, ..., 2^α. If the elements of x are indiscernible m-atoms, then (as it results from axiom $\equiv_5$, as we shall see), any two sub-quasi-sets with the same quasi-cardinality are indistinguishable, and the theory has no means to discern them by a formula, although they are not *the same* quasi-set.

8.2.5 Weak extensionality

The next postulate of $\mathfrak{Q}$ is the weak extensionality axiom, which intuitively says that qsets with "the same quantity" (expressed in terms of the quasi-cardinals) of elements of "the same kind" (related by $\equiv$) are indistinguishable (are themselves in the relation $\equiv$). In the statement of the postulate below, $Qsim(z, t)$ means that the elements of z and t are indiscernible and that they have the same quasi-cardinal; $x/{\equiv}$ stands for the quotient qset of x by the relation $\equiv$.[10] In symbols,

[Weak Extensionality Axiom]

$(\equiv_5)$ $\forall_Q x \forall_Q y((\forall z(z \in x/{\equiv} \to \exists t(t \in y/{\equiv} \wedge \wedge QSim(z, t)))) \wedge$
$\forall t(t \in y/{\equiv} \to \exists z(z \in x/{\equiv} \wedge \wedge QSim(t, z))) \to x \equiv y)$

The following theorem express the invariance under permutations in $\mathfrak{Q}$, and with this result we finish our revision:

Theorem 8.2.1 *Let x be a finite qset such that $\neg(x = [z]_t)$ for some t and let z be an m-atom such that $z \in x$. If $w \in t$, $w \equiv z$ and $w \notin x$, then there exists $[[w]]_t$ such that*

$$(x - [[z]]_t) \cup [[w]]_t \equiv x$$

Proof: (We shall suppress the subindices) Case 1: $t \in [[z]]$ does not belong to x. In this case, $x - [[z]] = x$ and so we

[10] Replacement axioms and the qset version of the axiom of choice may be taken exactly as in French & Krause 2006.

may admit the existence of $[[w]]$ such that its unique element s does belong to x (for instance, s may be z itself); then $(x - [[z]]) \cup [[w]] = x$. Case 2: $t \in [[z]]$ does belong to x. Then $qc(x - [[z]]) = qc(x) - 1$ (by a result not proven here).[11] We then take $[[w]]$ such that its element is w itself, and so it follows that $(x - [[z]]) \cap [[w]] = \emptyset$. Hence, by (qc_6), $qc((x - [[z]]) \cup [[w]]) = qc(x)$. This intuitively says that both $(x - [[z]]) \cup [[w]]$ and x have the same quantity of indistinguishable elements. The theorem follows from the week extensionality axiom. ∎

This above theorem is illustrated by the figure 8.4 below, where $[z]$ is the collection—or "quasi-class"—of all indiscernibles of z, while $[z]_t$ is given by the pair axiom and the separation schema.

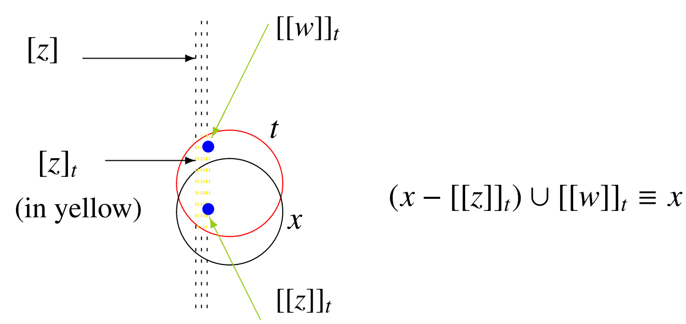

Figure 8.4: The invariance by permutations in $\mathfrak{Q}$. Two indiscernible elements from $z \in x$ and $w \notin x$, expressed by their quasi-singletons $[[z]]_t$ and $[[w]]_t$, are "interchanged" and the resulting qset x remains indiscernible from the original one. The hypothesis that $\neg(x = [z]_t)$ grants that there are indiscernible from z in t which do not belong to x.

[11] But see [28, p.293].

8.3 Deformable (not-rigid) structures in $\mathfrak{Q}$

Let us begin by giving another look at the quasi-set universe. Calling Q^m its 'pure' part, that is, that erected from m-atoms, we may define structures $\mathfrak{A} = \langle D, r_\iota \rangle$ which cannot be extended to undeformable (rigid) structures. Recall that in ZFC, any structure can be extended to a rigid one (see figure 8.5).

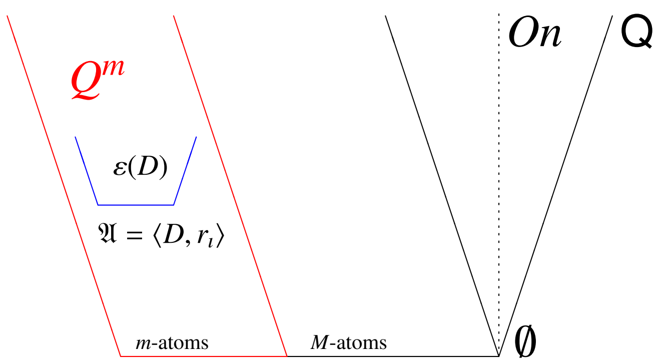

Figure 8.5: A structure $\mathfrak{A}$ and its scale $\varepsilon(D)$ in the 'pure' part Q^m of the quasi-set universe. Since identity does not hold for entities in this part, there are structures which cannot be make undeformable.

In order to understand what is happening, let us consider a quasi-set D whose elements involve a and b and let us suppose that $a \not\equiv b$ that is, they are not indiscernible. By the weak pair axiom, we can obtain $[a]_D$ and $[a, b]_D$. Again by the weak pair, we get $\langle a, b \rangle_D := [[a]_D, [a, b]_D]$ (figure 8.6). Thus we can define a particular binary quasi-relation $R = [\langle a, b \rangle_D : a, b \in D]$ containing just one pair for simplicity (the relation could contain more than one ordered pair of course). The ordered pair $\langle a, b \rangle_D$ has the following property. The qset $[a]_D$, for instance, may have more than one element, that is, $qc([a]_D) \geq 1$.

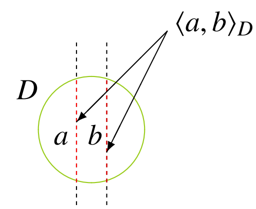

Figure 8.6: In a pure quasi-set D, the ordered pair $\langle a, b\rangle_D$ is formed by the elements of D that are indistinguishable from either a or b (in red), and may contain more than just two elements.

Then, speaking informally (and in the metalanguage we can use the notion of identity), there are precisely $qc([a]_D)$ elements of D that may appear in the relation R without altering its meaning, in the sense that the relation obtained by an exchange of one element by an indistinguishable one is indistinguishable from the original one, as entailed by the weak extensionality axiom (words such as 'another' may confound us, but we need to understand the result in terms of the theorem 8.2.1). Perhaps an analogy with a situation in chemistry helps us in understanding the motivations for these moves.

Let us take the ethylic alcohol to exemplify (Figure 8.7). In this case, where we have the symbols H, C and O, which should be seen (as they of course are) as *places* for carbon, hydrogen and oxygen atoms. In other words, what we are expressing with the Figure 8.7 is not something concerning a *set* of particular individual atoms; the ethylic molecule is not simple a collection (set) of certain atoms. A chemical substance like C_2H_6O may originate different chemical compounds, which are described by presenting their structural formulas; depending on the arrangement of the atoms (their *structure*, or *form*), the same collection of carbons, hydrogens and oxygens may produce quite different chemical substances, called isomers (and the same happens in several other cases). For instance, C_2H_6O may stand for both $CH_3 - CH_2 - OH$, the ethylic alcohol and $H_3C - O - CH_3$, the methylic ether (Figure 8.8). So, certain *structures* appear in these situations.

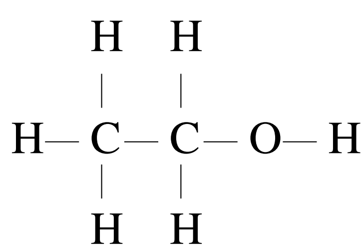

Figure 8.7: Ethylic Alcohol, C_2H_6O

But, while (as we have seem) in standard mathematics the simple substitution of the elements of A by other elements does not ensure that the relation is preserved,[12] in chemistry if we substitute the elements by similar ones, we do find the 'same' chemical element, regarded that H atoms are substituted by H atoms and so on (we are talking in general terms, so we shall not discuss how such a substitution can be performed, but simply assume that it can be done, even in ideal terms. Perhaps more adequate examples can be found by chemists or quantum physicists). This is an important point: we *can* (at least ideally) substitute the H, C and O atoms by 'other' H, C and O atoms respectively by 'preserving the structure'. This fact is nicely exemplified by Roger Penrose in the context of quan-

[12]For instance, take the set $A = \{a, b\}$ and an equivalence relation R on A, say $R = \{\langle a, a\rangle, \langle b, b\rangle, \langle a, b\rangle, \langle b, a\rangle\}$. Now consider the set $B = \{1, 2, 3\}$ and substitute 1 for a and 2 for b. This give us the relation $R' = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle\}$, which is not an equivalence relation on B.

tum mechanics (which of course would provide most adequate examples of what we are trying to say):

> ”according to the modern theory [quantum mechanics], if a particle of a person's body were exchanged with a similar particle in one of the bricks of his house then nothing would happened at all.” (Penrose 1989, p. 360)

Perhaps Pink Floyd is right in saying that we are nothing more than “another brick in the wall”. Back to science, in chemistry, the kind of ’substitutional’ property of atoms in a certain structure makes it in certain sense to be *independent* of the particular (individual) involved elements. If we imagine that the structural formula of the ethylic alcohol represents a certain ’relation’ among the H, C and O atoms (which is of course what is happening), then its chemical properties do not depend on the particular atoms being involved. So, there is a ’relation’ which can be said to be independent of the individual relata it links (except in what respects their ’nature’ –see below). Let us reinforce this idea of the independence of particular elements

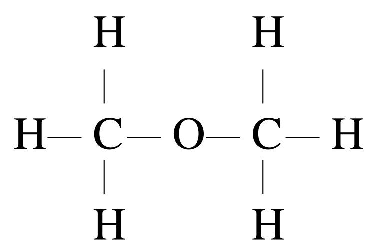

Figure 8.8: Methylic Ether, C_2H_6O

that enter in a certain effect by giving another example. Take for instance a simple chemical reaction

$$NO + O_3 \rightarrow NO_2 + O_2,$$

where one nitric oxide molecule reacts with one ozone molecule to produce one nitrogen dioxide molecule and one oxygen molecule. We remark that it is not important *what* particular oxygen atom (there are three) was captured by the nitric oxide molecule to form the nitrogen dioxide; the only relevant fact is that the captured element *must be* one oxygen atom. Thus Toraldo di Francia says: ”this enable us to put within parentheses the *true* nature of the entities and emphasize the *only* secure property: the number!” (1986, p. 122). Really, every oxygen atom of the ozone molecule plays the same role in the reaction. Only the quantity of them is important, and for sure the same holds in the quantum context in regarding elementary particles. An interesting point would be to capture these intuitions from a set-theoretical point of view, and quasi-set theory enables us to do that. So, what we are approaching is a characterization of a concept of structure which mirrors what happens in chemical (and of course in quantum) situations, but we shall try to do it in (quasi)set-theoretical terms. What we can do in the theory $\mathfrak{Q}$ is to prove that there are quasi-relations R so that when the elements of these pairs are ’exchanged’ by indistinguishable ones, the relation ’continues to hold’, which is of course contrary to the case involving standard relations of usual set theories, due to the axiom of extensionality, as we have seen.

To end our remarks, we need to justify why in quasi-set theory there are structures that cannot be extended to rigid or undeformable structures. Again, let us consider the structure $\mathfrak{A} = \langle D, r_\iota \rangle$ built in the 'pure' part of the quasi-set theory (see again the figure 8.5). Let us take further two elements $a, a' \in D$ such that $a \equiv a'$. In order to prove that $\mathfrak{A}$ cannot be extended to rigid structure, it suffices to prove that there are automorphisms of $\mathfrak{A}$, for since the notion of identity cannot be used in Q^m, these automorphisms cannot be the identity function. But this is trivial for whatever bijective quasi-function $h : D \mapsto D$ such that $h(a) = a'$ is an automorphism, as it results from the above considerations about quasi-relations, for $R(a, b)$ iff $R(a', b)$ (this can be generalized to n-ary relations). In fact, for any relation R, $[[a]_D, [a, b]_D] \in R$ iff $[[a']_D, [a', b]_D] \in R$.[13]

The existence of deformable (non-rigid) structures seem to be interesting from the quantum mechanics foundational point of view. In fact, if we are to accept the existence of 'legitimate' indiscernible objects, objects that in no way can be differentiated from one each other, than the existence of such structures are relevant. Thus, the 'pure' part of quasi-set theory seems to be the right place to find quantum structures that involve

[13]We remark that this result doesn't hold for whatever indistinguishable from a, but it is true for those indistinguishable from a that belong to D.

indiscernible objects without the needs of the standard trick of using symmetries for 'making' individuals indiscernible. In Domenech et al. 2008, 2009, the first steps in the direction of a construction of a quantum mechanics in this sense was advanced, but we shall not consider these results here. But these developments show that sometimes we get interesting intuitions and results going out from standard frameworks.

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