Universal Logic III
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Tutorial on Truth Values
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Lecture 2, Part I

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The set of classical truth values $\mathbf{2} = \{F, T\}$. A classical valuation v^2 (a 2-valuation) is a function from the set of atoms into $\mathbf{2}$.

Nuel Belnap (1977) suggested that the elements of $\mathcal{P}(\mathbf{2})$ may be viewed as four generalized truth values:

$\mathbf{N} = \varnothing$ – none ("told neither falsity nor truth");

$\mathbf{F} = \{F\}$ – plain falsehood ("told only falsity");

$\mathbf{T} = \{T\}$ – plain truth ("told only truth");

$\mathbf{B} = \mathbf{2} = \{F, T\}$ – both falsehood and truth ("told both falsity and truth").

$\mathbf{4} = \mathcal{P}(\mathbf{2}) = \{\mathbf{N}, \mathbf{F}, \mathbf{T}, \mathbf{B}\}$. A function v^4 from the set of propositional variables into $\mathbf{4}$ (a 4-valuation) is then a generalized truth value function.

Matthew Ginsberg (1988) introduced the notion of a bilattice and pointed out that Belnap's four truth values form the smallest non-trivial bilattice.

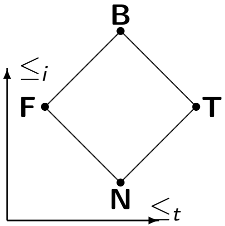


Figure: Bilattice $FOUR_2$

$\leq_t$ – a truth order ($x \leq_t y$ – y is as least as true as x)

$\leq_i$ – an information order ($x \leq_i y$ – y is as least as informative as x)

The information order is just set-inclusion, and $\leq_t$, it is claimed, represents increase in truth.

Lattice meet (greatest lower bound) and lattice join (least upper bound) with respect to $\leq_t$ give rise to a conjunction $\wedge$ and a disjunction $\vee$, and one can define a unary operation $-$ that satisfies $- - x = x$ and $\leq_t$-inversion and gives rise to a negation connective $\sim$.

Definition

$A \models^4 B$ iff $\forall v^4 \left(v^4 (A) \leq_t v^4 (B)\right)$.

This entailment relation can be axiomatized by the consequence system know as First Degree Entailment, the logic codifying, according to Belnap, how a computer should think.

The system is a pair $(\mathcal{L}, \vdash)$, where $\vdash$ is a binary relation (consequence) on the language $\mathcal{L}$ satisfying the following postulates (axiom schemes and rules of inference):

$A1.$ $A \wedge B \vdash A$
$A2.$ $A \wedge B \vdash B$
$A3.$ $A \vdash A \vee B$
$A4.$ $B \vdash A \vee B$
$A5.$ $A \wedge (B \vee C) \vdash (A \wedge B) \vee C$
$A6.$ $A \vdash \sim\sim A$
$A7.$ $\sim\sim A \vdash A$

$R1.$ $A \vdash B,\ B \vdash C \,/\, A \vdash C$
$R2.$ $A \vdash B,\ A \vdash C \,/\, A \vdash B \wedge C$
$R3.$ $A \vdash C,\ B \vdash C \,/\, A \vee B \vdash C$

But, after moving from **2** to $\mathcal{P}(\mathbf{2})$, should one stop at **4**?

An Argument:

Consider, e.g., the combination **TB** ($= \{\{T\}, \{F, T\}\}$) of **T** and **B**. This new truth value would then mean "true and true-and-false". But a repetition of truths gives us no new information and hence is superfluous. Thus, the meaning of **TB** collapses just into "true-and-false", and in this way we simply obtain **B**.

This argument is, of course, a non sequitur:
$\mathbf{TB} = \{\{T\}, \{F, T\}\}$ is distinct from $\mathbf{B} = \{T\} \cup \{F, T\}$.

A further generalization of the classical truth values results from taking the power-set of $\{\mathbf{N}, \mathbf{F}, \mathbf{T}, \mathbf{B}\}$ giving the set **16**:

1.	$\mathbf{N} = \varnothing$	9.	$\mathbf{FT} = \{\{F\}, \{T\}\}$
2.	$\boldsymbol{N} = \{\varnothing\}$	10.	$\mathbf{FB} = \{\{F\}, \{F, T\}\}$
3.	$\boldsymbol{F} = \{\{F\}\}$	11.	$\mathbf{TB} = \{\{T\}\}, \{F, T\}\}$
4.	$\boldsymbol{T} = \{\{T\}\}$	12.	$\mathbf{NFT} = \{\varnothing, \{F\}, \{T\}\}$
5.	$\boldsymbol{B} = \{\{F, T\}\}$	13.	$\mathbf{NFB} = \{\varnothing, \{F\}, \{F, T\}\}$
6.	$\mathbf{NF} = \{\varnothing, \{F\}\}$	14.	$\mathbf{NTB} = \{\varnothing, \{T\}, \{F, T\}\}$
7.	$\mathbf{NT} = \{\varnothing, \{T\}\}$	15.	$\mathbf{FTB} = \{\{F\}, \{T\}, \{F, T\}\}$
8.	$\mathbf{NB} = \{\varnothing, \{F, T\}\}$	16.	$\mathbf{A} = \{\varnothing, \{T\}, \{F\}, \{F, T\}\}$.

Assume that one source of information tells a computer that a sentence is true-only, while another informant supplies inconsistent data, namely that the sentence is both true and false: this is a clear case for **TB**.

If the computer is only told that a sentence is true-only this gives the hitherto unavailable value $\{\{T\}\} = \boldsymbol{T}$.

Support from the literature: Dunn and Hardegree (2001, p. 277): "[T]here can be states of information that are inconsistent, incomplete, *or both*".

We must, for example, distinguish between the following situations:

An informant gives only the information "The sentence is true": $\{T\}$

An informant gives only the information "The sentence is true-only": $\{\{T\}\}$

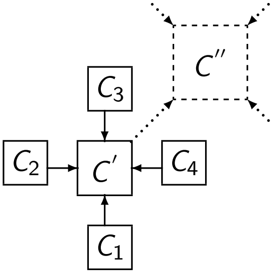


Figure: A computer network

How insane are they? Do they really suggest that the logic of C'' is an at least 65536-valued logic? Yes, they do, but ... before we shall return to this question, we first take a slightly closer look at a certain lattice structure defined on **16**.

Definition

An **n-dimensional multilattice** (or simply **n-lattice**) is a structure $\mathcal{M}_n = (S, \leq_1, \ldots, \leq_n)$ such that S is a non-empty set and $\leq_1, \ldots, \leq_n$ are partial orders defined on S such that $(S, \leq_1), \ldots, (S, \leq_n)$ are all distinct lattices.

Consider any two distinct partial orders defined on some non-empty set. We say that these relations are mutually independent with respect to these definitions (or are defined mutually independently) iff they are not inversions of each other and the only common terms that are used in both definitions, except of metalogical connectives and quantifiers, are the usual set theoretical terms.

Definition

A multilattice is called **proper** iff all its (pairs of) partial orders can be defined mutually independently.

Formal definition of the ordering relations $\leq_i$ and $\leq_t$ in $FOUR_2$.

$\leq_i$: for any $x, y \in \mathbf{4}$, $x \leq_i y$ iff $x \subseteq y$.

$\leq_t$: $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^f \subseteq x^f$,

where for each element of **4** its 'truth part' and its 'falsity part' is defined as follows as follows:

$$x^t \quad := \{z \in x \mid z = T\}\,; \quad x^f \quad := \{z \in x \mid z = F\}\,.$$

In $FOUR_2$, $\leq_t$ is not just a *truth* order but rather a '*truth-and-falsity* order' in the sense that in order to define $\leq_t$ we must refer to both T and F.

For every $x \in \mathbf{16}$ we denote by x^t the subset of x that contains exactly those elements of x which in turn contain T as an element and by x^{-t} the 'truthless' subset of x:

$$x^t \quad := \{y \in x \mid T \in y\}\,; \quad x^{-t} \quad := \{y \in x \mid T \notin y\}\,;$$

and analogously for falsity:

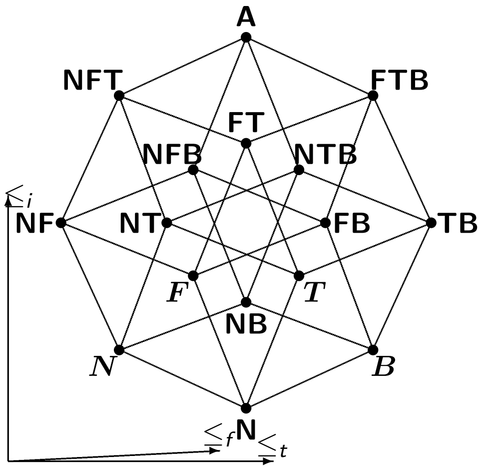
$$x^f \quad := \{y \in x \mid F \in y\}\,; \quad x^{-f} \quad := \{y \in x \mid F \notin y\}\,.$$

Definition

For every x, y in $\mathbf{16}$:

- $x \leq_i y$ iff $x \subseteq y$;
- $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
- $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

We obtain a structure that combines the three (complete) lattices $(\mathbf{16}, \leq_i)$, $(\mathbf{16}, \leq_t)$, and $(\mathbf{16}, \leq_f)$ into the *trilattice* $SIXTEEN_3 = (\mathbf{16}, \leq_i, \leq_t, \leq_f)$.



$SIXTEEN_3$ may also be represented as the structure $(\mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$.

On $SIXTEEN_3$ we might want to consider unary 'inversion' operations with the following properties:

1. $t\text{−}inversion\ (-_t)$:
$(a) a \leq_t b \Rightarrow -_t b \leq_t -_t a;$
$(b) a \leq_f b \Rightarrow -_t a \leq_f -_t b;$
$(c) a \leq_i b \Rightarrow -_t a \leq_i -_t b;$
$(d) -_t -_t a = a.$

2. $f\text{−}inversion\ (-_f)$:
$(a) a \leq_t b \Rightarrow -_f a \leq_t -_f b;$
$(b) a \leq_f b \Rightarrow -_f b \leq_f -_f a;$
$(c) a \leq_i b \Rightarrow -_f a \leq_i -_f b;$
$(d) -_f -_f a = a.$

3. $i\text{−}inversion\ (-_i)$:
$(a) a \leq_t b \Rightarrow -_i a \leq_t -_i b;$
$(b) a \leq_f b \Rightarrow -_i a \leq_f -_i b;$
$(c) a \leq_i b \Rightarrow -_i b \leq_i -_i a;$
$(d) -_i -_i a = a.$

4. $tf\text{−}inversion\ (-_{tf})$:
$(a) a \leq_t b \Rightarrow -_{tf} b \leq_t -_{tf} a;$
$(b) a \leq_f b \Rightarrow -_{tf} b \leq_f -_{tf} a;$
$(c) a \leq_i b \Rightarrow -_{tf} a \leq_i -_{tf} b;$
$(d) -_{tf} -_{tf} a = a.$

$$5. ti{-}inversion\ (-_{ti}):$$
$$(a) a \leq_t b \Rightarrow -_{ti} b \leq_t -_{ti} a;$$
$$(b) a \leq_f b \Rightarrow -_{ti} a \leq_f -_{ti} b;$$
$$(c) a \leq_i b \Rightarrow -_{ti} b \leq_i -_{ti} a;$$
$$(d) -_{ti} -_{ti} a = a.$$

$$6. fi{-}inversion\ (-_{fi}):$$
$$(a) a \leq_t b \Rightarrow -_{if} a \leq_t -_{if} b;$$
$$(b) a \leq_f b \Rightarrow -_{if} b \leq_f -_{if} a;$$
$$(c) a \leq_i b \Rightarrow -_{if} b \leq_i -_{if} a;$$
$$(d) -_{if} -_{if} a = a.$$

$$7. tfi{-}inversion\ (-_{tfi}):$$
$$(a) a \leq_t b \Rightarrow -_{tif} b \leq_t -_{tif} a;$$
$$(b) a \leq_f b \Rightarrow -_{tif} b \leq_f -_{tif} a;$$
$$(c) a \leq_i b \Rightarrow -_{tif} b \leq_i -_{tif} a;$$
$$(d) -_{tif} -_{tif} a = a.$$

In $SIXTEEN_3$ such inversion operations can be defined:

a	$-_t a$	$-_f a$	$-_i a$	$-_{tf} a$	$-_{ti} a$	$-_{fi} a$	$-_{tfi} a$
N	**N**	**N**	**A**	**N**	**A**	**A**	**A**
N	T	F	**NFT**	B	**NTB**	**NFB**	**FTB**
F	B	N	**NFB**	T	**FTB**	**NFT**	**NTB**
T	N	B	**NTB**	F	**NFT**	**FTB**	**NFB**
B	F	T	**FTB**	N	**NFB**	**NTB**	**NFT**
NF	**TB**	**NF**	**NF**	**TB**	**TB**	**NF**	**TB**
NT	**NT**	**FB**	**NT**	**FB**	**NT**	**FB**	**FB**
FT	**NB**	**NB**	**NB**	**FT**	**FT**	**FT**	**NB**
NB	**FT**	**FT**	**FT**	**NB**	**NB**	**NB**	**FT**
FB	**FB**	**NT**	**FB**	**NT**	**FB**	**NT**	**NT**
TB	**NF**	**TB**	**TB**	**NF**	**NF**	**TB**	**NF**
NFT	**NTB**	**NFB**	N	**FTB**	T	F	B
NFB	**FTB**	**NFT**	F	**NTB**	B	N	T
NTB	**NFT**	**FTB**	T	**NFB**	**N**	B	F
FTB	**NFB**	**NTB**	B	**NFT**	F	T	N
A	**A**	**A**	**N**	**A**	**N**	**N**	**N**

Table: Inversions in $SIXTEEN_3$

According to G. Priest, "[t]here is growing evidence that the logical paradoxes … are both true and false", and since he claims that "a sentence must have *some* value at least", Priest's preferred set of truth values is $\mathbf{3} = \{\mathbf{F}, \mathbf{T}, \mathbf{B}\}$.

Priest suggests considering 'higher-order' combinations of truth values from $\mathbf{3}$ and beyond. The motivation for this is a 'revenge Liar' argument, leading to so-called 'impossible values' or hyper-contradictions (inadmissible combinations of admissible values).

Consider the sentence: (*) *This sentence is false only*.

Against the background of **3**, (*) is either (i) true only, (ii) false only, or (iii) both true and false. (i): If (*) is true only, what (*) says is true and hence the sentence is true only *and* false only. In other words, (*) takes the impossible value $\boldsymbol{B} = \{\{F\}, \{T\}\}$ not available in **3**. (ii): If (*) is false only, what (*) says is not true, and thus the sentence is either true only or both true and false. Hence (*) is either false only and true only, or it is false only and both true and false. That is, (*) takes the impossible value $\boldsymbol{B}$ or the impossible value $\{\{F\}, \mathbf{B}\}$. (iii): Suppose (*) is both true and false. Then in particular it is true, and thus takes an impossible value $\{\{F\}, \mathbf{B}\}$.

It is well-known that the set $\mathbf{3}' = \{\mathbf{F}, \mathbf{T}, \mathbf{N}\}$ also gives rise to a revenge Liar.

Consider the sentence: (**) *This sentence is false or neither true nor false*.

Against the background of $\mathbf{3}'$, (**) is either (i) true, (ii) false, or (iii) neither true nor false. (i): If (**) is true, we have to consider two cases. If (**) is false, (**) takes the impossible value $\boldsymbol{B}$; if (**) is neither true nor false, it takes the impossible value $\{\{T\}, \mathbf{N}\}$. (ii): If (**) is false, what (**) says is not the case. Hence the sentence is true and takes the impossible value $\boldsymbol{B}$. (iii): Suppose (**) is neither true nor false. Then in particular it is not true, and hence, (**) takes the impossible value $\{\{T\}, \mathbf{N}\}$.

While according to Priest, a sentence always takes at least some value, and the paradoxes reveal that some sentences are both true and false, according to Keith Simmons (2002), "[t]he claim that Liar sentences are gappy seems natural enough – after all, the assumption that they are true or false leads to a contradiction."

In any case, both (*) and (**) show that admittedly the only way to escape the revenge Liar is to introduce higher-order truth values such as $\{\{F\}, \{T\}\}$, $\{\{T\}, \{F, T\}\}$ and so on.

Priest defines for any nonempty set of truth values S_n the corresponding higher-order set S_{n+1} as follows: $S_{n+1} = \mathcal{P}(S_n) \setminus \{\varnothing\}$ for all $n \in \omega$, where S_0 is just the set **2** of classical truth values ($= \{F, T\}$). Then he introduces the following definition for evaluating compound formulas on each level:

Definition

Given the classical truth value functions $\wedge_0$, $\vee_0$, and $\sim_0$ on S_0:

- $x \wedge_{n+1} y = \{z : \exists x' \in x \exists y' \in y \, (z = x' \wedge_n y')\}$;
- $x \vee_{n+1} y = \{z : \exists x' \in x \exists y' \in y \, (z = x' \vee_n y')\}$;
- $\sim_{n+1} x = \{z : \exists x' \in x \, (z = \sim_n x')\}$.

Next, Priest considers the map $\sigma(x) = \{x\}$ and shows that σ is an isomorphism between any S_n and $\sigma[S_n]$ $(= \{\sigma(x) \mid x \in S_n\})$. In virtue of this fact Priest identifies S_n with $\sigma[S_n]$, $\wedge_n$ with $\wedge_{n+1}$ restricted to $\sigma[S_n]$, etc.

Priest then defines the set $S = \bigcup_n S_n$ and introduces on S generalized logical operators $\wedge, \vee$ and $\sim$ in an analogous way (so that, e.g., $\wedge = \bigcup_n \wedge_n$, etc.).

Finally he singles out a set of designated values D so that a value is designated just if it contains T at some depth of membership.

Definition

$\Sigma \models A$ iff $\forall v : \exists B \in \Sigma \;\; v(B) \notin D$, or $v(A) \in D$.

The main result of (Priest 1984) is that $\models$ coincides with the consequence relation of Priest's (1979) Logic of Paradox, i.e. $\models = \models_1$. That is, Priest tells us, "hyper-contradictions make no difference: the first contradiction $\{1, 0\}$ of S_1 changes the consequence relation... Subsequent contradictions have no effect" (Priest 1984, p. 241).

Pragati Jain (1997) extends the result by Priest. She does not collect the S_n together to form the set S, but keeps each S_n distinct, and defines semantic consequence relations $\models_n$ for any n accordingly. Then she shows that if we define the sets D_n of designated values following Priest's definition, the following holds: for each n, $\models_n = \models_1$.

In (Shramko and Wansing 2005) it is shown that the logic of the truth (falsity) order of $SIXTEEN_3$ in the language of the truth (falsity) connectives coincides with the one of $FOUR_2$. It is First Degree Entailment.

We can extend this result to the infinite case and show that Belnap's strategy of generalizing the set $\mathbf{2} = \{T, F\}$ of classical truth values not only is coherent but stabilizes. At any stage, no matter how far it goes, the logic of the truth (falsity) order is again First Degree Entailment.

Let X be a basic set of truth values, $\mathcal{P}^1(X) := \mathcal{P}(X)$ and $\mathcal{P}^n(X) := \mathcal{P}(\mathcal{P}^{n-1}(X))$ for $1 < n$, $n \in \omega$.

We consider $\mathcal{P}^n(\mathbf{4})$. In order to define a truth ordering $\leq_t$ on $\mathcal{P}^n(\mathbf{4})$, we define for every $x \in \mathcal{P}^n(\mathbf{4})$ the set x^t of its 'truth-containing' elements and the set x^{-t} of its 'truthless' elements:

$$x^t := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \in y_{n-1}\}$$
$$x^{-t} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \in y_{n-1}\}$$

To define a falsity ordering $\leq_f$ on $\mathcal{P}^n(\mathbf{4})$, we define for every $x \in \mathcal{P}^n(\mathbf{4})$ the set x^f of its 'falsity-containing' elements and the set x^{-f} of its 'falsityless' elements analogously:

$$x^f := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\; F \in y_{n-1}\}$$
$$x^{-f} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\; F \in y_{n-1}\}$$

Definition

For every x, y in $\mathcal{P}^n(\mathbf{4})$:

- $x \leq_i y$ iff $x \subseteq y$;
- $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
- $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

Definition

A Belnap trilattice is a structure

$$\mathcal{M}_3^n := (\mathcal{P}^n(\mathbf{4}), \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f),$$

where $\sqcap_i$ ($\sqcap_t$, $\sqcap_f$) is the lattice meet and $\sqcup_i$ ($\sqcup_t$, $\sqcup_f$) is the lattice join with respect to the ordering $\leq_i$ ($\leq_t$, $\leq_f$) on $\mathcal{P}^n(\mathbf{4})$.

Thus, $SIXTEEN_3$ ($= \mathcal{M}_3^1$) is the smallest Belnap trilattice.

As to negation, it seems quite natural to assume that some partial order $\leq_u$ in a given lattice determines the corresponding object language negation operator ($\sim_u$) exactly when $\leq_u$ is equipped with a 1-1 lattice operation ($-_u$) of 'period two' which basically *inverts* this order. However, as we have seen already, in a multilattice with several partial orders the situation can be more intricate: the operation under consideration should not only invert the corresponding ordering, but also preserve all the other orders.

Definition

Let $\mathcal{M}_n = (S, \leq_1, \ldots, \leq_n)$ be a multilattice and $1 \leq j \leq n$. Then a unary operation $-_j$ on S is said to be a (pure) j-inversion iff the following conditions are satisfied:

$$\begin{array}{llcl}
(iso) & x \leq_1 y & \Rightarrow & -_j x \leq_1 -_j y; \\
 & & \vdots & \\
(anti) & x \leq_j y & \Rightarrow & -_j y \leq_j -_j x; \\
 & & \vdots & \\
(iso) & x \leq_n y & \Rightarrow & -_j x \leq_n -_j y; \\
(per2) & -_j -_j x & = & x.
\end{array}$$

Theorem

For any Belnap trilattice $\mathcal{M}_3^n$ there exist t-inversions and f-inversions on $\mathcal{P}^n(\mathbf{4})$.

Proof. For any $\mathcal{M}_3^n$ we can define an operation of t-inversion in a canonical way as follows. Let $x \in \mathcal{P}^n(\mathbf{4})$. If x is empty, we define $-_t x = x$. If $x \neq \varnothing$, we define $-_t x$ by considering the elements $y \in x$. Every $y \in x$ contains at some depth of nesting elements from **4**, i.e., $\varnothing$, $\{T\}$, $\{F\}$, or $\{T, F\}$. We replace these elements according to the following instruction:

$$\begin{array}{rll} \varnothing & \text{is replaced by} & \{T\} \\ \{T\} & \text{is replaced by} & \varnothing \\ \{F\} & \text{is replaced by} & \{F, T\} \\ \{F, T\} & \text{is replaced by} & \{F\} \end{array}$$

Example: *If x is the value $\{\{\varnothing, \{F\}, \{F, T\}\}, \{\{T\}, \{F, T\}\}\}$, then $-_t x = \{\{\{T\}, \{F, T\}, \{F\}\}, \{\varnothing, \{F\}\}\}$*

In other words, for every element of **4** at some depth of nesting, $-_t$ introduces the classical T, where it is absent, and excludes T from where it is present. Obviously, this definition of $-_t x$ preserves the information order $\leq_i$, since x and $-_t x$ have the same cardinality. The falsity ordering $\leq_f$ is preserved, too, because the inclusion or exclusion of T has no effect on the presence or absence of F. And clearly, the truth ordering $\leq_t$ is inverted by definition, as well as $-_t -_t x = x$.

The canonical definition of an f-inversion is analogous. ■

In what follows we can without loss of generality consider Belnap trilattices with t-inversions and f-inversions.

Next, we assume an infinite set of propositional variables and define the syntax of the languages $\mathcal{L}_t$, $\mathcal{L}_f$, and $\mathcal{L}_{tf}$ in Backus–Naur form as follows:

$$
\begin{array}{rll}
\mathcal{L}_t: & A ::= & p \mid \sim_t A \mid A \wedge_t A \mid A \vee_t A \\
\mathcal{L}_f: & A ::= & p \mid \sim_f A \mid A \wedge_f A \mid A \vee_f A \\
\mathcal{L}_{tf}: & A ::= & p \mid \sim_t A \mid \sim_f A \mid A \wedge_t A \mid A \vee_t A \mid A \wedge_f A \mid A \vee_f A
\end{array}
$$

An n-valuation is a map v^n from the set of propositional variables into $\mathcal{P}^n(\mathbf{4})$. Any n-valuation is extended to an interpretation of arbitrary formulas in $\mathcal{P}^n(\mathbf{4})$.

Definition

For any formula A and B:

$$\begin{array}{rclrcl}
v^n(A \wedge_t B) & = & v^n(A) \sqcap_t v^n(B); & v^n(A \wedge_f B) & = & v^n(A) \sqcup_f v^n(B); \\
v^n(A \vee_t B) & = & v^n(A) \sqcup_t v^n(B); & v^n(A \vee_f B) & = & v^n(A) \sqcap_f v^n(B); \\
v^n(\sim_t A) & = & -_t v^n(A); & v^n(\sim_f A) & = & -_f v^n(A).
\end{array}$$

We can define new notions of t-entailment and f-entailment for any n:

Definition

For all formulas A, B from $\mathcal{L}_{tf}$:

$$A \models^n_t B \text{ iff } \forall v^n \, (v^n(A) \leq_t v^n(B)).$$

Definition

For all formulas A, B from $\mathcal{L}_{tf}$:

$$A \models^n_f B \text{ iff } \forall v^n \, (v^n(B) \leq_f v^n(A)).$$

The (first degree) consequence relation $\vdash_t$ for $\mathcal{L}_t$ is defined by the following axioms and rules:

$a_t1.\ A \wedge_t B \vdash_t A$
$a_t2.\ A \wedge_t B \vdash_t B$
$a_t3.\ A \vdash_t A \vee_t B$
$a_t4.\ B \vdash_t A \vee_t B$
$a_t5.\ A \wedge_t (B \vee_t C) \vdash_t (A \wedge_t B) \vee_t C$
$a_t6.\ A \vdash_t \sim_t \sim_t A$
$a_t7.\ \sim_t \sim_t A \vdash_t A$

$r_t1.\ A \vdash_t B, B \vdash_t C / A \vdash_t C$
$r_t2.\ A \vdash_t B, A \vdash_t C / A \vdash_t B \wedge_t C$
$r_t3.\ A \vdash_t C, B \vdash_t C / A \vee_t B \vdash_t C$
$r_t4.\ A \vdash_t B / \sim_t B \vdash_t \sim_t A.$

We shall refer to this proof system as $\mathbf{FDE}^t_t = (\mathcal{L}_t, \vdash_t)$.

Theorem

For all A, B $\in \mathcal{L}_t$, $A \vdash_t B$ *iff* $A \models^n_t B$.

We obtain the system $\mathbf{FDE}^f_f = (\mathcal{L}_f, \vdash_f)$ by replacing $\wedge_t$, $\vee_t$, $\sim_t$ and $\vdash_t$ in the axioms and rules of $\mathbf{FDE}^t_t$ by $\wedge_f$, $\vee_f$, $\sim_f$ and $\vdash_f$, respectively.

Theorem

For any $A, B \in \mathcal{L}_f$: $A \vDash^n_f B$ *iff* $A \vdash_f B$.

The completeness proofs make use of a not unsophisticated canonical model construction.

Priest does not supply his definition of generalized connectives with a theoretical justification except of a short remark that this way of defining propositional connectives is "obvious" (Priest 1984, p. 237). However, its obviousness notwithstanding, Priest's Definition gives rise to some problems:

- The definition cannot be naturally extended to a construction that would allow the empty set to enter at every stage. If $S^g_{n+1} := \mathcal{P}(S_n)$, then, as Priest himself mentions, the definition gives the extension of any truth functor according to the rule "gap-in, gap-out". E.g., for S^g_1 so defined, $\varnothing \wedge_1 x = \varnothing \vee_1 x = \sim_1\varnothing = \varnothing$. But such an extension of S_1 would not be identical (as one could expect) with $FOUR_2$, where, e.g., $\mathbf{N} \wedge \mathbf{F}$ amounts to $\mathbf{F}$ and not to $\mathbf{N}$.

- The approach proposed by Priest cannot naturally be extended to the sets **4**, **16**, etc., but it also cannot be applied to the set $S'_1 = \mathbf{3}'$ $(= \{\mathbf{F}, \mathbf{T}, \mathbf{N}\})$ taken as the set of truth values of Kleene's *strong* three-valued logic and its possible generalizations.
  As P. Jain (1997, § 4) points out, this situation is caused by the fact that Priest's definition treats truth functions in terms of the members of each argument, but $\varnothing$ has no members.

Priest's $S_2 = \mathbf{7} = \{\mathbf{F}, \mathbf{T}, \mathbf{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB}\}$. By applying to this set the multilattice approach, its algebraic structure constitutes what can be called a *bi-and-a-half*-lattice $SEVEN_{2.5}$.

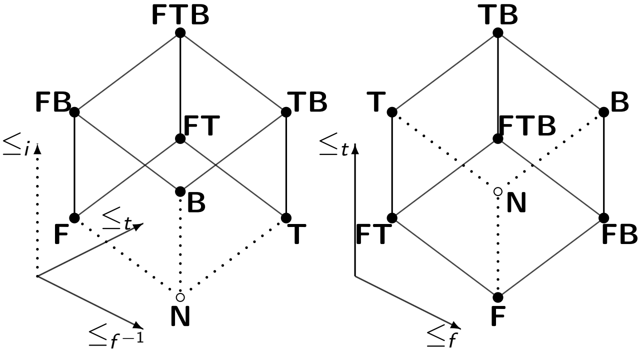


Figure: Bi-and-a-half-lattice $SEVEN_{2.5}$ and trilattice $EIGHT_3$

One can clearly observe here the complete lattices under $\leq_t$ and $\leq_f$, but the information order is merely a *semi*lattice with **FTB** as a top, but with no bottom. However, $SEVEN_{2.5}$ can be directly extended to a trilattice $EIGHT_3$ by adding **N** as a bottom element for $\leq_i$. The dotted lines in Figure 3 present the result of such an extension.

Note that $EIGHT_3$ is *not* a Belnap trilattice.

Proposition

It is impossible in $SEVEN_{2.5}$ to define pure t-inversion.

Analogously it is not difficult to show that there exists no pure f-inversion in $SEVEN_{2.5}$.

Following (Dunn and Hardegree, 2001) we call a unary operation $-_j$ a subminimal j-inversion iff it satisfies the earlier conditions (*anti*) and (*iso*). That is, a subminimal inversion, although it reverses the corresponding partial order, is not necessarily an involution.

Adding to (*anti*) and (*iso*) the condition $x \leq_j -_j -_j x$ would give a so-called quasiminimal j-inversion.

It turns out that a subminimal t-inversion and f-inversion can be defined in $SEVEN_{2.5}$ as presented in the following table:

a	$-_t a$	$-_f a$
F	**TB**	**F**
T	**T**	**FB**
B	**F**	**T**
FT	**TB**	**FB**
FB	**FB**	**T**
TB	**F**	**TB**
FTB	**FTB**	**FTB**

Table: Inversions in $SEVEN_{2.5}$

Consider again the language $\mathcal{L}_{tf}$. Valuations v^7 and their extension to compound formulas are introduced in the usual way, as well as corresponding entailment relations: $\models^7_t$ and $\models^7_f$. As a result we get logics $(\mathcal{L}_{tf}, \models^7_t)$ and $(\mathcal{L}_{tf}, \models^7_f)$ (semantically defined) as well as their fragments (e.g. the logic $(\mathcal{L}_t, \models^7_t)$ etc.).

Summary

- Revenge Liar arguments can be used to motivate the move from the set **2** of classical truth values to infinitely many generalized truth values obtained by a suitable generalization procedure such as the one suggested by Priest.
- We have considered iterated powerset formation applied to the set **4** and introduced Belnap trilattices. These structures give rise to relations of truth entailment and falsity entailment. Our main result is the observation that the logic of truth and the logic of falsity for every Belnap trilattice is one and the same, namely First Degree Entailment.

- We have seen that the lattice-based approach significantly differs from Priest's construction based on the sets of values $\mathcal{P}(S_n) \setminus \{\varnothing\}$. In particular, Priest's procedure for generalizing **3** cannot naturally be extended to **4** (or be applied to Kleene's set of truth values $\mathbf{3}'$ and its possible generalizations). Moreover, no pure t-inversion or f-inversion can be defined on the lattice structure of Priest's set of generalized truth values S_2 ($= \mathbf{7}$).

- Only in Belnap trilattices we have been able to define mutually independent truth and falsity orderings, and this gives us in a most natural way a richer logical vocabulary and as a result a richer logical landscape. First steps towards exploring this landscape have been taken in (Shramko ad Wansing 2005).