Universal Logic III
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Tutorial on Truth Values
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Lecture 2, Part II

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1. **Two ways of presenting a many-valued logic**

1. **Harmonious many-valued logics**

2. **Example where the two approaches might, perhaps, be expected to lead to the same logic**

3. **Entailment relations as truth values**

4. **Summary**

The standard way of presenting a many-valued logic is by singling out a non-empty set of designated values and defining entailment as the preservation of membership in this set of designated values from the premises to the conclusions.

In other words, a propositional many-valued logic according to this approach is given by a matrix, a non-empty set $\mathcal{V}$ of truth values (alias truth degrees) containing at least two elements, a nonempty set $\mathcal{D} \subseteq \mathcal{V}$ of designated truth values, and a set of truth functions $\{f_c \mid c \in \mathcal{C}\}$, where $\mathcal{C}$ is a finite set of primitive finitary connectives, f_c and c having the same arity.

It is well-known that the truth values of certain many-valued logics constitute lattices.

According to Arieli and Avron (1996) it is even the case that "[w]hen using multiple-valued logics, it is usual to order the truth values in a lattice structure".

A slightly less standard way of defining a many-valued logic proceeds by (i) defining a lattice order $\leq$ on the set $\mathcal{V}$, (ii) interpreting logical operations as operations on the lattice $(\mathcal{V}, \leq)$, and (iii) stipulating that a formula A entails a formula B iff for every homomorphic valuation function v, $v(A) \leq v(B)$.

The most well-known example of the latter approach is provided by the bilattice $FOUR_2$, which gives rise to the logic of first-degree entailment (FDE), also called the logic of "tautological entailment", or Belnap's and Dunn's useful four-valued logic.

In $FOUR_2$ the lattice order used to define entailment is interpreted as a truth order on the set of generalized truth values $\mathbf{4} = \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$, where $\mathbf{N} = \varnothing$, $\mathbf{T} = \{T\}$, $\mathbf{F} = \{F\}$, $\mathbf{B} = \{T, F\}$, and $\{T, F\}$ is the set of classical truth values *true* and *false*.

FDE has a matrix presentation, and the coincidence of the matrix and the lattice presentation may be seen as mutual support for the naturalness of both first-degree entailment and the two ways of presenting this logic.

Nevertheless, the two ways of defining a many-valued logic are generally non-equivalent in the sense that not every matrix presentation gives rise to an equivalent lattice presentation.

In the bilattice $FOUR_2$ the information order $\leq_i$ is set-inclusion, and the truth order ($\leq_t$) is based on assumptions about truth *and* falsity.

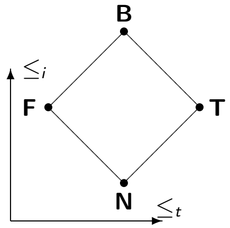

Figure: The bilattice $FOUR_2$

Conjunction $\wedge$ may be interpreted as lattice meet and disjunction $\vee$ as lattice join with respect to $\leq_t$. Negation $\sim$ may be interpreted as truth order inversion satisfying the double-negation laws. This interpretation is captured by the following tables for the corresponding truth functions:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

The relation $A \models_t^4 B$ between formulas A and B is defined by requiring that for every homomorphic valuation function v from the propositional language into **4** the following holds: $v(A) \leq_t v(B)$.

It is well-known that $\models_t^4$ is exactly FDE.

The relation $\models_t^4$ can also be described in terms of the preservation of designated truth values. Consider the matrix $B_4 = \langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, with its associated entailment relation $\models_4^+$ defined as:

$$A \models_4^+ B \text{ iff } \forall v\, (v(A) \in \mathcal{D} = \{\mathbf{T}, \mathbf{B}\} \text{ implies } v(B) \in \mathcal{D}).$$

The designated truth values are the values which contain the classical value T, and we have $\models_4^+ = \models_t^4$.

If we change the definition of designated values and put, e.g., $\mathcal{D} = \{\mathbf{T}\}$, the two ways of defining entailment do not result in one and the same relation. We obtain the *ex falso quodlibet* entailment $(A \wedge \sim A) \models_4^+ B$, for all formulas A and B.

In Wansing and Shramko (2008) we introduce a so-called "separated 4-valued propositional logic" B_4* which makes use of *antidesignated* values. The idea is to define a many-valued logic by introducing *two sets* of distinguished truth values, namely a set $\mathcal{D}^+$ of *designated* values associated with truth and another set $\mathcal{D}^-$ of *antidesignated* values associated with falsity.

Such a distinction is well-known. It leaves room for values that are *neither* designated *nor* antidesignated and for values that are *both* designated *and* antidesignated. Gottwald and also Malinowski impose the condition that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$.

An extensional n-valued ($2 \leq n \in \mathbb{N}$) propositional logic is a structure

$$\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle,$$

where $\mathcal{V}$ is a non-empty set containing n elements ($2 \leq n$), $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, $\mathcal{C}$ is the (non-empty, finite) set of (primitive) connectives of some propositional language $\mathcal{L}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. The elements of $\mathcal{V}$ are called truth values, and the elements of $\mathcal{D}$ are regarded as the *designated* truth values. A structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$ may be viewed as a logic, because the set of designated truth values determines an entailment relation $\models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$, where $\mathcal{P}(\mathcal{L})$ is the powerset of $\mathcal{L}$.

A valuation function v is a function from the set of all atomic formulas into $\mathcal{V}$. Every valuation function v is inductively extended to a function from the set of all $\mathcal{L}$-formulas into $\mathcal{V}$ by the following definition:

$$v(c(A_1, \ldots, A_m)) \;\; = \;\; f_c(v(A_1), \ldots, v(A_m)),$$

where c is an m-place connective from $\mathcal{C}$. A set of formulas Δ *entails* a set of formulas Γ ($\Delta \models \Gamma$) iff for every valuation function v the following holds true: if for every $A \in \Delta$, $v(A) \in \mathcal{D}$, then $v(B) \in \mathcal{D}$ for some $B \in \Gamma$.

An n-valued tautology is a formula A such that $\varnothing \models A$.

Definition

An *n*-valued propositional logic is a structure

$$\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$$

where $\mathcal{V} \neq \varnothing$ $(2 \leq |\,\mathcal{V}\,|)$, $\mathcal{D}^+, \mathcal{D}^- \subseteq \mathcal{V}$, $\varnothing \neq \mathcal{D}^+ \neq \mathcal{D}^- \neq \varnothing$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. Valuation functions v are defined as usual. For all sets of $\mathcal{L}$-formulas Δ, Γ, *two* semantic consequence relations $\models^+$ and $\models^-$ are defined as follows:

- $\Delta \models^+ \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}^+$, then $v(B) \in \mathcal{D}^+$ for some $B \in \Gamma$);
- $\Delta \models^- \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}^-$, then $v(B) \in \mathcal{D}^-$ for some $B \in \Delta$).

An n-valued tautology then is a formula A such that $\varnothing \models^+ A$, and an n-valued contradiction is a formula A such that $A \models^- \varnothing$.

Definition

Let $\Lambda = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$ be an n-valued propositional logic. Λ is called a *separated* n-valued logic, if $\mathcal{V} \setminus \mathcal{D}^+ \neq \mathcal{D}^-$ (that is if $\mathcal{V}$ is not partitioned into the non-empty sets $\mathcal{D}^+$ and $\mathcal{D}^-$), and Λ is said to be *refined*, if it is separated and $\models^+ \neq \models^-$.

Clearly, if an n-valued logic is not separated, the two entailment relations $\models^+$ and $\models^-$ coincide. In a refined n-valued propositional logic, however, neither "positive" entailment $\models^+$ nor "negative" entailment $\models^-$ enjoys a privileged status in comparison to each other.

One might expect that the relations $\models^+$ and $\models^-$ come with their own languages, $\mathcal{L}^+$ and $\mathcal{L}^-$, and one might be interested in n-valued logics in which these languages have the same signature.

Definition

Let $\mathcal{C}$ be a finite non-empty set of finitary connectives, let $\mathcal{L}^+$ be the language based on $\mathcal{C}^+ = \{c^+ \mid c \in \mathcal{C}\}$, and let $\mathcal{L}^-$ be the language based on $\mathcal{C}^- = \{c^- \mid c \in \mathcal{C}\}$. If A is an $\mathcal{L}^+$-formula, let A^- be the result of replacing every connective c^+ in A by c^-. If Δ is a set of $\mathcal{L}^+$ formulas, let $\Delta^- = \{A^- \mid A \in \Delta\}$. If the language $\mathcal{L}$ of a refined n-valued logic Λ is based on $\mathcal{C}^+ \cup \mathcal{C}^-$, then Λ is said to be *harmonious* iff (i) for all sets of $\mathcal{L}^+$-formulas Δ, Γ: $\Delta \models^+ \Gamma$ iff $\Delta^- \models^- \Gamma^-$, and (ii) for every $c \in \mathcal{C}$, $f_{c^+} \neq f_{c^-}$.

Definition

Kleene's strong 3-valued logic K_3 and Łukasiewicz's 3-valued logic Ł$_3$ are defined as follows:

- $K_3 = \langle \{T, \varnothing, F\}, \{T\}, \{\varnothing, F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined as follows:

$f_\sim$	
T	F
$\varnothing$	$\varnothing$
F	T

$f_\wedge$	T	$\varnothing$	F
T	T	$\varnothing$	F
$\varnothing$	$\varnothing$	$\varnothing$	F
F	F	F	F

$f_\vee$	T	$\varnothing$	F
T	T	T	T
$\varnothing$	T	$\varnothing$	$\varnothing$
F	T	$\varnothing$	F

$f_\supset$	T	$\varnothing$	F
T	T	$\varnothing$	F
$\varnothing$	T	$\varnothing$	$\varnothing$
F	T	T	T

- Ł$_3 = \langle \{T, \varnothing, F\}, \{T\}, \{\varnothing, F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined as in K_3 except that:

$f_\supset$	T	$\varnothing$	F
T	T	$\varnothing$	F
$\varnothing$	T	T	$\varnothing$
F	T	T	T

Definition

The separated 3-valued propositional logics K_3* and $Ł_3$* are defined as follows:

- $K_3{*} := \langle\{T, \varnothing, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined as in K_3;
- $Ł_3{*} := \langle\{T, \varnothing, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined as in $Ł_3$.

Observation

K_3 and $Ł_3$* are refined, i.e., the relations $\models^+$ and $\models^-$ do not coincide.*

Proof.

In both logics $A \wedge (A \supset B)$ has the same truth table:

	A	B	$A \wedge (A \supset B)$
	T	T	T
	T	$\varnothing$	$\varnothing$
	T	F	F
	$\varnothing$	T	$\varnothing$
	$\varnothing$	$\varnothing$	$\varnothing$
$*$	$\varnothing$	F	$\varnothing$
	F	T	F
	F	$\varnothing$	F
	F	F	F

Whereas $A \wedge (A \supset B) \models^+ B$, $A \wedge (A \supset B) \not\models^- B$. □

Although it is not surprising, perhaps, that in a separated *n*-valued logic the relations $\models^+$ and $\models^-$ need not coincide, there are separated *n*-valued logics which are not refined. Let $\mathbf{N} := \varnothing$, $\mathbf{T} := \{T\}$, $\mathbf{F} := \{F\}$ and $\mathbf{B} := \{T, F\}$.

Definition

The useful 4-valued logic of Dunn and Belnap is the structure $B_4 = \langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{N}, \mathbf{F}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as follows:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

Definition

The separated 4-valued propositional logic B_4* is the structure $\langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee, \}\}\rangle$, where the functions f_c are defined as in B_4.

The set $\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$ is also referred to as **4**. Note that in B_4* not only $\mathbf{4} \setminus \mathcal{D}^+ \neq \mathcal{D}^-$, but also $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$.

Observation (M. Dunn)

The separated logic B_4 is not refined: $\models^+ = \models^-$.*

That is, in *FDE* not only the matrix and the lattice presentations turn out to be coincident ($\models^+ = \models_t$), but so are the two entailment relations defined in terms of designated and antidesignated truth values.

The following questions naturally arise:

Are there harmonious finitely-valued propositional logics?

Given a set of truth values, how is the choice of the designated (and antidesignated) truth values justified?

A corresponding question concerning the lattice approach is:

Given a set of truth values, how is the definition of the lattice order(s) on this set justified?

Arieli and Avron (1996) explain that "[f]requently, in an algebraic treatment of the subject, the set of the designated values forms a filter, or even a prime (ultra-) filter, relative to some natural ordering of the truth values".

In $FOUR_2$ truth and falsity are not dealt with as independent of each other, because it is assumed that F by itself is less true than T, and hence, in $FOUR_2$ $\{T, F\}$ is taken to be less true than $\{T\}$.

The set of designated values of B_4 might be justified by pointing out that it is a prime filter with respect to $\leq_t$ in $FOUR_2$, but if the truth order $\leq_t$ itself is not convincingly justified, this criterion remains fairly technical.

If we move from the set **4** of generalized truth values to its powerset $\mathcal{P}(\mathbf{4}) = \mathbf{16}$, we may not only obtain a truth order on the underlying set of values that is better justified than the truth order of $FOUR_2$ but in addition also an independent and equally well-justified falsity order.

Against the background of these separate truth and falsity orderings the definition of sets of designated values in terms of containment of T and in terms of of containment of F appears to be quite natural. Interestingly, in this case the two ways of defining a many-valued propositional logic do not converge.

Recall the propositional language $\mathcal{L}_{tf}$:

$$\mathcal{L}_{tf}: \quad A ::= \quad p \mid \sim_t \mid \sim_f \mid \wedge_t \mid \vee_t \mid \wedge_f \mid \vee_f$$

Let v be a map from the set of propositional variables into **16**. The function v is recursively extended to a function from the set of all $\mathcal{L}_{tf}$-formulas into **16** as follows:

1. $v\,(A \wedge_t B) = v\,(A) \sqcap_t v\,(B)$;
2. $v\,(A \vee_t B) = v\,(A) \sqcup_t v\,(B)$;
3. $v\,(\sim_t A) = -_t v\,(A)$;
4. $v\,(A \wedge_f B) = v\,(A) \sqcup_f v\,(B)$;
5. $v\,(A \vee_f B) = v\,(A) \sqcap_f v\,(B)$;
6. $v\,(\sim_f A) = -_f v\,(A)$.

Following the lattice approach, the logic of the trilattice $SIXTEEN_3$ is semantically presented as a *bi-consequence system*, namely the structure $(\mathcal{L}_{tf}, \models_t, \models_f)$, where the two entailment relations $\models_t$ and $\models_f$ are defined with respect to the truth order $\leq_t$ and the falsity order $\leq_f$, respectively.

Definition

The set-to-set entailment relations $\models_t \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models_f \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\Delta \models_t \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcap_t \{v(A) \mid A \in \Delta\} \leq_t \textstyle\bigsqcup_t \{v(A) \mid A \in \Gamma\};$$

$$\Delta \models_f \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcup_f \{v(A) \mid A \in \Gamma\} \leq_f \textstyle\bigsqcap_f \{v(A) \mid A \in \Delta\}.$$

$(\mathcal{L}_{tf}, \models_t, \models_f)$ induces a 16-valued logic $(\mathcal{L}_{tf}, \models^+, \models^-)$ in the language $\mathcal{L}_{tf}$.

Definition

B_{16} is the logic

$$\langle \mathbf{16}, \{x \in \mathbf{16} \mid x^t \neq \varnothing\}, \{x \in \mathbf{16} \mid x^f \neq \varnothing\}, \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle$$

i.e., $\mathcal{D}^+ = \{x \in \mathbf{16} \mid x^t \neq \varnothing\ \}$, and $\mathcal{D}^- = \{x \in \mathbf{16} \mid x^f \neq \varnothing\ \}$. For all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, the relations $\models^+$ and $\models^-$ are canonically defined as follows:

- $\Delta \models^+ \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}^+$, then $v(B) \in \mathcal{D}^+$ for some $B \in \Gamma$);
- $\Delta \models^- \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}^-$, then $v(B) \in \mathcal{D}^-$ for some $B \in \Delta$).

Observation

B_{16} is refined.

Proof.

It can easily be seen that in B_{16} the relations $\models^+$ and $\models^-$ are distinct, e.g. in view of the following counterexample:
$(A \wedge_f B) \models^- A$ but $(A \wedge_f B) \not\models^+ A$ (since, e.g.,
$\mathbf{F} \sqcup_f \mathbf{FTB} = \mathbf{FB}$). □

Theorem

The 16-valued propositional logic B_{16} is harmonious.

We may note that the corresponding entailment relations of the logics $(\mathcal{L}_{tf}, \models_t, \models_f)$ and $(\mathcal{L}_{tf}, \models^+, \models^-)$ are distinct: $\models_t \neq \models^+$ and $\models_f \neq \models^-$.

We may, for example, observe that for every $\mathcal{L}_{tf}$-formula A, $A \models^+ \sim_f A$ and $A \models^- \sim_t A$, whereas there exists no atomic $\mathcal{L}_{tf}$-formula A, such that $A \models_t \sim_f A$ or $A \models_f \sim_t A$.

a	$-_t a$	$-_f a$	$-_i a$	$-_{tf} a$	$-_{ti} a$	$-_{fi} a$	$-_{tfi} a$
N	**N**	**N**	**A**	**N**	**A**	**A**	**A**
N	T	F	**NFT**	B	**NTB**	**NFB**	**FTB**
F	B	N	**NFB**	T	**FTB**	**NFT**	**NTB**
T	N	B	**NTB**	F	**NFT**	**FTB**	**NFB**
B	F	T	**FTB**	N	**NFB**	**NTB**	**NFT**
NF	**TB**	**NF**	**NF**	**TB**	**TB**	**NF**	**TB**
NT	**NT**	**FB**	**NT**	**FB**	**NT**	**FB**	**FB**
FT	**NB**	**NB**	**NB**	**FT**	**FT**	**FT**	**NB**
NB	**FT**	**FT**	**FT**	**NB**	**NB**	**NB**	**FT**
FB	**FB**	**NT**	**FB**	**NT**	**FB**	**NT**	**NT**
TB	**NF**	**TB**	**TB**	**NF**	**NF**	**TB**	**NF**
NFT	**NTB**	**NFB**	N	**FTB**	T	F	B
NFB	**FTB**	**NFT**	F	**NTB**	B	N	T
NTB	**NFT**	**FTB**	T	**NFB**	**N**	B	F
FTB	**NFB**	**NTB**	B	**NFT**	F	T	N
A	**A**	**A**	**N**	**A**	**N**	**N**	**N**

Table: Inversions in $SIXTEEN_3$

This observation poses the following problem:

Does the logic $(\mathcal{L}_{tf}, \models_t, \models_f)$ induced by the trilattice $SIXTEEN_3$ have an adequate matrix presentation?

Also, a more general question arises: Under which conditions does the lattice presentation of a many-valued logic allow an equivalent matrix presentation and *vice versa*?

$SIXTEEN_3$ is an example of a *Belnap trilattice*. Belnap trilattices are obtained by iterated powerset-formation applied to **4** and by generalizing the definitions of a truth order and a falsity order on **16**. If X is a set, let $\mathcal{P}^1(X) := \mathcal{P}(X)$ and $\mathcal{P}^n(X) := \mathcal{P}(\mathcal{P}^{n-1}(X))$ for $1 < n$, $n \in \mathbb{N}$. Each of the sets $\mathcal{P}^n(\mathbf{4})$ can be equipped with relations $\leq_i$, $\leq_t$, and $\leq_f$ as $SIXTEEN_3$ by defining for every x, y $\in \mathcal{P}^n(\mathbf{4})$ the sets x^t, x^{-t}, x^f and x^{-f} as follows:

$$x^t := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \in y_{n-1}\}$$

$$x^{-t} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \in y_{n-1}\}$$

$$x^f := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ F \in y_{n-1}\}$$

$$x^{-f} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ F \in y_{n-1}\}$$

Thus, $x^{-t} = x \setminus x^t$ and $x^{-f} = x \setminus x^f$.

We say that x is t-positive (t-negative, f-positive, f-negative) iff x^t (x^{-t}, x^f, x^{-f}) is non-empty.

Definition

For every x, y in $\mathcal{P}^n(\mathbf{4})$:

- $x \leq_i y$ iff $x \subseteq y$;
- $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
- $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

Definition

A Belnap trilattice is a structure

$$\mathcal{M}_3^n := (\mathcal{P}^n(\mathbf{4}), \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f),$$

where $\sqcap_i$ ($\sqcap_t$, $\sqcap_f$) is the lattice meet and $\sqcup_i$ ($\sqcup_t$, $\sqcup_f$) is the lattice join with respect to the ordering $\leq_i$ ($\leq_t$, $\leq_f$) on $\mathcal{P}^n(\mathbf{4})$, $n \geq 1$.

Thus, $SIXTEEN_3$ ($= \mathcal{M}_3^1$) is the smallest Belnap trilattice.

Observation

If $\mathcal{M}_3^n$ is a Belnap trilattice, then there exist operations of t-inversion and f-inversion on $\mathcal{P}^n(\mathbf{4})$.

We consider again the language $\mathcal{L}_{tf}$. An *n*-valuation is a function v^n from the set of atoms into $\mathcal{P}^n(\mathbf{4})$. This function can be extended to an interpretation of arbitrary formulas as before.

Definition

The relations $\models^n_t \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models^n_f \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\Delta \models^n_t \Gamma \quad \text{iff} \quad \forall v^n \ \textstyle\bigsqcap_t \{v^n(A) \mid A \in \Delta\} \leq_t \bigsqcup_t \{v^n(A) \mid A \in \Gamma\};$$

$$\Delta \models^n_f \Gamma \quad \text{iff} \quad \forall v^n \ \textstyle\bigsqcup_f \{v^n(A) \mid A \in \Gamma\} \leq_f \bigsqcap_f \{v^n(A) \mid A \in \Delta\}.$$

Semantically, the logic of a Belnap trilattice $\mathcal{M}^n_3$ is the bi-consequence system $(\mathcal{L}_{tf}, \models^n_t, \models^n_f)$.

We now define an infinite chain of separated finitely-valued logics.

Definition

Let $\sharp n$ be the cardinality of $\mathcal{P}^n(\mathbf{4})$. The $\sharp n$-valued logic $B_{\sharp n}$ is the structure $\langle \mathcal{P}^n(\mathbf{4}), \mathcal{D}^{n+}, \mathcal{D}^{n-}, \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle$, where $\mathcal{D}^{n+} := \{x \in \mathcal{P}^n(\mathbf{4}) \mid x$ is t-positive$\}$ and $\mathcal{D}^{n-} := \{x \in \mathcal{P}^n(\mathbf{4}) \mid x$ is f-positive$\}$. For every logic $B_{\sharp n}$, for all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, the semantic consequence relations $\models^{n+}$ and $\models^{n-}$ are defined as for (generalized) n-valued logics.

Observation

For every $n \in \mathbb{N}$, the logic $B_{\sharp n}$ is refined.

Theorem

For every $n \in \mathbb{N}$, the logic $B_{\sharp n}$ is harmonious.

G. Malinowski calls any structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set with at least two elements, $\mathcal{D}^+$ and $\mathcal{D}^-$ are distinct non-empty proper subsets of $\mathcal{V}$ such that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$, and every f_c is a function on $\mathcal{V}$ with the same arity as c, an n-valued q-matrix (quasi-matrix). If it is not required that $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$, we may talk of generalized q-matrices.

A valuation function v in $\mathfrak{M}$ is a function from $\mathcal{L}$ into the set of truth degrees $\mathcal{V}$, and we may restrict our attention to valuations which satisfy the recursive conditions stated above.

Malinowski also defines a kind of relation, called *q-entailment*, which depends on both sets $\mathcal{D}^+$ and $\mathcal{D}^-$. A q-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\} \rangle$ determines a q-entailment relation $\models_{\mathfrak{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by defining $\Delta \models_{\mathfrak{M}} A$ iff for every valuation v in $\mathfrak{M}$, $v(\Delta) \cap \mathcal{D}^- = \varnothing$ implies $v(A) \in \mathcal{D}^+$, where $v(\Delta) = \{v(B) \mid B \in \Delta\}$.

A q-entailment relation in general is not reflexive and does not admit of a reduction to a bivalent semantics.

Every q-matrix is an example of what is called a *ramified* matrix in (Wójcicki 1988). In a ramified matrix there are finitely many distinguished sets of values $\mathcal{D}_1$, ..., $\mathcal{D}_j$.

If we draw the distinction between a set of designated values $\mathcal{D}^+$ and antidesignated values $\mathcal{D}^-$ there are various alternatives for defining entailment. The following definitions can be found in the literature:

$$A \models_t B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+ \tag{1}$$

$$A \models_f B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \notin \mathcal{D}^- \tag{2}$$

$$A \models_q B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \in \mathcal{D}^+ \tag{3}$$

$$A \models_p B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \notin \mathcal{D}^- \tag{4}$$

We will refer to these relations as t-entailment, f-entailment, q-entailment and p-entailment, correspondingly.

Whereas t-entailment is the standard truth-preserving relation, f-entailment incorporates the idea of non-falsity preservation. The relation of q-entailment can be seen as reflecting a reasoning from hypotheses (understood as statements that merely are taken to be non-refuted). This relation has been introduced by Malinowski (together with the underlying concept of a q-matrix)
in order to provide a counterexample to Suszko's Thesis (the claim that every many-valued logic can be characterized by a two-valued semantics). And p-entailment (p for "plausibility") has been studied by Frankowski (2004), who tried to explicate "reasonings wherein the degree of strength of the conclusion (i.e. the conviction it is true) is smaller than that of the premisses".

Theorem

Recall that the Belnap generalized q-matrix B_4^* *is the structure* $\langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$ *(functions* f_c *being defined as in the usual Belnap matrix) and consider on this structure the above four entailment relations. Then* $\models_t = \models_f$ *and* $\models_q = \models_p$*. Moreover,* $\models_t$ *(alias* $\models_f$*)* $\neq \models_p$ *(alias* $\models_q$*).*

In our particular example of a *generalized four-valued q*-matrix, it turned out that $\models_t = \models_f$ and $\models_q = \models_p$. Interestingly, in a non-generalized setting the picture may become even more complex.

Let the (ordinary) *Kleene-Priest q-matrix* KP_3^* be the structure $\langle \{T, I, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as in K_3 and P_3, and let us consider the entailment relations defined by (1)–(4) with respect to this matrix.

Then, these relations are *all distinct*, $\models_t$ is the entailment relation of Kleene's logic, and $\models_f$ corresponds to the entailment of Priest's Logic of Paradox (cf. (Dunn 2000)). Moreover, the following proposition exposes some simple facts about the interconnections between the entailments:

Observation

In the Kleene-Priest q-matrix: (i) $\models_q \subseteq \models_t$*; (ii)* $\models_q \subseteq \models_f$*; (iii)* $\models_t \subseteq \models_p$*; (iv)* $\models_f \subseteq \models_p$*; (v)* $\models_q \subset \models_t \cap \models_f$*.*

Facts (i)–(iv) are mentioned in (Devyatkin 2007). Note, that they actually reveal that in KP_3^* our four entailment relations are ordered by means of $\subseteq$ to form a lattice with $\models_q$ as the bottom and $\models_p$ as the top. At this place an analogy comes to mind with the information ordering in bilattice $FOUR_2$ which is defined as set-inclusion. It turns out that it is not the only possible analogy one can draw here.

Let $\models_x^+$ ($x \in \{t, f, q, p\}$) stand for the part of $\models_x$ which only preserves designated values and $\models_x^-$ for the part of $\models_x$ that only preserves non-antidesignated values:

$$\models_x^+ := \{(A,B) \in \models_x \mid (\forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+ \text{ and } \exists v : v(A) \notin \mathcal{D}^- \text{ and } v(B) \in \mathcal{D}^-)\} \quad (5)$$

$$\models_x^- := \{(A,B) \in \models_x \mid (\forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \notin \mathcal{D}^- \text{ and } \exists v : v(A) \in \mathcal{D}^+ \text{ and } v(B) \notin \mathcal{D}^+)\} \quad (6)$$

Intuitively $\models_x^+$ and $\models_x^-$ can be seen as representing correspondingly the "pure truth content" and the "pure falsity content" of $\models_x$. One can define then a "truth order" between the entailment relations:

$$\models_x \leq_t \models_y \text{ iff } \models_x^+ \subseteq \models_y^+ \text{ and } \models_y^- \subseteq \models_x^- \tag{7}$$

The following proposition shows how the four entailment relations can be organized into a logical lattice with $\models_f$ as the bottom and $\models_t$ as the top:

Observation

In the Kleene-Priest q-matrix: (i) $\models_f \leq_t \models_q$; *(ii)* $\models_f \leq_t \models_p$; *(iii)* $\models_q \leq_t \models_t$; *(iv)* $\models_p \leq_t \models_t$.

By combining the previous observations we immediately see that the entailment relations defined in the q-matrix KP_3^* constitute a structure isomorphic to the bilattice $FOUR_2$, where $\models_t$ is analogous to $\mathbf{T}$, $\models_f$ plays the role of $\mathbf{F}$, $\models_q$ stands for $\mathbf{N}$, and $\models_p$ is like $\mathbf{B}$. In this way we obtain another representation of a four-valued logic whose values are formed by the entailment relations defined on the basis of a three-valued quasi-matrix.

A generalized q-matrix is *proper* iff it satisfies the conditions: $\mathcal{D}^+ \cup \mathcal{D}^- \neq \mathcal{D}$ and $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$.

We had in view the concrete q-matrix KP_3^* and the concrete generalized q-matrix B_4^*. Nevertheless, one can observe that the observation about entailment relations as truth values can be extended to *any* generalized q-matrix, provided it is proper and its truth functions allow dualization, i.e., for any valuation v there exists a valuation v^* subject to the following conditions (for any formula A):

$$v(A) \in \mathcal{D}^+ \text{ and } v(A) \notin \mathcal{D}^- \Leftrightarrow v^*(A) \in \mathcal{D}^+ \text{ and } v^*(A) \notin \mathcal{D}^- \quad (8)$$

$$v(A) \notin \mathcal{D}^+ \text{ and } v(A) \in \mathcal{D}^- \Leftrightarrow v^*(A) \notin \mathcal{D}^+ \text{ and } v^*(A) \in \mathcal{D}^- \quad (9)$$

$$v(A) \in \mathcal{D}^+ \text{ and } v(A) \in \mathcal{D}^- \Leftrightarrow v^*(A) \notin \mathcal{D}^+ \text{ and } v^*(A) \notin \mathcal{D}^- \quad (10)$$

$$v(A) \notin \mathcal{D}^+ \text{ and } v(A) \notin \mathcal{D}^- \Leftrightarrow v^*(A) \in \mathcal{D}^+ \text{ and } v^*(A) \in \mathcal{D}^- \quad (11)$$

The four truth values of B_4^* can generally be modeled by the following conditions: $v(A) = \mathbf{T}$ iff $v(A) \in \mathcal{D}^+$ and $v(A) \notin \mathcal{D}^-$; $v(A) = \mathbf{F}$ iff $v(A) \notin \mathcal{D}^+$ and $v(A) \in \mathcal{D}^-$; $v(A) = \mathbf{B}$ iff $v(A) \in \mathcal{D}^+$ and $v(A) \in \mathcal{D}^-$; $v(A) = \mathbf{N}$ iff $v(A) \notin \mathcal{D}^+$ and $v(A) \notin \mathcal{D}^-$. Then it is not difficult to rewrite the proof concerning entailment relations on B_4^* in a general form so that it holds for any proper generalized q-matrix (with dual valuations).

Summary

- There exist two ways of defining a many-valued logic: the matrix approach and the lattice approach.
- The general notion of a many-valued logic (the notion of a generalized q-matrix) leads to the notion of harmonious many-valued logics.
- Examples of harmonious many-valued logics are induced by Belnap trilattices.
- Generalized q-matrices suggest four natural notions of semantic consequence: t-entailment, f-entailment, q-entailment, and p-entailment.
- There exists a simple but fairly general method of constructing a generalized four-valued q-matrix by taking as its values the basic entailment relations defined on an arbitrary non-generalized q-matrix.