

Refutability and Post Completeness

TOMASZ SKURA

Abstract

The goal of this paper is to give a necessary and sufficient condition for a multiple-conclusion consequence relation to be Post complete by using proof/refutation systems.

Keywords. refutation systems, Post completeness, multiple-conclusion logic.

1 Introduction

When we deal with the set L of propositional formulas valid in some structures, we are usually interested in laws and inference rules generating valid formulas. However, we can also consider non-laws and refutation rules generating non-valid formulas. As a result, we have a pair $\mathbf{S} = (\mathcal{T}, \mathcal{F})$ of complementary inference systems (\mathcal{T} for L and \mathcal{F} for $-L$), which can be called a proof/refutation system. Such systems enable provability and refutability on the propositional level. In a manner of speaking, we have two engines rather than one.

The idea of syntactic refutability was introduced by Łukasiewicz [3] (but it was already known to Aristotle), and it is now used in computer science (see e.g. [5], [2], [1]). For an introduction to refutation systems see [11].

In this paper we show that this method provides a natural characterization of Post completeness.

Post completeness can be defined in a couple of ways. Here, in view of possible generalizations, it is convenient to define it as follows. A logic (or a logical theory¹) L is Post complete iff L is consistent and L has no consistent proper extension. This definition was adopted by (among others) McKinsey [4] and Segerberg [8]. (Note that this definition is different from that in [9].)

¹A logical theory is a set of formulas closed under a consequence relation.

The fact that Classical Logic (that is, the set CL of Boolean laws) is Post complete can be established by showing the following.

For any formula $A \notin CL$ there is a substitution s such that $s(A) \rightarrow p \in CL$, where p is a fixed propositional variable.

In other words, for every formula A

either $A \in CL$ or A is refutable in the Łukasiewicz refutation system:

Refutation axiom: p

Refutation rules:

(*reverse substitution*) $s(A)/A$, where $s(A)$ is a substitution instance of A

(*reverse modus ponens*) B/A , where $A \rightarrow B \in CL$

Thus, if this refutation system is complete for CL , then CL is Post complete. On the other hand, it can be shown that if CL is Post complete, then the system is complete for CL . This fact suggests a close relationship between Post completeness and syntactic refutability.

Post completeness is such a nice property. However, although there are infinitely many Post complete logics (see [4], [8]), it seems that there is no interesting standard non-classical logic that is Post complete. What happens if the term logic is construed more generally? There are two possibilities.

(1) A *logic* is a consequence relation \vdash between finite sets of formulas and formulas. Then the set $L_{\vdash} = \{A : \vdash A\}$ of theorems is a logical theory. We now say that a logic \vdash is Post complete iff \vdash is consistent and has no consistent proper extension. However, it turns out that \vdash is Post complete iff L_{\vdash} is Post complete (see e.g. [6]), so that in this case nothing happens.

(2) A *logic* is a consequence relation $\vdash\vdash$ between finite sets of formulas (or a multiple-conclusion consequence relation). Again, we say that $\vdash\vdash$ is Post complete iff $\vdash\vdash$ is consistent and has no consistent proper extension. Then the relation $\vdash = \{X/A : X \vdash\vdash A\}$ (where X is a set of formulas and A is a formula) is a logic in the sense defined in (1). Interestingly, the situation is now dramatically different. For example, consider any non-classical logic L that is not Post complete. Let $\vdash^L = \{X/Y : \text{If } X \subseteq L \text{ then } Y \cap L \neq \emptyset\}$ (so \vdash^L is the set of multiple-conclusion inferences preserving L). Then \vdash^L is Post complete (cf. Corollary 3.4).

In this paper we give a necessary and sufficient condition (in terms of proof/refutation systems) for a multiple-conclusion consequence relation to be Post complete.

2 Preliminaries

Let FOR be the set of formulas generated from the propositional variables p, p_1, p_2, \dots by standard connectives. By an *inference* we mean a pair X/A , where X is a finite set of formulas and $A \in FOR$. And by a *rule* \mathcal{R} we mean a set of inferences. Note that if $\mathcal{R}, \mathcal{R}'$ are rules, then so is $\mathcal{R} \cup \mathcal{R}'$, so that a number of rules can be presented as a single rule. An inference system is a pair (AX, RU) , where AX is a set of formulas (axioms) and RU is a set of rules. Since axioms A can be regarded as rules \emptyset/A , any inference system can be presented as a rule.

A *consequence relation* is a relation \vdash between finite sets of formulas and formulas satisfying the following conditions.

- $A \vdash A$.
- If $X \vdash A$ then $X' \vdash A$, where $X \subseteq X'$.
- If $X \vdash A$ and $X, A \vdash B$, then $X \vdash B$.

Every rule \mathcal{R} determines a consequence relation $\vdash_{\mathcal{R}}$ defined as follows.

$X \vdash_{\mathcal{R}} A$ iff A is derivable from X by \mathcal{R} , that is, there is a sequence A_1, \dots, A_n such that $A_n = A$ and each A_i is in X or is obtained by \mathcal{R} .

We say that a set Z of formulas is closed under a rule \mathcal{R} (or \mathcal{R} preserves Z) iff for all $X/A \in \mathcal{R}$, if $X \subseteq Z$ then $A \in Z$. We remark that if \mathcal{R} preserves Z , then so does $\vdash_{\mathcal{R}}$.

A *multiple-conclusion inference* (or a sequent) is a pair X/Y , where X, Y are finite sets of formulas. And a *multiple-conclusion rule* (or a sequent rule) is a set of sequents.

A *multiple-conclusion consequence relation* is a relation $\vdash\vdash$ between finite sets of formulas satisfying the following conditions.

- $X \vdash\vdash Y$ if $X \cap Y \neq \emptyset$.
- If $X \vdash\vdash Y$ then $X' \vdash\vdash Y'$, where $X \subseteq X', Y \subseteq Y'$.
- If $X \vdash\vdash A, Y$ and $X, A \vdash\vdash Y$, then $X \vdash\vdash Y$.

We say that $\vdash\vdash$ is *consistent* iff for no A both $\vdash\vdash A$ and $A \vdash\vdash$. And we say that $\vdash\vdash$ is *complete* iff for all A , either $\vdash\vdash A$ or $A \vdash\vdash$. Since $FOR \neq \emptyset$, we have: $\vdash\vdash$ is consistent iff $\emptyset \not\vdash\vdash \emptyset$.

By an *extension* (proper extension) of a consequence relation $\vdash\vdash$ we mean a consequence relation $\vdash\vdash'$ such that $\vdash\vdash \subseteq \vdash\vdash'$ ($\vdash\vdash \subset \vdash\vdash'$).

Every sequent rule Σ determines a multiple-conclusion consequence relation \vdash_{Σ} defined as follows. $X \vdash_{\Sigma} Y$ iff there is a finite sequence $\alpha_1, \dots, \alpha_n$ of sequents such that $\alpha_n = X/Y$ and each α_i is in Σ or is obtained from preceding sequents by one of the following rules.

- (R) $\frac{}{X/Y}$ where $X \cap Y \neq \emptyset$
(M) $\frac{X/Y}{X'/Y'}$ where $X \subseteq X', Y \subseteq Y'$
(C) $\frac{X/A, Y \quad X, A/Y}{X/Y}$

3 Consequence Relations

Lemma 3.1 (Scott [7]) *If a consequence relation \vdash is Post complete, then it is complete.*

PROOF Assume that \vdash is Post complete, but it is not complete. Since \vdash is consistent, there are finite sets $X_0, Y_0 \subseteq FOR$ such that $X_0 \not\vdash Y_0$. And since \vdash is not complete, there is $A \in FOR$ such that both $\not\vdash A$ and $A \not\vdash$. We now use the technique introduced by Scott in the proof of Proposition 1.3 [7]. We define \vdash_0, \vdash_1 as follows.

$X \vdash_0 Y$ iff $X \vdash A, Y$

$X \vdash_1 Y$ iff $X, A \vdash Y$

for all finite $X, Y \subseteq FOR$. It is easy to check that \vdash_0, \vdash_1 are consequence relations. Also, both \vdash_0 and \vdash_1 are proper consistent extensions of \vdash . Hence $X_0 \vdash_0 Y_0$ and $X_0 \vdash_1 Y_0$ (because \vdash is Post complete). But then $X_0 \vdash Y_0$ (by C), which is a contradiction. QED

Lemma 3.2 *Let \vdash be a consistent consequence relation.*

(i) *If both $\vdash A$ for all $A \in X$ and $B \vdash$ for all $B \in Y$, then $X \not\vdash Y$.*

(ii) *If \vdash is complete and $X \not\vdash Y$, then both $\vdash A$ for all $A \in X$ and $B \vdash$ for all $B \in Y$.*

PROOF (i) By induction on the number $|X \cup Y|$ of formulas in $X \cup Y$.

(1) $|X \cup Y| = 0$. Then $X = Y = \emptyset$, so $X \not\vdash Y$, and this is true.

(2) $|X \cup Y| > 0$. Then, say, $X \neq \emptyset$. (If $Y \neq \emptyset$, the argument is similar.)

So there is a formula $D \in X$. Suppose that $\vdash A$ for all $A \in X$ and $B \vdash$ for all $B \in Y$, but $X \not\vdash Y$. Then, in particular, $\vdash D$, so $X^- \vdash D, Y$ (by M),

where $X^- = X - \{D\}$. Also $X^-, D \vdash Y$. Hence $X^- \vdash Y$ (by C). Since $|X^- \cup Y| < |X \cup Y|$, by the induction hypothesis, we get $X^- \not\vdash Y$, which is a contradiction.

(ii) Assume that \vdash is complete and $X \not\vdash Y$, but either $\not\vdash A$ for some $A \in X$ or $B \not\vdash$ for some $B \in Y$. Then either $A \vdash$ for some $A \in X$ or $\vdash B$ for some $B \in Y$. Hence $X \vdash Y$ (by M), which is a contradiction. QED

Lemma 3.3 *Let \vdash be a consequence relation. If \vdash is consistent and complete, then it is Post complete.*

PROOF Assume that \vdash is consistent and complete, but it is not Post complete. then there is a consistent relation \vdash' such that $\vdash \subset \vdash'$, so there are finite sets $X, Y \subseteq FOR$ such that $X \vdash' Y$ and $X \not\vdash Y$. Since \vdash is complete (by Lemma 3.2 (ii)), we have: $\vdash A$ for all $A \in X$ and $B \vdash$ for all $B \in Y$, so

$\vdash' A$ for all $A \in X$ and $B \vdash'$ for all $B \in Y$ (because $\vdash \subseteq \vdash'$).

Hence, by Lemma 3.2 (i), $X \not\vdash' Y$, which is a contradiction. QED

Corollary 3.4 *A consequence relation \vdash is Post complete iff it is consistent and complete.*

PROOF From Lemma 3.1 and Lemma 3.3. QED

The following simple facts will be useful in Section 4.

Proposition 3.5 *Let \vdash be a consequence relation.*

(i) *If $X \vdash A_i$ for all $1 \leq i \leq n$ and $X, A_1, \dots, A_n \vdash Y$, then $X \vdash Y$.*

(ii) *If $X \vdash A_i$ for all $1 \leq i \leq n$ and $A_1, \dots, A_n \vdash Y$, then $X \vdash Y$.*

PROOF (i) By induction on n .

(1) $n = 1$. Assume $X \vdash A_1$ and $X, A_1 \vdash Y$. Then $X \vdash A_1, Y$ (by M), so $X \vdash Y$ (by C).

(2) $n > 1$. Assume $X \vdash A_i$ for all $1 \leq i \leq n$ and $X, A_1, A_2, \dots, A_n \vdash Y$. Then $X, A_2, \dots, A_n \vdash A_1, Y$ (by M), so $X, A_2, \dots, A_n \vdash Y$ (by C). Also $X \vdash A_i$ for all $2 \leq i \leq n$. Hence, by the induction hypothesis, we get $X \vdash Y$.

(ii) Assume $X \vdash A_i$ for all $1 \leq i \leq n$ and $A_1, \dots, A_n \vdash Y$. Then $X, A_1, \dots, A_n \vdash Y$ (by M), so $X \vdash Y$ (by (i)). QED

Proposition 3.6 *Let \vdash be a consequence relation.*

- (i) *If $X \vdash Y, A_1, \dots, A_n$ and $X, A_i \vdash Y$ for all $1 \leq i \leq n$, then $X \vdash Y$.*
- (ii) *If $X \vdash A_1, \dots, A_n$ and $A_i \vdash Y$ for all $1 \leq i \leq n$, then $X \vdash Y$.*

PROOF (i) By induction on n .

(1) $n = 1$. Then this holds by C .

(2) $n > 1$. Assume $X \vdash Y, A_1, \dots, A_n$ and $X, A_i \vdash Y$ for all $1 \leq i \leq n$.

Then $X, A_1 \vdash Y, A_2, \dots, A_n$ and $X \vdash A_1, Y, A_2, \dots, A_n$ (by M). Hence $X \vdash Y, A_2, \dots, A_n$ (by C). Also $X, B \vdash Y$ for all $B \in Y$ (by R). Therefore, by the induction hypothesis, $X \vdash Y$.

(ii) From (i). QED

Proposition 3.7 *Let \mathcal{R} be an inference rule.*

(i) *If $X \vdash_{\mathcal{R}} A$ then $X \vdash_{\mathcal{R}} A$.*

(ii) *If $X \vdash_{\mathcal{R}} A$ and $\mathcal{R}' = \{B/Y : Y/B \in \mathcal{R}\}$, then $A \vdash_{\mathcal{R}'} X$.*

PROOF (i) Assume that $X \vdash_{\mathcal{R}} A$. Then there is a derivation of A from X by \mathcal{R} , that is, a sequence A_1, \dots, A_n such that $A_n = A$ and each A_i is in X or is obtained by \mathcal{R} . We show, by induction on n , that there is a sequence $\alpha_1, \dots, \alpha_m$ such that $\alpha_m = X/A$ and each α_i is in \mathcal{R} or is obtained by R, M, C (so that $X \vdash_{\mathcal{R}} A$).

(1) $n = 1$. Then $A_1 = A$.

(Case 1) $A \in X$. Then $\alpha_1 = X/A$ is obtained by R .

(Case 2) A is obtained by \mathcal{R} . Then $\emptyset/A \in \mathcal{R}$ and $\alpha_1 = \emptyset/A \in \mathcal{R}$.

(2) $n > 1$ and we assume that every shorter derivation of B from X by \mathcal{R} has a sequent derivation of X/B from \mathcal{R} by R, M, C . We may also assume that $A \notin X$ (otherwise see (1)). Then A is obtained from some $B_1, \dots, B_k \in \{A_1, \dots, A_{n-1}\}$ and $B_1, \dots, B_k/A \in \mathcal{R}$. By the induction hypothesis, for every $1 \leq i \leq k$, X/B_i is derivable from \mathcal{R} by R, M, C . Hence, by Proposition 3.5 (ii), so is X/A , so that $X \vdash_{\mathcal{R}} A$.

(ii) The proof is similar to that of (i). The important modification is the following.

(Case 2) A is obtained from some $B_1, \dots, B_k \in \{A_1, \dots, A_{n-1}\}$ and $B_1, \dots, B_k/A \in \mathcal{R}$. We have $A/B_1, \dots, B_k \in \mathcal{R}'$. By the induction hypothesis, for every $1 \leq i \leq k$, B_i/X is derivable from \mathcal{R}' by R, M, C . Hence, by Proposition 3.6 (ii), so is A/X , so that $A \vdash_{\mathcal{R}'} X$. QED

4 Proof/Refutation Systems

Definition (i) Let $L \subseteq FOR$. A *proof/refutation system* for L is a pair

$$\mathbf{S} = (\mathcal{T}, \mathcal{F})$$

of inference rules such that \mathcal{T} preserves L and \mathcal{F} preserves $-L$.

(We also say that \mathbf{S} is a *proof/refutation system* iff \mathbf{S} is a proof/refutation system for some $L \subseteq FOR$.)

(ii) \mathbf{S} is *complete* for L iff for every $A \in FOR$ we have

$$\vdash_{\mathcal{T}} A \text{ or } \vdash_{\mathcal{F}} A$$

(iii) The *multiple-conclusion consequence relation* determined by \mathbf{S} is the relation \vdash_{Σ} , where

$$\Sigma = \mathcal{T} \cup \{A/X : X/A \in \mathcal{F}\}$$

Lemma 4.1 *Let $L \subseteq FOR$, and let \mathbf{S} be a complete proof/refutation system for L .*

- (i) $A \in L$ iff $\vdash_{\mathcal{T}} A$, and $A \notin L$ iff $\vdash_{\mathcal{F}} A$.
- (ii) \vdash_{Σ} preserves L .
- (iii) \vdash_{Σ} is consistent.
- (iv) $A \in L$ iff $\vdash_{\Sigma} A$, and $A \notin L$ iff $A \vdash_{\Sigma}$.
- (v) \vdash_{Σ} is complete.

PROOF (i) Since \mathcal{T} preserves L and \mathcal{F} preserves $-L$, we have:

If $\vdash_{\mathcal{T}} A$ then $A \in L$, and if $\vdash_{\mathcal{F}} A$ then $A \notin L$.

We show that if $A \in L$ then $\vdash_{\mathcal{T}} A$, and if $A \notin L$ then $\vdash_{\mathcal{F}} A$.

Suppose that $A \in L$ ($A \notin L$) but $\not\vdash_{\mathcal{T}} A$ ($\not\vdash_{\mathcal{F}} A$). Since \mathbf{S} is complete, this gives $\vdash_{\mathcal{F}} A$ ($\vdash_{\mathcal{T}} A$), so $A \notin L$ ($A \in L$), which is a contradiction.

(ii) This follows from the fact that Σ preserves L and this property is preserved by R, M, C . (For example, to check M , assume that X, Y preserves L . If $X' \subseteq L$, then $X \subseteq L$, so $Y \cap L \neq \emptyset$, so $Y' \cap L \neq \emptyset$, which means that X'/Y' preserves L .)

(iii) Suppose that $\vdash_{\Sigma} A$ and $A \vdash_{\Sigma}$ for some $A \in FOR$. Then $\emptyset \vdash \emptyset$, which contradicts (ii).

(iv) We first show that

(*) if $A \in L$ then $\vdash_{\Sigma} A$, and if $A \notin L$ then $A \vdash_{\Sigma}$.

Assume that $A \in L$ ($A \notin L$). Then, by (i), we have $\vdash_{\mathcal{T}} A$ ($\vdash_{\mathcal{F}} A$). Hence, by Proposition 3.7, we get $\vdash_{\mathcal{T}} A$ ($A \vdash_{\mathcal{F}}$), so $\vdash_{\Sigma} A$ ($A \vdash_{\Sigma}$), as required.

Now suppose that $\vdash_{\Sigma} A$ ($A \vdash_{\Sigma}$) but $A \notin L$ ($A \in L$). Then, by (*), $A \vdash_{\Sigma}$ ($\vdash_{\Sigma} A$), which contradicts (iii).

(v) For every $A \in FOR$ we have either $A \in L$ or $A \notin L$. Hence, by (iv), either $\vdash_{\Sigma} A$ or $A \vdash_{\Sigma}$. QED

Corollary 4.2 *If \mathbf{S} is a complete proof/refutation system, then \vdash_{Σ} is Post complete.*

PROOF From Lemma 3.3 and Lemma 4.1 (iii,v).

Lemma 4.3 *If \vdash is Post complete, then*

$$\vdash = \vdash_{\Sigma}$$

for some complete proof/refutation system \mathbf{S} .

PROOF Assume that \vdash is Post complete. Then \vdash is consistent (by definition) and complete (by Lemma 3.1). Let $\mathbf{S} = (\mathcal{T}, \mathcal{F})$, where

$$\mathcal{T} = \{\emptyset/A : \vdash A\} \text{ and } \mathcal{F} = \{\emptyset/B : B \vdash\}.$$

And let $L = \{A : \vdash A\}$. Then we have:

(1) \mathbf{S} is a proof/refutation system for L .

Indeed, if $\emptyset/A \in \mathcal{T}$, then $\vdash A$, so $A \in L$, so that \mathcal{T} preserves L .

And if $\emptyset/A \in \mathcal{F}$ (so $A \vdash$) but $A \in L$, then $\vdash A$, so \vdash is inconsistent, which is a contradiction, so that \mathcal{F} preserves $-L$.

(2) \mathbf{S} is complete.

Indeed, we have either $A \in L$ or $A \notin L$.

If $A \in L$, then $\vdash A$, so $\emptyset/A \in \mathcal{T}$, and thus $\vdash_{\mathcal{T}} A$.

And if $A \notin L$, then $\not\vdash A$, so $A \vdash$ (for \vdash is complete), so $\emptyset/A \in \mathcal{F}$, and thus $\vdash_{\mathcal{F}} A$.

(3) $\vdash = \vdash_{\Sigma}$.

(3.1) $\vdash_{\Sigma} \subseteq \vdash$.

Indeed, by Lemma 3.2, we have:

(\star) $X \vdash Y$ iff either $A \vdash$ for some $A \in X$ or $\vdash B$ for some $B \in Y$.

Note that Σ satisfies (\star) and the rules R, M, C preserve (\star). For example, to check C , assume that both $X, A \vdash Y$ and $X \vdash A, Y$ satisfy (\star), but

$X \vdash Y$ does not. Then $\vdash B$ for all $B \in X$ and $D \vdash$ for all $D \in Y$. Hence $A \vdash$ and $\vdash A$, which is impossible for \vdash is consistent.

(3.2) $\vdash \subseteq \vdash_{\Sigma}$.

Indeed, Let $X \vdash Y$. Then $A \vdash$ for some $A \in X$ or $\vdash B$ for some $B \in Y$. Thus, we have $A \vdash_{\Sigma}$ or $\vdash_{\Sigma} B$. Hence $X \vdash_{\Sigma} Y$ (by M), as required. QED

Theorem *A consequence relation \vdash is Post complete iff \vdash is determined by a complete proof/refutation system.*

PROOF From Corollary 4.2 and Lemma 4.3.

Remark Thus, Post complete logics and complete proof/refutation systems are two sides of one coin. Of course, there are various kinds of proof/refutation systems. The one given in the proof of Lemma 4.3 is rather trivial. Genuine proof systems (enabling proof search) for standard logics are well-known. And genuine refutation systems (enabling refutation search) are possible. For example, such systems for standard modal logics (and for Classical Logic) are given in [10, 12].

References

- [1] Fiorentini, C.: Terminating sequent calculi for proving and refuting formulas in S4. *Journal of Logic and Computation* (2012) doi: 10.1093/log-com/exs053
- [2] Goré, R. and L. Postniece. Combining derivations and refutations for cut-free completeness in bi-intuitionistic logic, *Journal of Logic and Computation* **20** (2010), 233-260
- [3] Łukasiewicz, J.: *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*. Oxford (1951)
- [4] McKinsey, J.C.C.: On the number of complete extensions of the Lewis systems of sentential calculus. *Journal of Symbolic Logic* **9** (1944), 42-45
- [5] Pinto, L. and R. Dyckhoff: Loop-free construction of counter-models for intuitionistic propositional logic. In: Behara, Fritsch, and Lintz (eds.) *Symposia Gaussiana* (1995), 225-232
- [6] Pogorzelski, W. and P. Wojtylak: *Elements of the Theory of Completeness in Propositional Logic*. Katowice (1982)

- [7] Scott, D.: Completeness and axiomatizability in many-valued logic. In: Proceedings of the Tarski Symposium, American Mathematical Society, Providence (1974)
- [8] Segerberg, K.: Post completeness in modal logic. *Journal of Symbolic Logic* **37** (1972), 711-715
- [9] Skura, T.: A refutation theory. *Logica Universalis* **3** (2009), 293-302
- [10] Skura, T.: On refutation rules. *Logica Universalis* **5** (2011), 249-254
- [11] Skura, T.: Refutation systems in propositional logic. In: Gabbay, D. and F. Guenther (eds.) *Handbook of Philosophical Logic*, vol. 16, 115-157 (2011)
- [12] Skura, T.: *Refutation Methods in Modal Propositional Logic*. Wydawnictwo Naukowe Semper, Warszawa (2013), forthcoming
- e-mail: T.Skura@ifil.uz.zgora.pl